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Estimation based on lower record values from exponentiated Pareto distribution†

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Abstract

In this paper, we aim to estimate two scale-parameters of exponentiated Pareto distribution (EPD) based on lower record values. Record values arise naturally in many real life applications involving data relating to weather, sport, economics and life testing studies. We calculate the Bayesian estimators for the two parameters of EPD based on lower record values. The Bayes estimators of two parameters for the EPD with lower record values under the squared error loss (SEL), linex loss (LL) and entropy loss (EL) functions are provided. Lindley's approximate method is used to compute these estimators. We compare the Bayesian estimators in the sense of the bias and root mean squared estimates (RMSE).

Keywords: Bayesian estimation, balanced loss function, exponentiated Pareto distribution, Lindley's approximation, lower record values.

1. Introduction

Record values arise naturally in many real life applications involving data relating to weather, sport, economics and life testing studies. Many authors have studied record values and associated statistics; for example, see Abd-El-Hakim and Sultan (2001), Arnold *et al.* (1998), Balakrishnan and Chan (1993), Sultan and Moshref (2000), Lee (2017) and Lee and Cho (2015). Arnold and Press (1983) proposed the Pareto distribution, named after the Italian civil engeneer, economist, and sociologist Vilfredo Pareto. Chandler (1952) defined the so-called record times and record values and gave groundwork for a mathematical theory of records. Arnold *et al.* (1998) summarized the study of record values and documented many of the basic properties of records as follows:

Suppose that $X_1, X_2, \cdots$ be an infinite sequence of independent and identically distributed (i.i.d) random variables with probability density function (pdf) $f(x)$ in equation (Eq.) (1.1),

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cumulatative density function (cdf) $F(x)$ in Eq. (1.2). Let $Y_j = max(X_1, X_2, \cdots, X_j)$ for $j \geq 1$. We say X_j an upper record if $Y_j > Y_{j-1}, j > 1$. Also, Let $Y_j = min(X_1, X_2, \cdots, X_j)$ for $j \geq 1$. We say X_j an lower record if $Y_j < Y_{j-1}, j > 1$. We will assume that X_j occurs at time j, then the record time sequence is defined as, $L(1) = 1$ and $L(m) = max(min)\left\{j : X_j > (<) X_{L(m-1)}\right\}$.

The Pareto distribution is a power law probability distribution that is used in description of social, scientific, geophysical, actuarial, and many other types of observable phenomena. Gupta *et al.* (1998) proposed to model failure data by $F^*(t) = [F(t)]^\theta$, where $F(t)$ is the baseline distribution function and θ is a positive real number. Mudholkar and Hutson (1996) studied the properties of the exponentiated weibull family, Nassar and Eissa (2003) studied Bayesian estimation for the exponentiated weibull model. In the same way Gupta and Kundu (1999, 2001a, 2001b, 2006), Raqab (2002, 2004), Raqab and Ahsanullah (2001) and Zheng (2002) studied the proposed model with exponential as the baseline distribution which is the generalized exponential family (or exponentiated exponential distribution).

Gupta *et al.* (1998) showed that the exponentiated Pareto distribution (EPD) can be used quite effectively in analyzing many lifetime data. Nadarajah (2005) introduced various exponentiated Pareto distribution and derive several of their properties including the moment generating function, expectation, variance, skewness, kurtosis, Shannon entropy, and the Renyi entropy. The probability density function (pdf) and cumulative density function (cdf) of EPD with two shape parameter θ and λ is

$$f(x) = \theta\lambda[1-(1+x)^{-\lambda}]^{\theta-1}(1+x)^{-(\lambda+1)}, x > 0, \lambda > 0, \theta > 0, \tag{1.1}$$

$$F(x) = [1-(1+x)^{-\lambda}]^{\theta}, x > 0, \lambda > 0, \theta > 0. \tag{1.2}$$

For more properties, see Shawky and Abu-Zinadah (2008). Shawky and Abu-Zinadah (2008) studied characterizations and maximum likelihood estimation of the exponentiated Pareto distribution based on record values.

The purpose of this paper is to consider Bayesian estimation for the exponentiated Pareto distribution under lower record values. We show Bayesian estimation using three balanced loss functions with Lindley's approximation. Finally, we compared Bayesian estimation using balanced loss functions with maximum likelihood estimation.

This paper is organized as follows: In Section 2, we review the maximum likelihood estimation (MLE) using Shawky and Abu-Zinadah (2008). In Section 3, the Bayesian estimation under balanced loss function using Lindley's approximation is obtained. The format of the Lindley's approximation is followed Kim *et al.* (2011), obtained the Bayes estimators of exponentiated weibull model based on progressive type II censoring scheme. Calulating Bayesian estimation is difficult. Thus we use Lindley's method to approximate all the Bayesian estimates. We analyzed and estimated real data set in Section 4. In Section 5, Bayesian estimation is computed using Monte Carlo method.

