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SIMULATION FUNCTIONS OVER M-METRIC SPACES

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Abstract. In this paper, existence of fixed point of certain operators imbedded in simulation function has been investigated in context of a complete M-metric spaces.

1. Introduction

Inspired from the notion of partial metric, introduced by Matthews [7], Asadi *et al.* [5] proposed the concept of a M-metric which refine the notion of partial metric. Like standard metric, M-metric has a topology and produce useful basic topological concepts.

Recently, Khojasteh *et al.*. proposed the notion of simulation function to unify the several existing fixed point results in the literature. In this paper, we investigate the existence and uniqueness of fixed points of certain mappings via simulation functions in the context of complete M-metric spaces. We shall also indicate that several results in the literature can be derived from our main results.

Definition 1. (see [2]) A function $\sigma : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is said to be simulation if it fulfils:
(σ_1) $\sigma(0, 0) = 0$;
(σ_2) $\sigma(t, u) < u - t$ for all $t, u > 0$;
(σ_3) if $\{t_n\}$, $\{u_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} u_n > 0$, then

$$\limsup_{n\to\infty} \sigma(t_n, u_n) < 0. \tag{1}$$

Let Σ be the collection of all simulation functions $\sigma : [0, \infty) \times [0, \infty) \to \mathbb{R}$. On account of the property (σ_2), we conclude that

$$\sigma(t, t) < 0 \text{ for all } t > 0. \tag{2}$$

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Example 1.1. *Let $\sigma : [0,\infty)\times[0,\infty) \to \mathbb{R}$ be a mapping such that $\sigma(t,u) = \frac{u}{2} - t$ for all $t, u \in [0,\infty)$. It is obvious that σ is a simulation function. For more examples of simulation functions in* [2, 3].

Suppose (X, d) is a metric space, T is a self-mapping on X and $\sigma \in \Sigma$. We say that T is a Σ-contraction with respect to σ [2], if

$$\sigma(d(Tx, Ty), d(x, y)) \geq 0 \quad \text{for all } x, y \in X. \tag{3}$$

For all distinct $x, y \in X$, by (σ_2), we have the inequality below

$$d(Tx, Ty) \neq d(x, y). \tag{4}$$

Thus we deduce that whenever a Σ-contraction T in a metric space has a fixed point, then it is necessarily unique.

Theorem 1.2. *Every Σ-contraction on a complete metric space has a unique fixed point.*

If an auxiliary non-decreasing function $\vartheta : [0,\infty) \to [0,\infty)$ fulfils that there exists $p_0 \in \mathbb{N}$ and $a \in (0, 1)$ and a convergent series of nonnegative terms $\Sigma_{p=1}^{\infty} v_p$ that

$$\vartheta^{p+1}(s) \leq a\vartheta^p(s) + v_p, \quad \text{for } p \geq p_0 \text{ and any } s \in [0,\infty),$$

then, the function ϑ is called (c)-comparison and denoted as $\vartheta \in \Psi$ (see e.g. [4]).

Lemma 1.3. (see e.g [4]) If $\vartheta \in \Psi$, such then the following hold:

(i) $(\vartheta^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \in (0,\infty)$;
(ii) $\vartheta(s) < s$, for any $t \in (0,\infty)$;
(iii) ϑ is continuous at 0;
(iv) the series $\Sigma_{p=1}^{\infty} \vartheta^p(s)$ converges for any $s \in (0,\infty)$.

In what follows we recall the notion of (triangular) α-orbital admissible, introduced by Popescu [6], that is inspired from [1].

Definition 2. [6] For a fixed mapping $\alpha : M \times M \to [0,\infty)$, we say that a self-mapping $T : M \to M$ is an α-orbital admissible if

$$(O1) \ \ \alpha(u, Tu) \geq 1 \Rightarrow \alpha(Tu, T^2u) \geq 1.$$

Let $\mathcal{A}$ be the collection of all α-orbital admissible $T : M \to M$.

