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ON gpr^{μ}-CLOSED MAPPINGS AND gpr^{μ}-OPEN MAPPINGS

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Abstract. In this paper we define new types of mappings known as gpr^{μ}-closed mappings, gpr^{μ}-open mappings and discuss its properties. Furthermore we introduce the concepts of supra pre*-normal spaces and also investigate its properties.

1. Introduction

In 1982, S.R.Malghan [5] introduced the notion of closed mappings in topology. Many researchers extended the work and investigated different types of closed and open functions in topological spaces. In 1983, Mashhour et.al.[6] introduced the notion of supra topological spaces and discussed the study of S-continuous maps and S^*-continuous maps in supra topological spaces. In 1999, Won Keun Min and Hong Soon Chang [13] introduced M-open mappings in supra topological spaces. In 2010, O.R.Sayed [9] investigated a class of sets called supra pre-open sets and discussed a class of maps called supra pre-open maps. In 2011, Ravi et.al. [7] introduced a class of sets called supra g-closed sets and analysed a class of maps called supra g-open and supra g-closed maps. Then in 2012, the same authors [4] used supra g-closed sets to define supra sg-closed sets and also introduced supra sg-closed mappings. In 2016, Vidhya Menon et.al. [10] made an extensive study of gpr^{μ}-closed sets. In this paper gpr^{μ}-closed mappings and gpr^{μ}-open mappings are introduced and their properties are studied. Also the notion of supra pre*-normal spaces is defined and studied.

Throughout this paper (X,τ), (Y,σ) and (Z,η) denote topological spaces on which no separation axioms are assumed unless explicitly stated. A subcollection $\mu \subset P(X)$ is called a supra topology on X if $X \in \mu$ and μ is closed under arbitrary union. (X,μ) is called a supra topological space. The elements of μ are said to be supra open in (X,μ) and complements of supra open sets are called supra closed sets. The supra closure of a set A, denoted by $cl^{\mu}(A)$, is the intersection of supra closed sets including A. The supra interior of a set

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A, denoted by $int^{\mu}(A)$, is the union of supra open sets included in A. We call μ a supra topology associated with τ if $\tau \subset \mu$. A subset A of X is called supra pre-closed [9] (resp. supra α-closed [1], supra regular closed, supra b-closed [8]) if $cl^{\mu}(int^{\mu}(A)) \subseteq A$ (resp. $cl^{\mu}(int^{\mu}(cl^{\mu}(A))) \subseteq A$, $A = cl^{\mu}(int^{\mu}(A))$, $int^{\mu}(cl^{\mu}(A)) \cap cl^{\mu}(int^{\mu}(A)) \subseteq A$). The complement of a supra pre-closed set (resp. supra α-closed set, supra regular closed set, supra b-closed set) is called supra pre-open set [9] (resp. supra α-open set [1], supra regular open set, supra b-open set [8]).
The collection of all supra pre-open, supra pre-closed, supra regular open, supra generalized pre-regular closed and supra generalized pre-regular open subsets of X are denoted by $PO^{\mu}(X)$, $PC^{\mu}(X)$, $RO^{\mu}(X)$, $GPRC^{\mu}(X)$, $GPRO^{\mu}(X)$.

2. Preliminaries

Definition 1. Let A be a subset of (X, μ). Then

(i) the supra pre-closure [12] of a set A, denoted by $pcl^{\mu}(A) = \bigcap \{B : B$ is a supra pre-closed set and $A \subseteq B\}$.
(ii) the supra pre-interior [12] of a set A, denoted by $pint^{\mu}(A) = \bigcup \{B : B$ is a supra pre-open set and $B \subseteq A\}$.
(iii) the supra gpr-closure of a set A, denoted by $gpr^{\mu}\text{-}cl(A) = \bigcap \{B : B$ is a gpr^{μ}-closed set and $A \subseteq B\}$.
(iv) the supra gpr-interior of a set A, denoted by $gpr^{\mu}\text{-}int(A) = \bigcup \{B : B$ is a gpr^{μ}-open set and $B \subseteq A\}$.
(v) the supra α-closure [1] of a set A, denoted by supra $\alpha cl(A) = \bigcap \{B : B$ is a supra α-closed set and $A \subseteq B\}$.
(vi) the supra b-closure [8] of a set A, denoted by $cl^{\mu}_{b}(A) = \bigcap \{B : B$ is a supra b-closed set and $A \subseteq B\}$.

Definition 2. A subset A of a space (X, μ) is called

(i) supra generalized closed [7] (briefly g^{μ}-closed) if $cl^{\mu}(A) \subseteq U$ whenever $A \subseteq U$ and U is supra open in (X, μ).
(ii) supra generalized pre-closed [12] (briefly gp^{μ}-closed) if $pcl^{\mu}(A) \subseteq U$ whenever $A \subseteq U$ and U is supra open in (X, μ).
(iii) supra generalized pre-regular closed [12] (briefly gpr^{μ}-closed) if $pcl^{\mu}(A) \subseteq U$ whenever $A \subseteq U$ and U is supra regular open in (X, μ).
(iv) supra generalized α-closed [11] (briefly $g\alpha^{\mu}$-closed) if $\alpha cl^{\mu}(A) \subseteq U$ whenever $A \subseteq U$ and U is supra α-open in (X, μ).
(v) supra α-generalized closed [11] (briefly αg^{μ}-closed) if $\alpha cl^{\mu}(A) \subseteq U$ whenever $A \subseteq U$ and U is supra open in (X, μ).
(vi) supra regular generalized closed [12] (briefly rg^{μ}-closed) if $cl^{\mu}(A) \subseteq U$ whenever $A \subseteq U$ and U is supra regular open in (X, μ).
(vii) supra generalized b-regular closed [11] (briefly $g^{\mu}br$-closed) if $bcl^{\mu}(A) \subseteq U$ whenever $A \subseteq U$ and U is supra regular open in (X, μ).