2. Maximum likelihood estimation

Shawky and Abu-Zinadah (2008) studied maximum likelihood estimation (MLE) of two shape parameters of the exponentiated Pareto distribution based on lower record values using fixed point solution as follows: The log-likelihood function is given by

$$l = lnL = mln\lambda + mln\theta - \sum_{i=1}^{m} ln[1 - V_i^{-\lambda}] - (\lambda + 1)\sum_{i=1}^{m} lnV_i + \theta ln[1 - V_m^{-\lambda}].$$

The normal equations become:

$$\frac{\partial l}{\partial \theta} = m + \theta ln(1 - V_m^{-\lambda}) = 0 \tag{2.1}$$

and

$$\frac{\partial l}{\partial \lambda} = m - \lambda \sum_{i=1}^{m} \left(\frac{lnV_i}{1 - V_i^{-\lambda}} \right) + \lambda\theta \left(\frac{V_m^{-\lambda} lnV_m}{1 - V_m^{-\lambda}} \right) = 0. \tag{2.2}$$

From Eq. (2.1), we obtain the MLE of θ as a function of λ, say $\hat{\theta}(\lambda)$, where

$$\hat{\theta}(\lambda) = \frac{-m}{ln[1 - V_m^{-\lambda}]}. \tag{2.3}$$

Therefore, MLE of λ, say $\hat{\lambda}_M$, can be obtained by solving Eq. (2.3) for respect to λ. It can be shown that the maximum of Eq. (2.2) can be obtained as fixed point solution of the following equation:

$$h(\lambda) = \lambda,$$

where

$$h(\lambda) = \left[m \sum_{i=1}^{m} \frac{lnV_i}{1 - V_i^{-\lambda}} - \theta \frac{V_m^{-\lambda} lnV_m}{1 - V_m^{-\lambda}} \right]^{-1}.$$

3. Bayes estimation

It is assumed here that the parameters θ and λ are independent. Each prior for θ and λ is gamma distribution as follows:

$$\pi_\theta(\theta) \propto \theta^{a_1 - 1} e^{-b_1\theta},\ a_1 > 0,\ b_1 > 0 \ \text{ and } \ \pi_\lambda(\lambda) \propto \lambda^{a_2 - 1} e^{-b_2\lambda},\ a_2 > 0,\ b_2 > 0,$$

where a_i and b_i are known for $i = 1, 2$.

So, the joint prior distribution of θ and λ is of the form

$$\pi(\theta, \lambda) \propto \theta^{a_1 - 1} e^{-b_1\theta} \lambda^{a_2 - 1} e^{-b_2\lambda},\ \theta > 0,\ \lambda > 0.$$

Based on the likelihood function and the joint prior distribution, the joint density of the θ, λ and $X = (X_1, X_2, \cdots, X_m)$, can be written as follows:

$$\pi(\theta, \lambda, X) \propto \theta^{a_1 + m - 1} \lambda^{a_2 + m - 1} e^{-b_1\theta - b_2\lambda} \frac{\prod_{i=1}^{m} [1 - V_i^{-\lambda}]^{\theta - 1} V_i^{-(\lambda + 1)}}{\prod_{i=1}^{m} [1 - V_i^{-\lambda}]^{\theta}}.$$

Then the posterior distribution of θ and λ, given X is obtained as

$$\pi(\theta,\lambda|X) \propto \frac{\pi(\theta,\lambda,X)}{\int_0^\infty \int_0^\infty \pi(\theta,\lambda,X)d\theta d\lambda}.$$

Now, we obtain Bayesian estimates of θ and λ about the loss functions when the prior distribution is taken to be $\pi(\theta,\lambda)$. Bayesian estimates of $g(\theta,\lambda)$ obtained as:

$$E\left[g(\theta,\lambda)|X\right] = \frac{\int_0^\infty g(\theta,\lambda)\pi(\theta,\lambda,X)d\theta d\lambda}{\int_0^\infty \int_0^\infty \pi(\theta,\lambda,X)d\theta d\lambda}.$$

3.1. Balanced loss function

In this section, we derive the Bayes estimators for parameters θ and λ of EPD under symmetric as well asymmetric loss functions. A very well known symmetric loss function is the squared error loss (SEL) function which is defined as $L_1[d(\theta),\hat{d}(\theta)] = [\hat{d}(\theta) - d(\theta)]^2$ with $\hat{d}(\theta)$ being an estimate of $d(\theta)$. Here $d(\theta)$ denotes some parametric function of θ. For this situation the Bayesian estimate, say $\hat{d}_s(\theta)$, is given by the posterior mean of $d(\theta)$.