In addition, T is called triangular α-orbital admissible if T is α-orbital admissible and

$$(O2) \ \ \alpha(u, v) \geq 1 \text{ and } \alpha(v, Tv) \geq 1 \Rightarrow \alpha(u, Tv) \geq 1.$$

Let $\mathcal{O}$ be the collection of all triangular α-orbital admissible $T : M \to M$.

Definition 3. ([5]) For a given non empty set X, we say that a function $\mu : X \times X \to [0,\infty)$ is an M-metric if

(m1) $\mu(x, x) = \mu(y, y) = \mu(x, y) \Leftrightarrow x = y$,

(m2) $m_{xy} \leq \mu(x,y)$, where $m_{xy} := \min\{\mu(x,x), \mu(y,y)\}$,
(m3) $\mu(x,y) = \mu(y,x)$,
(m4) $(\mu(x,y) - m_{xy}) \leq (\mu(x,z) - m_{xz}) + (\mu(z,y) - m_{zy})$.

In this case, the pair (X,μ) is called an M-metric space.

In the following example we present an example of an M-metric.

Example 1.4. *Let $X = \{a,b,c\} \cup [0,\infty)$ with $a,b,c \notin \{a,b,c\}$. Define*

$$\mu(a,b) = \mu(b,a) = \mu(a,a) = 8,$$

$$\mu(a,c) = \mu(c,a) = \mu(c,b) = \mu(b,c) = 7 \quad \mu(b,b) = 9 \quad \mu(c,c) = 5,$$

and $\mu(x,y) = |x-y|$ otherwise. So μ is M-metric. If $D(x,y) = \mu(x,y) - m_{x,y}$, then $\mu(a,b) = m_{a,b} = 8$ but it means $D(a,b) = 0$ while $a \neq b$ which means D is not metric.

Remark 1. ([5]) For every $x,y \in X$

(1) $0 \leq M_{xy} + m_{xy} = \mu(x,x) + \mu(y,y)$.
(2) $0 \leq M_{xy} - m_{xy} = |\mu(x,x) - \mu(y,y)|$.
(3) $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$.

For more examples and for the topology of M-metric space, we refer for example [5]. Like in standard metric space topology, the set

$$\{B_\mu(x,\varepsilon) : x \in X, \varepsilon > 0\},$$

forms a base for the topology of M-metric μ, where where

$$B_\mu(x,\varepsilon) = \{y \in X : \mu(x,y) < m_{x,y} + \varepsilon\},$$

for all $x \in X$ and $\varepsilon > 0$.

Definition 4. ([5]) Let (X,μ) be an M-metric space and $\{x_n\}$ be a sequence in (X,μ). Then,

(1) $\{x_n\}$ converges to a point $x \in X$ if

$$\lim_{n\to\infty} (\mu(x_n,x) - m_{x_n,x}) = 0. \tag{5}$$

(2) $\{x_n\}$ is called a m-Cauchy sequence if

$$\lim_{n,m\to\infty} (\mu(x_n,x_m) - m_{x_n,x_m}) \quad \text{and} \quad \lim_{n,m\to\infty} (M_{x_n,x_m} - m_{x_n,x_m}) \tag{6}$$

there exist (and are finite).

(3) (X,μ) is called complete if every m-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that

$$\left(\lim_{n\to\infty} (\mu(x_n,x) - m_{x_n,x}) = 0 \quad \text{and} \quad \lim_{n\to\infty} (M_{x_n,x} - m_{x_n,x}) = 0\right).$$

Lemma 1.5. ([5]) *Suppose that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an M-metric space (X, μ). Then, we have*

$$\lim_{n\to\infty} (\mu(x_n, y_n) - m_{x_n,y_n}) = \mu(x, y) - m_{xy}$$

and also

$$\lim_{n\to\infty} (\mu(x_n, y) - m_{x_n,y}) = \mu(x, y) - m_{x,y},$$

for all $y \in X$. Moreover, $\mu(x, y) = m_{xy}$. Further if $\mu(x, x) = \mu(y, y)$, then $x = y$.