Definition 3. Let (X, τ) and (Y, σ) be two topological spaces with supra topology μ associated with σ. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) supra closed [7] if the image of each closed set in X is supra closed in Y.
(ii) supra pre-closed [9] if the image of each closed set in X is supra pre-closed in Y.
(iii) supra regular closed if the image of each closed set in X is supra regular closed in Y.
(iv) g^{μ}-closed [7] if the image of each closed set in X is g^{μ}-closed in Y.
(v) gp^{μ}-closed if the image of each closed set in X is gp^{μ}-closed in Y.
(vi) $g\alpha^{\mu}$-closed if the image of each closed set in X is $g\alpha^{\mu}$-closed in Y.
(vii) αg^{μ}-closed if the image of each closed set in X is αg^{μ}-closed in Y.
(viii) rg^{μ}-closed if the image of each closed set in X is rg^{μ}-closed in Y.

The respective open functions are defined in a similar manner.

Definition 4. Let (X, τ) and (Y, σ) be two topological spaces with supra topology μ associated with τ. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) supra-continuous [7] if $f^{-1}(V)$ is supra closed (resp. supra open) in X for every closed set (resp. open set) V of Y.
(ii) gpr^{μ}-continuous [12] if $f^{-1}(V)$ is gpr^{μ}-closed (resp. gpr^{μ}-open) in X for every closed set (resp. open set) V of Y.

Definition 5. Let (X, τ) and (Y, σ) be two topological spaces with supra topology μ associated with σ. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) strongly supra continuous if $f^{-1}(V)$ is closed (resp. open) in X for every supra closed set (resp. supra open set) V of Y.
(ii) strongly gpr^{μ}-continuous if $f^{-1}(V)$ is closed (resp. open) in X for every gpr^{μ}-closed set (resp. gpr^{μ}-open set) V of Y.

Definition 6. Let (X, τ) and (Y, σ) be two topological spaces with supra topologies λ and μ associated with τ and σ respectively. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) supra irresolute [4] if $f^{-1}(V)$ is supra closed (resp. supra open) in X for every supra closed set (resp. supra open set) V of Y.
(ii) supra pre-irresolute if $f^{-1}(V)$ is supra pre-closed (resp. supra pre-open) in X for every supra pre-closed set (resp. supra pre-open set) V of Y.
(iii) supra *gpr*-irresolute if $f^{-1}(V)$ is gpr^{λ}-closed (resp. gpr^{λ}-open) in X for every gpr^{μ}-closed set (resp. gpr^{μ}-open set) V of Y.

Definition 7. (i) A space (X, τ) is called a $T_{1/2}$ space [2] if every g-closed set is closed.
(ii) A space (X, μ) is called a supra pre-regular $T_{1/2}$ space [12] if every gpr^{μ}-closed set is supra pre-closed.

3. gpr^{μ}-closed map & gpr^{μ}-open map

Definition 8. Let (X, τ) and (Y, σ) be two topological spaces with supra topology μ associated with σ. A function $f : (X, \tau) \to (Y, \sigma)$ is called gpr^{μ}-closed (resp. gpr^{μ}-open) if the image of each closed (resp. open) set in X is gpr^{μ}-closed (resp. gpr^{μ}-open) in Y.

Example 1
(i) Let $X = Y = \{a, b, c\}$, with the topology $\tau = \{\phi, X, \{a\}\}$, $\sigma = \{\phi, Y, \{b, c\}\}$ and the supra topology $\mu = \{\phi, Y, \{a, c\}, \{b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined as $f(a) = a$, $f(b) = b$, $f(c) = c$. Then f is gpr^{μ}-closed.
(ii) Let $X = Y = \{0, 1, 2\}$, with the topology $\tau = \{\phi, X, \{1\}, \{1, 2\}\}$, $\sigma = \{\phi, Y, \{1\}\}$ and the supra topology $\mu = \{\phi, Y, \{1\}, \{0, 2\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map defined as $f(0) = 1$, $f(1) = 2$, $f(2) = 1$. Then f is gpr^{μ}-open.

Definition 9. Let (X, τ) and (Y, σ) be two topological spaces with supra topology μ associated with σ. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) p-gpr^{μ}-open (resp. p-gpr^{μ}-closed) if the image of each pre-open (resp. pre-closed) set in X is gpr^{μ}-open (resp. gpr^{μ}-closed) in Y.
(ii) $g^{\mu}br$-closed (resp. $g^{\mu}br$-open) if the image of each closed (resp. open) set in X is $g^{\mu}br$-closed (resp. $g^{\mu}br$-open) in Y.