One of the most commonly used asymmetric loss function is the linex loss (LL) function which is defined by

$$L_2[d(\theta),\hat{d}(\theta)] = exp[h(\hat{d}(\theta) - d(\theta))] - h[\hat{d}(\theta) - d(\theta)] - 1,\ h \neq 0.$$

In this case Bayes estimate of $d(\theta)$ is obtained as

$$\hat{d}_L(\theta) = -\frac{1}{h} ln[E_\theta(exp(-h\theta)|X)],$$

provided the above expectation exists. The sign and magnitude of the shape parameter h represents the direction and degree of symmetry respectively. For more details about LL see Calabria and Pulcini (1996).

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\hat{d}(\theta)/d(\theta)$. In this case, a useful asymmetric loss function is the general entropy loss (EL) function proposed by Calabria and Pulcini (1996) :

$$L_3[d(\theta),\hat{d}(\theta)] \propto \left[\frac{\hat{d}(\theta)}{d(\theta)}\right]^q - qln\left[\frac{\hat{d}(\theta)}{d(\theta)}\right] - 1,\ q \neq 0.$$

In this case Bayes estimate of $d(\theta)$ is obtained as

$$\hat{d}_E(\theta) = [E_\theta(\theta^{-q})|X)]^{-1/q}$$

provided the above expectation exists.

From a Bayesian perspective, choice of loss function is an essential part in the estimation and prediction problems. Recently, a more generalized loss function called the balanced loss function (Jozani *et al.*, 2012) of the form

$$L_{\rho,\omega,\delta_0}(\theta,\delta) = \omega\rho(\theta,\delta_0) + (1-\omega)\rho(\theta,\delta), \tag{3.1}$$

where ρ is an arbitrary loss function, while δ_0 is a chosen a prior target estimator of θ, obtained for instance using the criterion of maximum likelihood estimator and the weight ω takes values in $[0, 1)$. A general development with regard to Bayesian estimators under L_{ρ,ω,δ_0} is given, namely by relating such estimators to Bayesian solutions to the unbalanced case, i.e., L_{ρ,ω,δ_0} with $\omega = 0$. L_{ρ,ω,δ_0} can be specialized to various choices of loss function, such as for SEL, LL and EL function. (Ahmadi *et al.*, 2009)

Let L_4, L_5 and L_6 are Balanced loss functions against SEL function L_1, LL function L_2 and EL function L_3. By choosing $\rho(\sigma, \delta) = (\delta - \sigma)^2$, Eq. (3.1) reduced to the balanced squared error loss (BSEL) function, in the form

$$L_4(d(\theta), \hat{d}(\theta)) = \omega \left(\hat{d}(\theta) - d_0(\theta)\right)^2 + (1-\omega)\left(\hat{d}(\theta) - d(\theta)\right)^2$$

and the corresponding Bayes estimate of the $d(\theta)$ is obtained as

$$\hat{d}_{BS}(\theta) = \omega\theta_0 + (1-\omega)E[\theta|Y]. \tag{3.2}$$

By choosing $\rho(\theta, \delta) = exp[h(\delta-\theta)] - h(\delta-\theta) - 1;\ h \neq 0$, Eq. (3.1) reduced to the balanced linex loss (BLL) function, in the form

$$\begin{aligned} L_5\left(d(\theta), \hat{d}(\theta)\right) &= \omega\left[exp\left\{h\left(\hat{d}(\theta) - d_0(\theta)\right)\right\} - \left\{h\left(\hat{d}(\theta) - d_0(\theta)\right)\right\}\right] \\ &\quad + (1-\omega)\left[exp\left\{h\left(\hat{d}(\theta) - d(\theta)\right)\right\} - \left\{h\left(\hat{d}(\theta) - d(\theta)\right)\right\}\right] \end{aligned}$$

and the corresponding Bayes estimate of the $d(\theta)$ is obtained as

$$\hat{d}_{BL}(\theta) = -\frac{1}{h} ln\left\{\omega exp(-h\theta_0) + (1-\omega)E[exp(-h\theta)|Y]\right\}. \tag{3.3}$$