In this paper, we consider the existence of fixed point of certain operators, defined via simulation function, in e very general setting, m-metric spaces.

2. Main result and fixed point theorems

We, first, define the following contractive mapping:

Definition 5. Let T be a self-mapping defined on an M-metric space (X, μ). If there exist $\sigma \in \Sigma$ and $\alpha : X \times X \to [0.\infty)$ such that

$$\sigma(\alpha(x, y)\mu(Tx, Ty), \mu(x, y)) \geq 0 \quad \text{for all } x, y \in X, \tag{7}$$

then we say that T is an α-admissible Σ-contraction with respect to σ.

If $\alpha(x, y) = 1$, then T turns into a Σ-contraction with respect to σ.

Lemma 2.1. *Let T is an α-admissible Σ-contraction with respect to σ in M-metric space (X, μ) and $x, y \in X$ such that $\mu(x, y) > 0$ then*

$$\alpha(x, y)\mu(Tx, Ty) < \mu(x, y). \tag{8}$$

Proof. Assume that $x, y \in X$ such that $\mu(x, y) > 0$. If $\mu(Tx, Ty) = 0$, then $\alpha(x, y)\mu(Tx, Ty) = 0 < \mu(x, y)$. Otherwise, $\mu(Tx, Ty) > 0$. If $\alpha(x, y) = 0$, then the inequality is satisfied trivially. So assume that $\alpha(x, y) > 0$ and applying (σ_2) with (7), we derive that

$$0 \leq \sigma(\alpha(x, y)\mu(Tx, Ty), \mu(x, y)) < \mu(x, y) - \alpha(x, y)\mu(Tx, Ty),$$

so (8) holds. □

We can now state the main result of this paper.

Theorem 2.2. *Let (X, μ) be a complete M-metric space and let $T : X \to X$ be a continuous α-admissible Σ-contraction with respect to σ. If $T \in \mathcal{O}$ and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then there exists $u \in X$ such that $Tu = u$.*

Proof. Due to the assumption of the theorem, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Set-up an iterative sequence $\{x_n\}$ in X by letting $x_{n+1} = Tx_n$ for all $n \geq 0$. We want to prove that $\mu(x_n, x_{n+1}) \to 0$, as $n \to \infty$.
If $\mu(x_{n_0}, x_{n_0+1}) = 0$, for some $n_0 \in \mathbb{N}$, then we have $\mu(x_{n_0+1}, x_{n_0+2}) = 0$.

Suppose to the contrary that $\mu(x_{n_0+1}, x_{n_0+2}) > 0$, so by the property of T and (σ_2)

$$\begin{aligned} 0 &\leq \sigma(\alpha(x_{n_0}, x_{n_0+1})\mu(Tx_{n_0}, Tx_{n_0+1}), \mu(x_{n_0}, x_{n_0+1})) \\ &< \mu(x_{n_0}, x_{n_0+1}) - \alpha(x_{n_0}, x_{n_0+1})\mu(Tx_{n_0}, Tx_{n_0+1}) \\ &= 0 - \alpha(x_{n_0}, x_{n_0+1})\mu(Tx_{n_0}, Tx_{n_0+1}). \end{aligned}$$

Now since $\mu(x_{n_0}, x_{n_0+1}) \geq 1$ so by inequality we obtain that $\mu(Tx_{n_0}, Tx_{n_0+1}) < 0$. Which is a contradiction, hence we have $\mu(x_n, x_{n+1}) = 0$ for all $n \geq n_0$. Consequently, we shall assume that

$$\mu(x_n, x_{n+1}) > 0, \qquad \text{for all } n = 0, 1, 2, \ldots. \tag{9}$$

Regarding that T is α-admissible, we derive

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Recursively, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1 \qquad \text{for all } n = 0, 1, \ldots. \tag{10}$$