Theorem 3.1. *Every closed map is gpr^{μ}-closed.*

Proof. Let (X, τ) and (Y, σ) be two topological spaces with supra topology μ associated with σ. Let $f : (X, \tau) \to (Y, \sigma)$ be a closed map and A be a closed set in X. Then $f(A)$ is closed in Y. Since μ associated with σ, $\sigma \subset \mu$. Therefore $f(A)$ is supra closed in Y which is gpr^{μ}-closed in Y. Hence f is gpr^{μ}-closed. □

Theorem 3.2. *Every supra closed (resp. supra pre-closed, g^{μ}-closed, gp^{μ}-closed, $g\alpha^{\mu}$-closed, αg^{μ}-closed, rg^{μ}-closed) map is gpr^{μ}-closed.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a supra closed map with $\sigma \subset \mu$ and V be a closed set in X. Then $f(V)$ is supra closed (resp. supra pre-closed, g^{μ}-closed, gp^{μ}-closed, $g\alpha^{\mu}$-closed, αg^{μ}-closed, rg^{μ}-closed) in Y. Since every supra closed (resp. supra pre-closed, g^{μ}-closed, gp^{μ}-closed, $g\alpha^{\mu}$-closed, αg^{μ}-closed, rg^{μ}-closed) set is gpr^{μ}-closed, $f(V)$ is gpr^{μ}-closed in Y. □

Converse of the theorem need not be true as seen from the following example:

Example 2:
(i) Let $X = Y = \{a, b, c\}$, with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{\phi, Y, \{b, c\}\}$ and the supra topology $\mu = \{\phi, Y, \{a, b\}, \{b, c\}\}$. Define the map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$. Then the function f is gpr^{μ}-closed but not supra closed as $f(\{b, c\}) = \{b, c\}$ is not supra closed in Y.

(ii) Let $X = Y = \{a, b, c\}$, with the topology $\tau = \{\phi, X, \{b, c\}\}$, $\sigma = \{\phi, Y, \{a\}\}$ and the supra topology $\mu = \{\phi, Y, \{a\}, \{a, b\}\}$. Define the map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$. Then the function f is gpr^μ-closed but not supra pre-closed as $f(\{a\}) = \{a\}$ is not supra pre-closed in Y.
(iii) Let $X = Y = \{a, b, c\}$, with the topology $\tau = \{\phi, X, \{a\}, \{b, c\}\}$, $\sigma = \{\phi, Y, \{b, c\}\}$ and the supra topology $\mu = \{\phi, Y, \{a\}, \{a, b\}, \{b, c\}\}$. Define the map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Then the function f is gpr^μ-closed but not g^μ-closed as $f(\{a\}) = \{b\}$ is not g^μ-closed in Y.
(iv) Let $X = Y = \{a, b, c\}$, with the topology $\tau = \{\phi, X, \{a\}, \{b, c\}\}$, $\sigma = \{\phi, Y, \{b, c\}\}$ and the supra topology $\mu = \{\phi, Y, \{a\}, \{a, b\}, \{b, c\}\}$. Define the map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c$, $f(b) = a$, $f(c) = b$. Then the function f is gpr^μ-closed but not gp^μ-closed as $f(\{b, c\}) = \{a, b\}$ is not gp^μ-closed in Y.
(v) Let $X = Y = \{a, b, c, d\}$, with the topology $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, $\sigma = \{\phi, Y, \{b, c, d\}\}$ and the supra topology $\mu = \{\phi, Y, \{a, b, d\}, \{b, c, d\}\}$. Define the map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$. Then the function f is gpr^μ-closed but not αg^μ-closed as $f(\{b, c, d\}) = \{b, c, d\}$ is not αg^μ-closed in Y.
(vi) Let $X = Y = \{a, b, c, d\}$, with the topology $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, $\sigma = \{\phi, Y, \{b, c, d\}\}$ and the supra topology $\mu = \{\phi, Y, \{a, b, d\}, \{b, c, d\}\}$. Define the map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$. Then the function f is gpr^μ-closed but not $g\alpha^\mu$-closed as $f(\{b, d\}) = \{b, d\}$ is not $g\alpha^\mu$-closed in Y.
(vii) Let $X = Y = \{a, b, c, d\}$, with the topology $\tau = \{\phi, X, \{a\}, \{b, c\}, \{c\}, \{a, c\}, \{a, c, d\}, \{a, b, c\}\}$, $\sigma = \{\phi, Y, \{a\}\}$ and the supra topology $\mu = \{\phi, Y, \{a\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$. Define the map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$. Then the function f is gpr^μ-closed but not rg^μ-closed as $f(\{b\}) = \{b\}$ is not rg^μ-closed in Y.

Theorem 3.3. *Every gpr^μ-closed map is $g^\mu br$-closed.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a gpr^μ-closed map with supra topology μ associated with σ and V be a closed set in X. Then $f(V)$ is gpr^μ-closed in Y. By theorem 3.1 [3], every gpr^μ-closed set is $g^\mu br$-closed. Hence $f(V)$ is $g^\mu br$-closed in Y. □

However the converse need not be true.

Example 3: Let $X = Y = \{a, b, c, d\}$, with the topology $\tau = \{\phi, X, \{a\}, \{b, c, d\}\}$, $\sigma = \{\phi, Y, \{a\}\}$ and the supra topology $\mu = \{\phi, Y, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$. Define the map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$. Then the function f is $g^\mu br$-closed but not gpr^μ-closed as $f(\{a\}) = \{a\}$ is not gpr^μ-closed in Y.