By choosing $\rho(\theta, \delta) = (\theta/\delta)^q - qln(\theta/\delta) - 1,\ q \neq 0$, Eq. (3.1) reduced to the balanced general entropy loss (BEL) function, in the form

$$L_6(d(\theta), \hat{d}(\theta)) = \omega\left[\left(\frac{\hat{d}(\theta)}{d_0(\theta)}\right)^q - qln\left(\frac{\hat{d}(\theta)}{d_0(\theta)}\right) - 1\right] + (1-\omega)\left[\left(\frac{\hat{d}(\theta)}{d(\theta)}\right)^q - qln\left(\frac{\hat{d}(\theta)}{d(\theta)}\right) - 1\right]$$

and the corresponding Bayes estimate of the $d(\theta)$ is obtained as

$$\hat{d}_{BE}(\theta) = \left\{\omega\theta_0^{-q} + (1-\omega)E[\theta^{-q}|Y]\right\}^{-1/q}. \tag{3.4}$$

It is clear that the balanced loss functions are more general, which include the maximum likelihood estimate and both symmetric and asymmetric Bayes estimates as special cases.

Based on BSEL function, given by Eq. (3.2), the approximate Bayes estimates of the θ and λ are give, respectively by

$$\hat{\theta}_{BS_\omega} = \omega\hat{\theta}_M + (1-\omega)\hat{\theta}_S \quad \text{and} \quad \hat{\lambda}_{BS_\omega} = \omega\hat{\lambda}_M + (1-\omega)\hat{\lambda}_S.$$

Also, based on BLL function, given by Eq. (3.3), the approximate Bayes estimates of the θ and λ are give, respectively by

$$\hat{\theta}_{BL_w} = -\frac{1}{h} ln[\omega exp(-h\hat{\theta}_M) + (1-\omega)\hat{\theta}_L]$$

and

$$\hat{\lambda}_{BL_w} = -\frac{1}{h} ln[\omega exp(-h\hat{\lambda}_M) + (1-\omega)\hat{\lambda}_L].$$

Further, based on BEL function, given by Eq. (3.4), the approximate Bayes estimates of the θ and λ are give, respectively by

$$\hat{\theta}_{BE_\omega} = [\omega\hat{\theta}_M^{-q} + (1-\omega)\hat{\theta}_E^{-q}]^{-1/q} \text{ and } \hat{\lambda}_{BE_\omega} = [\omega\hat{\lambda}_M^{-q} + (1-\omega)\hat{\lambda}_E^{-q}]^{-1/q}.$$

3.2. Lindley's approximation

In this subsection, based on a lower record values we obtained several Bayesian estimates of θ and λ of a EPD. These Bayesian estimates are derived against SEL, LL and EL functions. It is easily observed that all these estimates are in the form of ratio of two integrals for which simplified closed forms are not available. Thus we use Lindley's method to approximate all the Bayesian estimates.

For the two-parameter case, using notation $(\lambda_1, \lambda_2) = (\theta, \lambda)$, using Lindley's approximation, it follows from Kim *et al.* (2011) that

$$\hat{k} = k(\hat{\lambda}_1, \hat{\lambda}_2) + \frac{1}{2}\left(A + l_{30}B_{12} + l_{03}B_{21} + l_{21}C_{12} + l_{12}C_{21}\right) + p_1A_{12} + p_2A_{21}, \tag{3.5}$$

where

$$A = \sum_{i=1}^{2}\sum_{j=1}^{2}\omega_{ij}\tau_{ij},\ l_{ij} = \frac{\partial^{i+j}L(\lambda_1,\lambda_2)}{\partial\lambda_1^i\partial\lambda_2^j},\ i,j = 0,1,2,3 \ and \ i+j=3,$$
$$p_i = \frac{\partial p}{\partial\lambda_i},\ \omega_i = \frac{\partial k}{\partial\lambda_i},\ \omega_{ij} = \frac{\partial^2 k}{\partial\lambda_i\partial\lambda_j},\ p = ln\pi(\lambda_1,\lambda_2),$$
$$A_{ij} = \omega_i\tau_{ii} + \omega_j tau_{ji},\ B_{ij} = (\omega_i\tau_{ii} + \omega_j\tau_{ij})\tau_{ii},\ C_{ij} = 3\omega_i\tau_{ii}\tau_{ij} + \omega_j(\tau_{ii}\tau_{jj} + 2\tau_{ij}^2).$$