From (7) and (10), it follows that for all $n \geq 1$, we have

$$\begin{aligned} 0 &\leq \sigma(\alpha(x_n, x_{n-1})\mu(Tx_n, Tx_{n-1}), \mu(x_n, x_{n-1})) \\ &= \sigma(\alpha(x_n, x_{n-1})\mu(x_{n+1}, x_n), \mu(x_n, x_{n-1})) \\ &< \mu(x_n, x_{n-1}) - \alpha(x_n, x_{n-1})\mu(x_{n+1}, x_n). \end{aligned}$$

Consequently, we derive that

$$\mu(x_n, x_{n+1}) \leq \alpha(x_n, x_{n-1})\mu(x_n, x_{n+1}) < \mu(x_n, x_{n-1}) \qquad \text{for all } n = 1, 2, \ldots. \tag{11}$$

Hence, we conclude that the sequence $\{\mu(x_n, x_{n-1})\}$ is non-decreasing and bounded from below by zero. Consequently, there exists $m \geq 0$ such that $\lim_{n\to\infty} \mu(x_n, x_{n-1}) = m \geq 0$. We shall prove that

$$\lim_{n\to\infty} \mu(x_n, x_{n-1}) = 0. \tag{12}$$

Suppose, on the contrary that $m > 0$. Note that from the inequality (11), we derive that

$$\alpha(x_n, x_{n-1})\mu(x_n, x_{n+1}) = m. \tag{13}$$

Letting $s_n = \alpha(x_n, x_{n-1})\mu(x_n, x_{n+1})$ and t $t_n = \mu(x_n, x_{n-1})$ and taking $(\sigma_3\ )$ into account, we get that

$$0 \leq \limsup_{n\to\infty} \sigma(\alpha(x_n, x_{n-1})\mu(x_{n+1}, x_n), \mu(x_n, x_{n-1})) < 0, \tag{14}$$

which is a contradiction. Thus, we have $m = 0$. that means

$$\lim_{n\to\infty} \mu(x_n, x_{n-1}) = 0. \tag{15}$$

Now we prove that $\{x_n\}$ is M-Cauchy sequence in (X, μ). We have

$$\lim_{n\to\infty} \mu(x_n, x_{n+1}) = 0,$$

$$0 \leq m_{x_n,x_{n+1}} \leq \mu(x_n, x_{n+1}) \Rightarrow \lim_{n\to\infty} m_{x_n,x_{n+1}} = 0,$$

and

$$m_{x_n,x_{n+1}} = \min\{\mu(x_n, x_n), \mu(x_{n+1}, x_{n+1})\} \Rightarrow \lim_{n\to\infty} \mu(x_n, x_n) = 0.$$

On the other hand

$$m_{x_n,x_m} = \min\{\mu(x_n, x_n), \mu(x_m, x_m)\} \Rightarrow \lim_{n,m\to\infty} m_{x_n,x_m} = 0,$$

so

$$\lim_{n,m\to\infty} (M_{x_n,x_m} - m_{x_n,x_m}) = 0.$$

We show

$$\lim_{n,m\to\infty} (\mu(x_n, x_m) - m_{x_n,x_m}) = 0.$$

Define

$$M^*(x, y) := \mu(x, y) - m_{x,y}, \quad \forall x, y \in X.$$

If $\lim_{n,m\to\infty} M^*(x_n, x_m) \neq 0$, there exist $\varepsilon > 0$ and $\{l_k\} \subset \mathbb{N}$ such that

$$M^*(x_{l_k}, x_{n_k}) \geq \varepsilon.$$

Suppose that k is the smallest integer which satisfies above equation such that

$$M^*(x_{l_k-1}, x_{n_k}) < \varepsilon.$$

Now by (m4) we have

$$\varepsilon \leq M^*(x_{l_k}, x_{n_k}) \leq M^*(x_{l_k}, x_{l_k-1}) + M^*(x_{l_k-1}, x_{n_k}) < M^*(x_{l_k}, x_{l_k-1}) + \varepsilon.$$