Theorem 3.4. *Let (X,τ) and (Y,σ) be the two topological spaces with supra topology μ associated with σ.*

(i) *If $f:(X,\tau)\to(Y,\sigma)$ is gpr^μ-closed then gpr^μ-$cl(f(A))\subset f(cl(A))$ for each set A in X.*
(ii) *If $f:(X,\tau)\to(Y,\sigma)$ is gpr^μ-open then $f(int(A))\subset gpr^\mu$-$int(f(A))$ for each set A in X.*

Proof. (i) Let $A\subset X$ and $f:(X,\tau)\to(Y,\sigma)$ be a gpr^μ-closed map. Then $f(cl(A))$ is gpr^μ-closed in Y. Also $f(A)\subset f(cl(A))$. This implies gpr^μ-$cl(f(A))\subset gpr^\mu$-$cl(f(cl(A)))$. That is gpr^μ-$cl(f(A))\subset f(cl(A))$.

(ii) Let $A\subset X$ and $f:(X,\tau)\to(Y,\sigma)$ be a gpr^μ-open map. Then $f(int(A))$ is gpr^μ-open in Y. Also $f(int(A))\subset f(A)$. This implies gpr^μ-$int(f(int(A)))\subset gpr^\mu$-$int(f(A))$. As $f(int(A))$ is gpr^μ-open, $f(int(A))\subset gpr^\mu$-$int(f(A))$.

□

Theorem 3.5. *Let (X,τ) and (Y,σ) be the two topological spaces with supra topology μ associated with σ. A function $f:(X,\tau)\to(Y,\sigma)$ is gpr^μ-closed iff for each subset S of Y and each open set U containing $f^{-1}(S)$ there is a gpr^μ-open set V of Y such that $S\subset V$ and $f^{-1}(V)\subset U$.*

Proof. Let f be gpr^μ-closed, $S\subset Y$ and U be an open set of X such that $f^{-1}(S)\subset U$. Then $f(X-U)\in GPRC^\mu(Y)$ and $V=Y-f(X-U)\in GPRO^\mu(Y)$. $f^{-1}(S)\subset U$ implies $S\subset V$ and $f^{-1}(V)=X-f^{-1}(f(X-U))\subset X-(X-U)=U$.
Conversely let F be closed in X, then $f^{-1}(Y-f(F))\subset X-F$. By hypothesis there exist $V\in GPRO^\mu(Y)$ such that $Y-f(F)\subset V$ and $f^{-1}(V)\subset X-F$ and so $F\subset X-f^{-1}(V)$. Hence $Y-V\subset f(F)\subset f(X-f^{-1}(V))\subset Y-V$. This implies $f(F)\subset Y-V$. Thus $f(F)=Y-V$. Thus $f(F)$ is gpr^μ-closed in Y and therefore the function f is gpr^μ-closed. □

Theorem 3.6. *Let $f:(X,\tau)\to(Y,\sigma)$ be a bijective map with supra topology μ associated with σ. Then the following are equivalent:*

(i) *f is a gpr^μ-open map*
(ii) *f is a gpr^μ-closed map*
(iii) *f^{-1} is a gpr^μ-continuous map.*

Proof. (i) ⇒ (ii) Suppose B is closed in X. Then $X-B$ is open in X. By (i) $f(X-B)$ is a gpr^μ-open set in Y. Since f is bijective, $f(X-B)=Y-f(B)$. Hence $f(B)$ is a gpr^μ-closed set in Y. Therefore f is a gpr^μ-closed map.

(ii) ⇒ (iii) Let f be a gpr^μ-closed map and B be a closed set in X. Since $(f^{-1})^{-1}=f$ and $f(B)$ is gpr^μ-closed in Y, $(f^{-1})^{-1}(B)=f(B)$ is gpr^μ-closed set in Y. Hence by definition 4 (ii) f^{-1} is a gpr^μ-continuous map.

(iii) $\Rightarrow$ (i) Let A be an open set in X. Since f^{-1} is gpr^{μ}-continuous map, $(f^{-1})^{-1}(A) = f(A)$ is a gpr^{μ}-open set in Y. Hence f is a gpr^{μ}-open map.

□

Theorem 3.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be a map with supra topology μ associated with σ. If a function $f : (X, \tau) \to (Y, \sigma)$ is strongly supra continuous, gpr^{μ}-closed and A is g-closed set in X, then $f(A)$ is gpr^{μ}-closed in Y.*

Proof. Let $f(A) \subseteq O$ where O is a supra regular open set of Y. Then $f^{-1}(O)$ is an open set of X containing A. Since A is g-closed, we have $cl(A) \subseteq f^{-1}(O)$. Now, f is gpr^{μ}-closed implies $f(cl(A))$ is gpr^{μ}-closed in Y. That is $pcl^{\mu}(f(cl(A))) \subseteq O$, where O is supra regular open in Y. Thus $pcl^{\mu}(f(A)) \subseteq pcl^{\mu}(f(cl(A))) \subseteq O$. Hence $pcl^{\mu}(f(A)) \subseteq O$. Therefore $f(A)$ is a gpr^{μ}-closed set in Y. □

Definition 10. Let (X, τ) and (Y, σ) be two topological spaces with supra topologies μ and λ associated with τ and σ respectively. A function $f : (X, \tau) \to (Y, \sigma)$ is called *p-gpr^{μ}*-continuous if $f^{-1}(F)$ is gpr^{μ}-closed in X for every supra pre-closed set F of Y.