Based on Eq. (3.5), the approximate Bayes estimate of θ and λ under SEL function is given by

$$\hat{\theta}_S = E[\hat{\theta}_M|X] = \hat{\theta}_M + \frac{1}{2}[l_{30}\tau_{11}^2 + l_{03}\tau_{22}\tau_{21} + l_{12}(\tau_{22}\tau_{11} + 2\tau_{21}^2)] + p_1\tau_{11} + p_2\tau_{12}$$

and

$$\hat{\lambda}_S = E[\hat{\lambda}_M|X] = \hat{\lambda}_M + \frac{1}{2}[l_{03}\tau_{22}^2 + l_{30}\tau_{12}\tau_{11} + 3l_{12}\tau_{22}\tau_{21}] + p_1\tau_{21} + p_2\tau_{22},$$

where

$$p_1 = \frac{a_1-1}{\theta} - b_1,\ p_2 = \frac{a_2-1}{\lambda} - b_2,\ l_{30} = \frac{\partial^3 l}{\partial\theta^3} = \frac{2m}{\theta^3},\ l_{12} = \frac{\partial^3 l}{\partial\theta\partial\lambda^2} = -\frac{V_m^{-\lambda}(lnV_m)^2}{(1-V_m^{-\lambda})^2},$$
$$l_{03} = \frac{\partial^3 l}{\partial\lambda^3} = \frac{2m}{\lambda^3} - \sum_{i=1}^{m}\frac{V_i^{-\lambda}(lnV_i)^3}{(1-V_i^{-\lambda})^2}\left(1 + 2\frac{V_i^{-\lambda}}{1-V_i^{-\lambda}}\right) + \theta\frac{V_m^{-\lambda}(lnV_m)^3}{(1-V_m^{-\lambda})^2}\left(1 + 2\frac{V_m^{-\lambda}}{1-V_m^{-\lambda}}\right).$$

Based on Lindley's approximation, the approximate Bayes estimate of θ and λ under LL function is given by

$$\hat{\theta}_L = -\frac{1}{h} lnE[e^{-h\hat{\theta}_M}|X] \text{ and } \hat{\lambda}_L = -\frac{1}{h} lnE[e^{-h\hat{\lambda}_M}|X].$$

where

$$\begin{aligned}E[e^{-h\theta}|X] =& e_M^{-h\hat{\theta}} + \frac{1}{2}[h^2 e_M^{-h\hat{\theta}}\tau_{11} + (-he_M^{-h\hat{\theta}})(l_{30}\tau_{11}^2 + l_{03}\tau_{22}\tau_{21} \\ &+ l_{12}(\tau_{22}\tau_{11} + 2\tau_{21}^2) + 2p_1\tau_{11} + 2p_2\tau_{12})], \\ E[e^{-h\lambda}|X] =& e_M^{-h\hat{\lambda}} + \frac{1}{2}[h^2 e_M^{-h\hat{\lambda}}\tau_{22} + (-he_M^{-h\hat{\lambda}})(l_{03}\tau_{22}^2 + l_{30}\tau_{12}\tau_{11} \\ &+ 3l_{12}\tau_{22}\tau_{21} + 2p_1\tau_{21} + 2p_2\tau_{22})].\end{aligned}$$

Based on Lindley's approximation, the approximate Bayes estimate of θ and λ under EL function is given by

$$\hat{\theta}_E = \left\{E[\theta^{-q}|X]\right\}^{-1/q} \text{ and } \hat{\lambda}_E = \left\{E[\lambda^{-q}|X]\right\}^{-1/q},$$

where

$$\begin{aligned}E[\theta^{-q}|X] =& \hat{\theta}_M^{-q} + \frac{1}{2}\bigg[q(q+1)\hat{\theta}_M^{-(q+2)}\tau_{11} + (-q\hat{\theta}_M^{-(q+1)})(l_{30}\tau_{11}^2 \\ &+ l_{03}\tau_{21}\tau_{22} + l_{12}(\tau_{22}\tau_{11} + 2\tau_{21}^2) + 2p_1\tau_{11} + 2p_2\tau_{12})\bigg], \\ E[\lambda^{-q}|X] =& \hat{\lambda}_M^{-q} + \frac{1}{2}\bigg[q(q+1)\hat{\lambda}_M^{-(q+2)}\tau_{22} + (-q\hat{\lambda}_M^{-(q+1)})(l_{03}\tau_{22}^2 \\ &+ l_{30}\tau_{12}\tau_{11} + 3l_{12}\tau_{22}\tau_{21} + 2p_1\tau_{21} + 2p_2\tau_{22})\bigg].\end{aligned}$$

4. Illustrative examples

In this section, we use a real data set to show that the EPD based on the Pareto distribution. The data are the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958-1984, rounded to one decimal place. This data were analyzed by Choulakian and Stephens (2001) and are given in Table 4.1.