Thus

$$\lim_{k\to\infty} M^*(x_{l_k}, x_{n_k}) = \varepsilon,$$

which means

$$\lim_{k\to\infty} (\mu(x_{l_k}, x_{n_k}) - m_{x_{l_k},x_{n_k}}) = \varepsilon.$$

On the other hand

$$\lim_{k\to\infty} m_{x_{l_k},x_{n_k}} = 0,$$

so we have

$$\lim_{k\to\infty} \mu(x_{l_k}, x_{n_k}) = \varepsilon. \tag{16}$$

Again by (m4) we have

$$M^*(x_{l_k}, x_{n_k}) \leq M^*(x_{l_k}, x_{l_k+1}) + M^*(x_{l_k+1}, x_{n_k+1}) + M^*(x_{n_k+1}, x_{n_k}),$$

and

$$M^*(x_{l_k+1}, x_{n_k+1}) \leq M^*(x_{l_k}, x_{l_k+1}) + M^*(x_{l_k}, x_{n_k}) + M^*(x_{n_k+1}, x_{n_k}),$$

taking the limit as $k \to +\infty$, together with (15) and (16) we have

$$\lim_{k\to\infty} \mu(x_{l_k+1}, x_{n_k+1}) = \varepsilon. \tag{17}$$

Particularly, there exists $n_1 \in \mathbb{N}$ such that for all $k \geq n_1$ we have

$$\mu(x_{l_k}, x_{n_k}) > \frac{\varepsilon}{2} \text{ and } \mu(x_{l_k+1}, x_{n_k+1}) > \frac{\varepsilon}{2} > 0. \tag{18}$$

Moreover, since T is triangular α-orbital admissible, we have

$$\alpha(x_{l_k}, x_{n_k}) \geq 1. \tag{19}$$

Regarding the fact T is an α-admissible Σ-contraction with respect to σ, together with (18) and (19) we get that

$$\begin{array}{rcl} 0 & \leq & \sigma(\alpha(x_{l_k}, x_{n_k})\mu(Tx_{l_k}, Tx_{n_k}), \mu(x_{l_k}, x_{n_k})) \\ & = & \sigma(\alpha(x_{l_k}, x_{n_k})\mu(x_{l_k+1}, x_{n_k+1}), \mu(x_{l_k}, x_{n_k})) \\ & < & \mu(x_{l_k}, x_{n_k}) - \alpha(x_{l_k}, x_{n_k})\mu(x_{l_k+1}, x_{n_k+1}), \end{array}$$

for all $k \geq n_1$. Consequently, we have

$$0 < \mu(x_{l_k+1}, x_{n_k+1}) \leq \alpha(\alpha(x_{l_k}, x_{n_k})\mu(x_{l_k+1}, x_{n_k+1}) < \mu(x_{l_k}, x_{n_k}),$$

for all $k \geq n_1$. From above inequality, together with(16) and(17), we conclude that $s_n = \alpha(x_{l_k}, x_{n_k})\mu(x_{l_k+1}, x_{n_k+1}) \to \varepsilon$ as $t_n = \mu(x_{l_k}, x_{n_k}) \to \varepsilon$. On account of the above observations and regarding the condition (σ_3), we deduce that

$$0 \leq \limsup_{k\to\infty} \sigma(\alpha(x_{l_k}, x_{n_k})\mu(x_{l_k+1}, x_{n_k+1}), \mu(x_{l_k}, x_{n_k})) < 0,$$

which is a contradiction, and therefore $\{x_n\}$ is an M-Cauchy sequence. Now by completeness of X, $x_n \to u$, for some $u \in X$ in τ_m topology that means,