Theorem 3.8. *Let (X, τ) and (Y, σ) be two topological spaces with supra topologies μ and λ associated with τ and σ respectively. If $f : (X, \tau) \to (Y, \sigma)$ is p-gpr^{μ}-continuous and Y is supra pre-regular $T_{1/2}$ then f is supra gpr-irresolute.*

Proof. Let F be any gpr^{λ}-closed set of Y. Since Y is supra pre-regular $T_{1/2}$, then F is supra pre-closed in Y. By hypothesis f is *p-gpr^{μ}*-continuous, so $f^{-1}(F)$ is gpr^{μ}-closed in X. Hence f is supra *gpr*-irresolute. □

Definition 11. Let (X, τ) and (Y, σ) be two topological spaces with supra topologies μ and λ associated with τ and σ respectively. A function $f : (X, \tau) \to (Y, \sigma)$ is called supra r^*-closed (resp. supra r^*-open) if the image of each supra regular closed (resp. supra regular open) set in X is supra regular closed (resp. supra regular open) in Y.

Remark 1. Let (X, τ) and (Y, σ) be two topological spaces with supra topologies μ and λ associated with τ and σ respectively. A function $f : (X, \tau) \to (Y, \sigma)$ is supra r^*-closed iff for each subset S of Y and each supra regular open set U containing $f^{-1}(S)$ there is a supra regular open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Theorem 3.9. *Let (X, τ) and (Y, σ) be two topological spaces with supra topologies λ and μ associated with τ and σ respectively. If $f : (X, \tau) \to (Y, \sigma)$ is p-gpr^{λ}-continuous, supra r^*-closed and K is a gpr^{μ}-closed set of Y, then f is supra gpr-irresolute.*

Proof. Let K be any gpr^{μ}-closed set of Y and U be a supra regular open set containing $f^{-1}(K)$. Since f is supra r^*-closed, by remark 1 there exist a supra regular open set V of Y such that $K \subset V$ and $f^{-1}(V) \subset U$. Since K is gpr^{μ}-closed in Y, $pcl^{\mu}(K) \subset V$. This implies $f^{-1}(pcl^{\mu}(K)) \subset f^{-1}(V) \subset U$. Now, f is p-gpr^{λ}-continuous implies $f^{-1}(pcl^{\mu}(K))$ is gpr^{λ}-closed in X and hence $pcl^{\lambda}(f^{-1}(K)) \subset pcl^{\lambda}(f^{-1}(pcl^{\mu}(K))) \subset U$. Hence $f^{-1}(K)$ is gpr^{λ}-closed. □

Theorem 3.10. *Let (X, τ) and (Y, σ) be two topological spaces with the supra topologies λ and μ associated with τ and σ respectively.*

(i) *If the bijection $f : (X, \tau) \to (Y, \sigma)$ is supra r^*-open and p-gpr^{λ}-continuous, then it is supra gpr-irresolute.*
(ii) *If the bijection $f : (X, \tau) \to (Y, \sigma)$ is supra r^*-open and supra pre-irresolute, then it is supra gpr-irresolute.*

Proof. (i) Let A be a gpr^{μ}-closed in Y. Let $f^{-1}(A) \subseteq O$ where O is supra regular open in X. Since f is supra r^*-open map, $f(O)$ is supra regular open in Y. Thus $A \subseteq f(O)$ implies $pcl^{\mu}(A) \subseteq f(O)$. That is $f^{-1}(pcl^{\mu}(A)) \subseteq O$. Since f is p-gpr^{λ}-continuous, $pcl^{\lambda}(f^{-1}(pcl^{\mu}(A))) \subseteq O$ and hence $pcl^{\lambda}(f^{-1}(A)) \subseteq O$. Hence $f^{-1}(A)$ is gpr^{λ}-closed in X. Hence f is supra *gpr*-irresolute.
(ii) Let B be a gpr^{μ}-closed in Y and let $f^{-1}(B) \subseteq O$ where O is supra regular open in X. Since f is supra r^*-open map, $f(O)$ is supra regular open in Y. Thus $B \subseteq f(O)$ implies $pcl^{\mu}(B) \subseteq f(O)$. That is $f^{-1}(pcl^{\mu}(B)) \subseteq O$. Since f is supra pre-irresolute, $f^{-1}(pcl^{\mu}(B))$ is a supra pre-closed set in X. Thus $pcl^{\lambda}(f^{-1}(B)) \subseteq pcl^{\lambda}(f^{-1}(pcl^{\mu}(B))) = f^{-1}(pcl^{\mu}(B)) \subseteq O$. Hence $f^{-1}(B)$ is gpr^{λ}-closed in X. Hence f is supra *gpr*-irresolute. □

Remark 2. Composition of two gpr^{μ}-closed maps need not be gpr^{μ}-closed.

Example 4: Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two maps with the supra topologies μ, λ associated with σ, η respectively. Let $X = Y = \{0, 1, 2\}$, $Z = \{0, 1, 2, 3\}$, $\tau = \{\phi, X, \{0\}\}$, $\sigma = \{\phi, Y, \{0, 1\}\}$, $\mu = \{\phi, Y, \{0, 1\}, \{1, 2\}, \{1\}\}$, $\eta = \{\phi, Z, \{0\}\}$ and $\lambda = \{\phi, Z, \{0, 2\}, \{2, 3\}, \{0\}, \{0, 2, 3\}\}$. Let f be a identity mapping. Define the map $g : (Y, \sigma) \to (Z, \eta)$ by $g(0) = 1$, $g(1) = 2$, $g(2) = 3$. Then f and g are gpr^{μ}-closed mappings but $(g \circ f)(\{1, 2\})$ is not a gpr^{μ}-closed map.