Table 4.1 Exceedances of Wheaton river flood data

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	13.0	12.0	9.3	1.4	18.7
8.5	25.5	11.6	2.2	39.0	0.3	15.0	14.1	22.1	1.1	2.5	14.4	1.7
37.6	0.6	11.0	7.3	22.9	1.7	0.1	1.1	0.6	9.0	1.7	7.0	20.1
0.4	14.1	9.9	10.4	10.7	30.0	3.6	5.6	30.8	13.3	4.2	25.5	3.4
11.9	21.5	27.6	36.4	2.7	64.0	1.5	2.5	27.4	1.0	27.1	20.2	16.8
5.3	9.7	27.5	2.5	27.0	1.9	2.8						

Before lower recording further, Bourguignon *et al*. (2013) has examined the goodness-of-fit of the previous data to EPD. We consider the case when the data are lower recorded as follows :

$X_{L(1)}$	$X_{L(2)}$	$X_{L(3)}$	$X_{L(4)}$	$X_{L(5)}$
1.7	1.1	0.4	0.3	0.1

The hyper parameters are assigned values of $a_1 = a_2 = b_1 = b_2 = 0.0001$. In all cases, Bayesian estimates against the LL and BLL function are evaluated for $h = 2$ while for the EL and BEL function we take $q = 2$. The MLE's and Bayes estimates corresponding real data set are summarized as follows:

Table 4.2 MLE's and Bayes estimates corresponding real data set for $m = 5$

	$\omega = 0$		$\omega = 0.2$		$\omega = 0.5$		$\omega = 0.8$		$\omega = 1$	
	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
BSEL	2.374	1.414	2.420	1.463	2.488	1.537	2.557	1.611	2.602	1.661
BLL	2.908	1.903	2.871	1.865	2.785	1.807	2.615	1.782	2.602	1.661
BEL	2.371	0.965	2.436	1.016	2.527	1.355	2.583	1.565	2.602	1.661

5. Simulation study

In this section, we perform a numerical study to compare the proposed estimates of θ and λ. For each simulation, we consider $\theta = 3$ and $\lambda = 2.5$. We consider four different sample sizes, $m = 3, 5, 7, 9$. First we compute the MLEs using the methods described in Section 2. We report the average bias and RMSEs over 1,000 replications. The Bayes estimators cannot be found in closed form. Therefore, we use the Lindley's approximation method to compute Bayes estimates. The hyperparameters are assigned values of $a_1 = a_2 = b_1 = b_2 = 0.0001$. In all cases, Bayesian estimates against the LL and BLL function are evaluated for $h = -0.5,\ 1,\ 0.5$ while for the EL and BEL function we take $q = -0.5,\ 1,\ 0.5$. This allows us to consider the Bayes estimators under both symmetric and asymmetric loss functions.

In Table 5.1 $\sim$ 5.3, the following general observations can be made. The RMSEs and biases decrease as record times m increases. We tabulates average biases and RMSEs of the respective MLEs.

In Table 5.1, Bayesian estimates against the LL and BLL function are evaluated for $h = 1$ and $q = 1$. In Table 5.2, Bayesian estimates against the LL and BLL function are evaluated for $h = 0.5$ and $q = 0.5$. In Table 5.3, Bayesian estimates against the LL and BLL function are evaluated for $h = -0.5$ and $q = -0.5$.

In general, we observed that Bayes estimates are superior to the respective MLEs in terms of biases and RMSEs. In Table 5.1, for estimating both the unknown parameters, BLL Bayes estimates of θ better than the different estimates and EL Bayes estimates of λ are better than the different estimates. BLL Bayes estimates of θ of when $\omega = 0.2$ better than the BLL Bayes estimates of θ of where different weights.

In Table 5.2, for estimating both the unknown parameters, BLL Bayes estimates of θ better than the different estimates and EL Bayes estimates of λ are better than the different estimates. BLL Bayes estimates of θ of when $\omega = 0.2$ better than the BLL Bayes estimates of θ of where different weights.

Table 5.1 The average biases and the RMSEs of MLEs and Bayes estimators for different choices of m, $h=1,\ q=1$