$$\lim_{n\to\infty} (\mu(x_n, u) - m_{x_n,u}) = 0.$$

And

$$\lim_{n\to\infty} (M_{x_n,u} - m_{x_n,u}) = 0.$$

But we have $\lim_{n\to\infty} m_{x_n,u} = 0$, hence $\lim_{n\to\infty} \mu(x_n, u) = 0$ and by Remark 1

$$\mu(u, u) = 0.$$

T is continuous so

$$\lim_{n\to\infty} (\mu(Tx_n, Tu) - m_{Tx_n,Tu}) = 0,$$

that means

$$\lim_{n\to\infty} (\mu(x_{n+1}, Tu) - m_{x_{n+1},Tu}) = 0,$$

and similar to the above, we have $\lim_{n\to\infty} m_{x_{n+1},Tu} = 0$, hence $\lim_{n\to\infty} \mu(x_{n+1}, Tu) = 0$ and by Remark 1, $\mu(Tu, Tu) = 0$. On the other hand, $x_n \to u$ as $n \to \infty$ so by Lemma 1.5, we get

$$(\mu(x_n, Tu) - m_{x_n,Tu}) \to (\mu(u, Tu) - m_{u,Tu}) = \mu(u, Tu) \quad \text{as} \quad n \to \infty,$$

but we have

$$(\mu(x_n, Tu) - m_{x_n,Tu}) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$\mu(u, Tu) = 0,$$

therefore $\mu(u, Tu) = \mu(Tu, Tu) = \mu(u, u) = 0$ and by (m1) we get

$$Tu = u.$$

□

We say that (X, μ) is regular, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

Theorem 2.3. *Let (X, μ) be a complete M-metric space and let $T : X \to X$ be an α-admissible Σ-contraction with respect to σ. Suppose that (X, μ) is regular. If $T \in \mathcal{O}$ and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then there exists $u \in X$ such that $Tu = u$.*

Proof. Following the proof of Theorem 2.2, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges for some $u \in X$. From (10) and (X, μ) is regular , there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, u) \geq 1$ for all k. Applying (7), for all k, we get that

$$\begin{array}{rcl} 0 & \leq & \sigma(\alpha(x_{n_k}, u)\mu(Tx_{n_k}, Tu), \mu(x_{n_k}, u)) \\ & = & \sigma(\alpha(x_{n_k}, u)\mu(x_{n_k+1}, Tu), \mu(x_{n_k}, u)) \\ & < & \mu(x_{n_k}, u) - \alpha(x_{n_k}, u)\mu(x_{n_k+1}, Tu), \end{array}$$

which is equivalent to

$$0 \leq \mu(x_{n_k+1}, Tu) = \mu(Tx_{n_k}, Tu) \leq \alpha(x_{n_k}, u)\mu(Tx_{n_k}, Tu) \leq \mu(x_{n_k}, u). \quad (20)$$

Letting $k \to \infty$ in the above equality, we have

$$\mu(x_{n_k+1}, Tu) \to 0 \text{ as } k \to \infty.$$

Therefore as in proof of Theorem 2.2 we have $Tu = u$. □

For the uniqueness of a fixed point of an α-admissible Σ-contraction with respect to σ, we shall suggest the following hypothesis.
(U) For all $x, y \in \mathrm{Fix}(T)$, we have $\alpha(x, y) \geq 1$.
Here, $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 2.4. *Adding condition (U) to the hypotheses of Theorem 2.2 (resp. Theorem 2.3), we obtain that u is the unique fixed point of T.*