Theorem 3.11. *Let (X, τ), (Y, σ) and (Z, η) be the topological spaces with supra topology μ associated with η. If $f : (X, \tau) \to (Y, \sigma)$ is closed and $g : (Y, \sigma) \to (Z, \eta)$ is gpr^{μ}-closed map then $g \circ f$ is gpr^{μ}-closed.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a closed map, that is for each closed set $a \in X$, $f(a)$ is closed in Y. Let $g : (Y, \sigma) \to (Z, \eta)$ be a gpr^{μ}-closed map that is for each closed set $y \in Y$, $g(y)$ is gpr^{μ}-closed in Z. Also $f(a) \in Y$. Let $U = f(a)$. Now $(g \circ f)(a) = g(f(a)) = g(U)$ is gpr^{μ}-closed in Z. Hence $g \circ f$ is gpr^{μ}-closed. □

Theorem 3.12. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two maps with supra topologies λ and μ associated with σ and η respectively.*

(i) *If $g \circ f$ is gpr^{μ}-closed and f is continuous surjective, then g is gpr^{μ}-closed.*

(ii) *If $g \circ f$ is closed and g is gpr^{λ}-continuous injective, then f is gpr^{λ}-closed.*

Proof. (i) Let A be a closed set in Y. Then $f^{-1}(A)$ is a closed set in X. Since $g \circ f$ is a gpr^{μ}-closed map, $(g \circ f)(f^{-1}(A)) = g(f(f^{-1}(A)) = g(A)$ (because f is surjective) is a gpr^{μ}-closed set in Z. Hence g is gpr^{μ}-closed.
(ii) Let A be a closed set in X. Then $g(f(A))$ is closed in Z. Therefore $g^{-1}(g(f(A))) = f(A)$ (because g is injective) is a gpr^{λ}-closed set in Y. Hence f is a gpr^{λ}-closed map. □

Theorem 3.13. *Let (X, τ), (Y, σ) and (Z, η) be topological spaces with supra topologies λ and μ associated with σ and η respectively. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be the two functions such that $g \circ f : X \to Z$ is gpr^{μ}-closed. If g is supra pre-irresolute, supra r^*-closed injection then f is gpr^{μ}-closed.*

Proof. Let F be a closed set of X and U be a supra regular open set of Y containing $f(F)$. Then $(g \circ f)(F)$ is gpr^{μ}-closed in Z. Let $V = Y - U$. Then V is supra regular closed set in Y. Since g is supra r^*-closed injective, $g(V)$ is supra regular closed in Z and $g(V) \cap (g \circ f)(F) = \phi$. Hence $g(V) \cap pcl^{\mu}(g \circ f)(F) = \phi$ that is $pcl^{\mu}((g \circ f)(F)) \subset Z - g(V)$. Since g is supra pre-irresolute, $pcl^{\lambda}(f(F)) = pcl^{\lambda}(g^{-1}(g \circ f)(F)) \subset pcl^{\lambda}(g^{-1}(pcl^{\mu}(g \circ f)(F)))$ $= g^{-1}(pcl^{\mu}(g \circ f)(F)) \subset Y - V = U$. Thus f is gpr^{λ}-closed in Y. □

Theorem 3.14. *Let (X, τ), (Y, σ) and (Z, η) be the topological spaces with supra topologies μ, λ and α associated with τ, σ and η respectively. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be the two functions such that*

(i) *if f is supra gpr-irresolute and g is p-gpr^{λ}-continuous then $g \circ f : X \to Z$ is p-gpr^{μ}-continuous.*
(ii) *if f is p-gpr^{μ}-continuous and g is supra pre-irresolute then $g \circ f : X \to Z$ is p-gpr^{μ}-continuous.*

Proof. (i) Let V be a supra pre-closed set in Z. Since g is p-gpr^{λ}-continuous, $g^{-1}(V)$ is gpr^{λ}-closed in (Y, σ). As f is supra gpr-irresolute, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is gpr^{μ}-closed in (X, τ). Hence $g \circ f$ is p-gpr^{μ}-continuous.
(ii) Let V be a supra pre-closed set in Z. Since g is supra pre-irresolute, $g^{-1}(V)$ is supra pre-closed in (Y, σ). As f is p-gpr^{μ}-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is gpr^{μ}-closed in (X, τ). Hence $g \circ f$ is p-gpr^{μ}-continuous.