	ω	$m=3$		$m=5$		$m=7$		$m=9$	
		$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
MLE		2.0693	15.3493	1.8954	11.0786	1.5511	6.8747	1.1366	3.1314
		(0.0085)	(5.9981)	(-0.4473)	(5.9736)	(-0.0424)	(3.2596)	(0.2320)	(1.4072)
SEL		2.3563	12.8655	2.0187	9.6150	1.5879	6.1610	1.0900	2.8134
		(-0.0769)	(4.5767)	(-0.6023)	(4.8271)	(-0.2139)	(2.6692)	(0.0649)	(1.0662)
LL		1.7033	13.2267	1.6844	8.7822	1.3202	5.2134	0.9014	2.1117
		(-0.8011)	(3.8501)	(-1.2517)	(3.4952)	(-0.7987)	(1.4829)	(-0.4657)	(0.0951)
EL		1.7464	8.9235	1.6690	6.9667	1.3060	4.6496	0.9357	2.1428
		(-0.7380)	(2.4473)	(-1.2205)	(2.7262)	(-0.7087)	(1.3506)	(-0.3399)	(0.2021)
BSEL	0.2	2.2944	13.3568	1.9930	9.9058	1.5789	6.3024	1.0974	2.8755
		(-0.0598)	(4.8609)	(-0.5713)	(5.0564)	(-0.1796)	(2.7873)	(0.0983)	(1.1344)
	0.5	2.2054	14.0993	1.9554	10.3439	1.5669	6.5157	1.1103	2.9701
		(-0.0341)	(5.2874)	(-0.5248)	(5.4003)	(-0.1282)	(2.9644)	(0.1484)	(1.2367)
	0.8	2.1218	14.8476	1.9190	10.7841	1.5568	6.7303	1.1253	3.0664
		(-0.0085)	(5.7138)	(-0.4783)	(5.7443)	(-0.0767)	(3.1415)	(0.1986)	(1.3390)
BLL	0.2	1.6884	13.2975	1.6408	8.8774	1.2868	5.2908	0.8795	2.1578
		(-0.6998)	(4.0081)	(-1.1513)	(3.6763)	(-0.6953)	(1.6414)	(-0.3646)	(0.2386)
	0.5	1.6907	13.4524	1.5930	9.0870	1.2614	5.4665	0.8810	2.2843
		(-0.5190)	(4.3172)	(-0.9716)	(4.0374)	(-0.5125)	(1.9490)	(-0.1879)	(0.5094)
	0.8	1.7664	13.7682	1.6188	9.5157	0.3144	5.8296	0.9654	2.5695
		(-0.2765)	(4.8246)	(-0.7278)	(4.6500)	(-0.2753)	(2.4389)	(0.0347)	(0.9107)
BEL	0.2	1.7508	9.7370	1.6567	7.5315	1.3110	4.9689	0.9430	2.2716
		(-0.6325)	(2.8779)	(-1.1066)	(3.1525)	(-0.6009)	(1.6210)	(-0.2424)	(0.3784)
	0.5	1.7856	11.2770	1.6735	8.5661	1.3515	5.5433	0.9812	2.5181
		(-0.4444)	(3.7036)	(-0.9047)	(3.9472)	(-0.4184)	(2.1090)	(-0.0818)	(0.6920)
	0.8	1.8964	13.4011	1.7652	9.9178	1.4469	6.2712	1.0591	2.8486
		(-0.2022)	(4.8760)	(-0.6530)	(5.0184)	(-0.2054)	(2.7340)	(0.0986)	(1.0847)

Table 5.2 The average biases and the RMSEs of MLEs and Bayes estimators for different choices of m, $h=0.5,\ q=0.5$

	ω	$m=3$		$m=5$		$m=7$		$m=9$	
		$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
LL		1.6260	12.1107	1.5277	7.8115	1.1737	4.6530	1.3243	4.8521
		(-0.7181)	(3.3417)	(-1.1690)	(2.9554)	(-0.6649)	(1.3193)	(-0.6275)	(1.5032)
EL		1.7639	9.2575	1.6748	7.2661	1.3243	4.8521	0.9472	2.2319
		(-0.6662)	(2.5831)	(-1.1401)	(2.9119)	(-0.6275)	(1.5032)	(-0.2653)	(0.3125)
BLL	0.2	1.6253	12.2470	1.5055	7.9961	1.1714	4.8067	0.8449	2.0334
		(-0.6126)	(3.5741)	(-1.0620)	(3.2383)	(-0.5636)	(1.5401)	(-0.2163)	(0.3425)
	0.5	1.6587	12.5451	1.5145	8.4002	1.2096	5.1409	0.9006	2.2680
		(-0.4343)	(4.0146)	(-0.8802)	(3.7805)	(-0.3963)	(1.9471)	(-0.0643)	(0.6502)
	0.8	1.7802	13.1397	1.6321	9.1836	1.3361	5.7538	1.0138	2.6617
		(-0.2135)	(4.6966)	(-0.6553)	(4.6266)	(-0.2021)	(2.5309)	(0.1051)	(1.0449)
BEL	0.2	1.7741	10.1306	1.6742	7.8501	1.3381	5.1721	0.9614	2.3631
		(-0.5604)	(3.0544)	(-1.0266)	(3.3668)	(-0.5254)	(1.7808)	(-0.1755)	(0.4901)
	0.5	1.8212	11.7026	1.7066	8.8694	1.3854	5.7221	1.0038	2.6011
		(-0.3790)	(3.9165)	(-0.8355)	(4.1720)	(-0.3593)	(2.2573)	(-0.0322)	(0.7907)
	0.8	1.9337	13.6963	1.7951	10.1037	1.4704	6.3727	1.0739	2.8968
		(-0.1622)	(5.0394)	(-0.6145)	(5.1686)	(-0.1758)	(2.8222)	(0.1222)	(1.1407)