Proof. Suppose that $u, v \in X$ are two fixed points of T. we have $\mu(u, u) = 0$, let in a contrary $\mu(u, u) > 0$, so by Lemma 2.1

$$\mu(u, u) = \mu(Tu, Tu) \leq \alpha(u, u)\mu(Tu, Tu) < \mu(u, u).$$

Which is a contradiction so we have $\mu(u, u) = 0$. By similar way we have $\mu(v, v) = 0$ and $\mu(u, v) = 0$, hence by (m_1)

$$u = v.$$

□

Example 2.5. *Let $X = [0, 1]$ and $\mu : X \to [0, \infty)$ defined by*

$$\mu(x, y) = \frac{x + y}{2}$$

be an M-metric on X, clearly (X, μ) is a complete M-metric space. Suppose that $T : X \to X$ be a mapping defined by

$$Tx = \frac{x^2}{3},$$

we prove that T is continuous in (X, μ). Assume that $x_0 \in X$ and $\varepsilon > 0$ be arbitrary we want to show that there exist $\delta > 0$, such that if $\mu(x_0, y) - \mu_{x_0,y} < \delta$, then $\mu(Tx_0, Ty) - \mu_{Tx_0,Ty} < \varepsilon$. Let $y \in X$, and $x_0 > y$, then $Tx_0 > Ty$, hence we get

$$\mu_{x_0,y} = \min\{\mu(x_0, x_0), \mu(y, y)\} = y,$$

and

$$\mu_{Tx_0,Ty} = \min\{\mu(Tx_0, Tx_0), \mu(Ty, Ty)\} = \frac{y^2}{3}.$$

So

$$\begin{aligned} &\mu(Tx_0, Ty) - \mu_{Tx_0,Ty} < \varepsilon \\ \Rightarrow &\frac{\frac{x_0^2}{3} + \frac{y^2}{3}}{2} - \frac{y^2}{3} < \varepsilon \\ \Rightarrow &\frac{x_0^2 - y^2}{6} < \varepsilon \\ (*) \qquad \Rightarrow &\frac{x_0 - y}{6}(x_0 + y) < \varepsilon, \end{aligned}$$

now let $\delta = \frac{1}{2}$ and $\mu(x_0, y) - \mu_{x_0,y} < \delta = \frac{1}{2}$, so

$$\begin{aligned} &0 \leq \mu(x_0, y) - \mu_{x_0,y} < \frac{1}{2} \\ \Rightarrow &0 \leq \frac{x_0 + y}{2} - y < \frac{1}{2} \\ \Rightarrow &0 \leq \frac{x_0 - y}{2} < \frac{1}{2} \\ \Rightarrow &0 \leq x_0 - y < 1 \\ (**) \qquad \Rightarrow &2x_0 \geq x_0 + y > -1 + 2x_0. \end{aligned}$$

By using $(**)$ in $(*)$ we get

$$\frac{x_0 - y}{6} < 2\varepsilon x_0 \Rightarrow \frac{x_0 - y}{2} < 6\varepsilon x_0,$$

hence we get

$$\mu(x_0, y) - \mu_{x_0,y} < 6\varepsilon x_0$$

therefore we let

$$\delta = \min\{\frac{1}{2}, 6\varepsilon x_0\}.$$

In similar way for $x_0 < y$ we get $\delta > 0$ such that $\mu(x_0, y) - \mu_{x_0,y} < \delta$. Hence T is continuous in arbitrary $x_0 \in X$. Let $\alpha : X \times X \to [0, \infty)$ be a function defined by

$$\alpha(x, y) = \begin{cases} 1, & x, y = 0; \\ \frac{x+y}{x^2+y^2}, & \text{otherwise.} \end{cases}$$

Then T is an α-admissible mapping on X and $T \in \mathcal{O}$, since for all $x, y \in X$ we have

$$\alpha(x, y) \geq 1.$$

Now we define $\sigma : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\sigma(t, u) = \frac{u}{2} - t.$$