□

Theorem 3.15. *Let (X, τ), (Y, σ) and (Z, η) be the topological spaces and μ be the supra topology associated with η. Let $f : (X, \tau) \to (Y, \sigma)$ be a g-closed map, $g : (Y, \sigma) \to (Z, \eta)$ be a gpr^{μ}-closed map and (Y, σ) be a $T_{1/2}$ space. Then $g \circ f : (X, \tau) \to (Z, \eta)$ is a gpr^{μ}-closed map.*

Proof. Let A be a closed set of (X,τ). By assumption, $f(A)$ is g-closed in Y. Since (Y,σ) is $T_{1/2}$ space, $f(A)$ is closed in Y and $g(f(A))$ is gpr^{μ}-closed in (Z,η). Thus $(g\circ f)(A)$ is gpr^{μ}-closed in (Z,η). Hence $g\circ f$ is gpr^{μ}-closed in (Z,η). □

Theorem 3.16. *Let (X,τ), (Y,σ) and (Z,η) be the topological spaces with supra topologies λ and μ associated with respect to σ and η respectively. Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be two maps and $g\circ f$ be a p-gpr^{μ}-open function. Then*

(i) *if f is surjective pre-irresolute map, then g is p-gpr^{μ}-open.*
(ii) *if g is an injective supra gpr-irresolute map, then f is p-gpr^{λ}-open.*

Proof. (i) Let V be a pre-open set in Y. Then $f^{-1}(V)$ is pre-open in X. Since $g\circ f$ is *p-gpr*$^{\mu}$-open function, $(g\circ f)(f^{-1}(V)) = g(V)$ is gpr^{μ}-open in Z. Hence g is *p-gpr*$^{\mu}$-open map.

(ii) Let U be pre-open subset of X. Since $g\circ f$ is *p-gpr*$^{\mu}$-open function, $(g\circ f)(U)$ is gpr^{μ}-open in Z. Also by assumption g is supra *gpr*-irresolute injective, $g^{-1}(g\circ f)(U) = f(U)$ is gpr^{λ}-open in Y. Hence f is *p-gpr*$^{\lambda}$-open map.

□

Definition 12. X is said to be supra pre*-normal if for each pair of disjoint supra regular closed sets A and B, there exist disjoint supra pre-open sets U and V such that $A\subset U$ and $B\subset V$.

Theorem 3.17. *A space X is supra pre*-normal iff for every supra regular closed set A and every supra regular open set B containing A, there exist a supra pre-open set U such that $A\subset U\subset pcl^{\mu}(U)\subset B$.*

Proof. Necessity : Let A be a supra regular closed set in X and B be a supra regular open set containing A. Then A and $X-B$ are disjoint supra regular closed sets in X. Since X is supra pre*-normal there exist disjoint supra pre-open sets U and V such that $A\subset U$ and $X-B\subset V$. Thus $A\subset U\subset X-V\subset B$. Since $X-V$ is supra pre-closed, $A\subset U\subset pcl^{\mu}(U)\subset X-V\subset B$.
Sufficiency: Let A and B be any two disjoint supra regular closed sets in X. That is $A\subset X-B$. Since $X-B$ is supra regular open, there exist a supra pre-open set U such that $A\subset U\subset pcl^{\mu}(U)\subset X-B$. Let $V = X - pcl^{\mu}$ (U) $= pint^{\mu}(X-U)$. Thus, V is the largest supra pre-open set contained in $X-U$. Also $B\subset V$. Thus $A\subset U$, $B\subset V$ and $U\cap V=\phi$. Hence X is supra pre*-normal. □

Theorem 3.18. *Let (X,τ) be a topological space and μ be the associated supra topology with τ. Then the following statements are equivalent:*

(i) *X is supra pre*-normal.*
(ii) *for any pair of disjoint supra regular closed sets A and B of X, there exist disjoint gpr^{μ}-open sets U and V of X such that $A\subset U$ and $B\subset V$.*

(iii) *for each supra regular closed set A and each supra regular open set B containing A, there exist a* gpr^{μ}*-open set U such that* $cl^{\mu}(A) \subset U \subset pcl^{\mu}(U) \subset B$.

Proof. (i) $\Rightarrow$ (ii) Let A and B be disjoint supra regular closed subsets of X. Since X is supra pre*-normal, for each pair of disjoint supra regular closed sets A and B there exist disjoint supra pre-open sets U and V such that $A \subset U$ and $B \subset V$. Also each supra pre-open set is gpr^{μ}-open. Thus we have for each pair of disjoint supra regular closed sets A and B, there exist disjoint gpr^{μ}-open sets U and V such that $A \subset U$ and $B \subset V$.
(ii) $\Rightarrow$ (iii) Let A be a supra regular closed subset of X and B be a supra regular open set such that $A \subset B$. Then A and $X - B$ are disjoint supra regular closed subsets of X. By (ii) there exist disjoint gpr^{μ}-open sets U and V such that $A \subset U$ and $X - B \subset V$. This implies $A \subset U \subset X - V \subset B$. Since B is supra regular open and $X - V$ is gpr^{μ}-closed, $pcl^{\mu}(X - V) \subset B$. Thus $A \subset U \subset pcl^{\mu}(X - V) \subset B$. Hence $cl^{\mu}(A) \subset U \subset pcl^{\mu}(U) \subset B$.
(iii) $\Rightarrow$ (i) Let A and $X - B$ be two disjoint supra regular closed subsets of X. Then B is supra regular open such that $A \subset B$. Therefore there exists a gpr^{μ}-open set U such that $A \subset U \subset pcl^{\mu}(U) \subset B$. Since U is gpr^{μ}-open, $A \subset pint^{\mu}(U)$. Let $V = pint^{\mu}(U)$, then V is supra pre-open and $A \subset V \subset pcl^{\mu}(V) \subset B$. Thus $A \subset V$, $X - B \subset X - pcl^{\mu}(V)$ which is supra pre-open and $V \cap (X - pcl^{\mu}(V)) = \phi$. Hence X is supra pre*-normal. ☐

Theorem 3.19. *Let* (X, τ) *be a topological space and* μ *be the associated supra topology with* τ*. If every supra closed (resp. supra open) set is supra regular closed (resp. supra regular open), then the following are equivalent:*

(i) *X is supra pre*-normal.*
(ii) *for each supra regular closed set A and each* g^{μ}*-open set B containing A, there exist a supra pre-open set U such that* $A \subset U \subset pcl^{\mu}(U) \subset int^{\mu}(B)$.
(iii) *for each* g^{μ}*-closed set A and each supra regular open set B containing A, there exist a supra pre-open set U such that* $cl^{\mu}(A) \subset U \subset pcl^{\mu}(U) \subset B$.
(iv) *for each* g^{μ}*-closed set A and each supra regular open set B containing A, there exist a* gpr^{μ}*-open set G such that* $cl^{\mu}(A) \subset G \subset pcl^{\mu}(G) \subset B$.