Table 5.3 The average biases and the RMSEs of MLEs and Bayes estimators for different choices of m, $h=-0.5$, $q=-0.5$

		$m=3$		$m=5$		$m=7$		$m=9$	
	ω	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
LL		3.0254	18.7550	2.7809	14.4232	2.1906	9.1025	1.5413	4.3237
		(0.4736)	(8.2828)	(0.0007)	(8.5701)	(0.3000)	(4.8224)	(0.5038)	(2.2970)
EL		1.9933	11.1055	1.8266	8.5522	1.4599	5.5980	1.0213	2.5579
		(-0.3388)	(3.5330)	(-0.8285)	(3.9098)	(-0.3768)	(2.1521)	(-0.0602)	(0.7426)
BLL	0.2	2.9298	18.5737	2.6847	14.1873	2.1174	8.9097	1.4801	4.1748
		(0.4150)	(8.0896)	(-0.0582)	(8.3275)	(0.2464)	(4.6454)	(0.4562)	(2.1677)
	0.5	2.7499	18.2062	2.5038	13.7133	1.9832	8.5324	1.3758	3.9035
		(0.3125)	(7.7220)	(-0.1626)	(7.8633)	(0.1573)	(4.3196)	(0.3798)	(1.9433)
	0.8	2.4740	17.5445	2.2335	12.8892	1.7899	7.9143	1.2474	3.5275
		(0.1755)	(7.1472)	(-0.3016)	(7.1387)	(0.0514)	(3.8517)	(0.2954)	(1.6607)
BEL	0.2	2.0033	11.8816	1.8323	9.0251	1.4712	5.8384	1.0378	2.6620
		(-0.2718)	(3.9843)	(-0.7548)	(4.2978)	(-0.3115)	(2.3637)	(-0.0029)	(0.8704)
	0.5	2.0230	13.1137	1.8481	9.7651	1.4947	6.2131	1.0689	2.8284
		(-0.1690)	(4.7004)	(-0.6418)	(4.9031)	(-0.2121)	(2.6904)	(0.0841)	(1.0669)
	0.8	2.0487	14.4276	1.8733	10.5412	1.5260	6.6043	1.1072	3.0065
		(-0.0634)	(5.4634)	(-0.5260)	(5.5361)	(-0.1109)	(3.0282)	(0.1724)	(1.2692)

In Table 5.3, for estimating both the unknown parameters, EL Bayes estimates of θ and λ better than the different estimates. Overall, the balanced loss estimates of θ for the choices $\omega = 0.2$ provide better estimates compared with among balanced loss estimates and EL Bayes estimates of λ are better.

6. Conclusions

In this paper, estimation from exponentiated Pareto distribution based on lower record values has been considered. Both the classical and Bayesian inference of the unknown parameters are provided. It is observed that the MLEs of the unknown parameters cannot be obtained in closed form, hence fixed point solution have been considered. We also consider the Bayes estimates of the unknown parameters based on different loss functions, and it is observed that they cannot be obtained in explicit forms, hence Lindley's approximation has been considered.

When $h = 1$ and $q = 1$, for estimating both the unknown parameters, BLL Bayes estimates of θ better than the different estimates and EL Bayes estimates of λ are better than the different estimates. BLL Bayes estimates of θ of when $\omega = 0.2$ better than the BLL Bayes estimates of θ of where different weights.

When $h = 0.5$ and $q = 0.5$, for estimating both the unknown parameters, BLL Bayes estimates of θ better than the different estimates and EL Bayes estimates of λ are better than the different estimates. BLL Bayes estimates of θ of when $\omega = 0.2$ better than the BLL Bayes estimates of θ of where different weights.

When $h = -0.5$ and $q = -0.5$, for estimating both the unknown parameters, EL Bayes estimates of θ and λ better than the different estimates. Overall, the balanced loss estimates of θ for the choices $\omega = 0.2$ provide better estimates compared with among balanced loss estimates and EL Bayes estimates of λ are better. In this paper, although we have mainly considered lower record values case, the same method can be extended for other record values and distributions also.

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