So $\sigma \in \Sigma$. We have also $\alpha(x_0, Tx_0) \geq 1$, for an arbitrary $x_0 \in X$, on the other hand T is Σ-contraction with respect to σ. since for all $x, y \in X$ if $x, y \neq 0$,

$$\begin{aligned} \sigma(\alpha(x, y)\mu(Tx, Ty), \mu(x, y)) &= \sigma(\frac{x+y}{x^2+y^2}\frac{\frac{x^2}{3}+\frac{y^2}{3}}{2}, \frac{x+y}{2}) \\ \sigma(\frac{x+y}{6}, \frac{x+y}{2}) &= \frac{x+y}{4} - \frac{x+y}{6} \geq 0. \end{aligned}$$

If $x, y = 0$, we have, $\sigma(\alpha(x, y)\mu(Tx, Ty), \mu(x, y)) = \sigma(0, 0) = 0$. Hence for all $x, y \in X$,

$$\sigma(\alpha(x, y)\mu(Tx, Ty), \mu(x, y)) \geq 0.$$

Hence T is satisfied in the assumptions of Theorem 2.2 with respect to the defined functions α, σ and m as a metric on X and we have $T0 = 0$.

3. Consequences

In this section, we shall illustrate that several existing fixed point results in the literature can be derived from our main results by regarding Example 1.1 and also Example 12 - Example 19 in [3].

Theorem 3.1. [1] *Let $T : X \to X$ be an $\alpha - \vartheta$-contractive mapping where (X, d) is a complete metric space. Suppose that $T \in \mathcal{A}$ and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. If, either, T is continuous, or (X, d) is regular, then, there exists $u \in X$ such that $Tu = u$.*

Theorem 3.2. *Adding to the hypotheses of Theorem 3.1 the condition: For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, we obtain uniqueness of the fixed point.*

We conclude that the main result of Samet et al. [1] can be expressed as a corollary of our main result.

Theorem 3.3. *Theorem 3.1 is a consequence of Theorem 2.4.*

Proof. Taking $\sigma_E(t,s) = \vartheta(s) - t$ for all $s,t \in [0,\infty)$, in Theorem 2.4, we get that

$$\alpha(x,y)d(Tx,Ty) \leq \vartheta(d(x,y)), \text{ for all } x,y \in X.$$

We skip the details. □

Hence, all consequences, including the famous fixed point theorem of Banach, can be expressed easily from the above theorem as in [1]. We derive that the main result of Khojasteh et al. [2] can be expressed as a corollary of our main result.

Theorem 3.4. *Theorem 1.2 is a consequence of Theorem 2.4.*

Proof. It is enough to take $\alpha(x,y) = 1$ for all $x,y \in X$. □

4. Conclusion

It is clear that we can list several consequences of our main results by defining the mapping σ in a proper way like in the Example 1.1 and examples in [3] . In particular, we are able to get several existing fixed point theorems in the various settings (in the context of partially ordered set endowed with a metric, in the setting of cyclic contraction etc.) regarding Theorem (and hence Theorem 3.1). We omit the details since they are obvious.

References

[1] B. Samet, C. Vetro, P. Vetro, *Fixed point theorem for $\alpha - \psi$-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154 – 2165.

[2] F.Khoasteh, S. Shukla, S. Radenović, *A new approach to the study of fixed point theorems via simulation functions*, Filomat **6** (2015), no. 29, 1189–1194.

[3] H.H. Alsulami, E. Karapınar, F. Khojasteh, A.F. Roldán-López-de-Hierro, *A proposal to the study of contractions in quasi-metric spaces*, Discrete Dynamics in Nature and Society 2014, Article ID 269286, 10 pages.

[4] I. A.Rus, *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, 2001.

[5] M. Asadi, E. Karapınar, and P. Salimi, *New Extension of p-Metric Spaces with Some fixed point Results on M-metric spaces*, Journal of Inequalities and Applications 2014, 2014:18.

[6] O. Popescu, *Some new fixed point theorems for α-Geraghty contractive type maps in metric spaces*, Fixed Point Theory Appl.2014, 2014:190

[7] S.G. Matthews, *Partial metric topology*, Ann. New York Acad. Sci. **728** (1994), 183–197.

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