Proof. (i) $\Rightarrow$ (ii) Let A be a supra regular closed subset of X and B be an g^{μ}-open set such that $A \subset B$. Then by theorem 3.18 [7], $A \subset int^{\mu}(B)$. Now, by hypothesis X is supra pre*-normal and $int^{\mu}(B)$ is supra regular open. Therefore by theorem 3.17, there exist a supra pre-open set U such that $A \subset U \subset pcl^{\mu}(U) \subset int^{\mu}(B)$.
(ii) $\Rightarrow$ (iii) Let A be any g^{μ}-closed subset of X and B be a supra regular open set such that $A \subset B$. Then $cl^{\mu}(A) \subset B$. Since every supra regular open set is g^{μ}-open, B is g^{μ}-open. By hypothesis $cl^{\mu}(A)$ is supra regular closed. Therefore, there exist a supra pre-open set U such that $cl^{\mu}(A) \subset U \subset pcl^{\mu}(U) \subset int^{\mu}(B) \subset B$.

(iii) $\Rightarrow$ (iv) Let A be a g^μ-closed set and B be a supra regular open set containing A. Then there exist a supra pre-open set U such that $cl^\mu(A) \subset U \subset pcl^\mu(U) \subset B$. Since each supra pre-open set is gpr^μ-open, there exist a gpr^μ-open set G such that $cl^\mu(A) \subset G \subset pcl^\mu(G) \subset B$.
(iv) $\Rightarrow$ (i) Let A and B be two disjoint supra regular closed subsets of X. Then A is g^μ-closed and $A \subset X - B$. By hypothesis there exists a gpr^μ-open set G such that $A \subset cl^\mu(A) \subset G \subset pcl^\mu(G) \subset X - B$. Since G is gpr^μ-open, $A \subset pint^\mu(G)$. Let $V = pint^\mu(G)$, then V is supra pre-open and $A \subset V \subset pcl^\mu(V) \subset X - B$. Thus $A \subset V$, $B \subset X - pcl^\mu(V)$ which is supra pre-open and $V \cap (X - pcl^\mu(V)) = \phi$. Therefore X is supra pre*-normal. □

References

[1] R.Devi, S.Sampathkumar and M.Caldas, *On Supra α-open sets and $S\alpha$-continuous functions*, General Mathematics **16(2)**(2008), 77-84.

[2] W.Dunham, *$T_{\frac{1}{2}}$-spaces*, Kyungpook Math.J. **17**(1977), 161-169.

[3] Gnanambal Ilango and Vidhya Menon, *Characterizations of supra generalized pre-regular closed sets*, Int. Jr. of Engineering, Contemporary Mathematics and Sciences **1(2)**(2015),7-12.

[4] M.Kamraj, G.Ramkumar, O.Ravi and M.L.Thivagar, *Supra sg-closed sets and related maps*, Global Journal of Advances in Pure and Applied Mathematics **1(1)**(2012), 9-23.

[5] S.R.Malghan, *Generalized closed maps*, J. Karnatak Univ. Sci. **27**(1982), 82-88.

[6] A.S.Mashhour, A.A.Allam, F.S.Mahmoud and F.H.Khedr, *On supra topological spaces*, Indian J. Pure Appl. Math. **14(4)**(1983), 502-510.

[7] O.Ravi, G.Ramkumar and M.Kamaraj, *On supra g-closed sets*, International Journal of Advances in Pure and Applied Mathematics **1(2)**(2011), 52-66.

[8] O.R.Sayed and Takashi Noiri, *On supra b-open sets and supra b-continuity on topological spaces*, European Journal of Pure and Applied Mathematics **3(2)** (2010), 295-302.

[9] O.R.Sayed, *Supra pre-open sets and supra pre-continuity on topological spaces*, Scientific Studies and Research Series Mathematics and Informatics **20(2)**(2010), 79-88.

[10] O.R.Sayed, Gnanambal Ilango and Vidhya Menon, *An extensive study of supra generalized pre-regular closed sets*, International Journal of Pure and Applied Mathematics **106(6)**(2016), 57-73.

[11] M.Trinita Pricilla and I.Arockiarani, *On generalized b-regular closed sets in supra topological spaces*, Asian Journal of Current Enigineering and Maths **1(1)**(2012), 1-4.

[12] Vidhya Menon, *On generalized preregular closed sets in supra topological spaces*, International Journal Of Scientific & Engineering Research **3(11)**(2012), 6-11.

[13] Won Keun Min and Hong Soon Chang, *On M-open mappings*, Kangweon-Kyungki Math.Jour. **7(1)**(1999), 117-121.

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