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APPLICATIONS OF TOPOLOLOGICAL METHODS TO THE SEMILINEAR BIHARMONIC PROBLEM WITH DIFFERENT POWERS

Tacksun Jung† and Q-Heung Choi*

Abstract. We prove the existence of multiple solutions for the fourth order nonlinear elliptic problem with fully nonlinear term. Our method is based on the critical point theory; the variation of linking method and category theory.

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ and let $b \in R$ be a constant. Let $\lambda_k(k = 1, 2, \cdots)$ denote the eigenvalues and $\phi_k(k = 1, 2, \cdots)$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω with $u = 0$ on $\partial\Omega$,where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $\lambda_1 < \lambda_2 \leq \lambda_3 \ldots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$.

We investigate the existence of the nontrivial solutions for the following fourth order semilinear elliptic equation with fully nonlinear term

$$\Delta^2 u + c\Delta u + bu^+ = (u^+)^{p-1} - (u^-)^{q-1} \qquad \text{in } \Omega, \tag{1.1}$$

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∗ Corresponding author.
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$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

where $c \in R$, $u^+ = \max\{u, 0\}$ and $p, q > 2(p \neq q)$.

Jung and Choi [4] investigated, by a linking argument, the existence and the multiplicity of the solutions for the following fourth order semilinear elliptic equation with Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = b((u+1)^+ - 1) \qquad \text{in } \Omega, \tag{1.2}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

where $c \in R$ and $u^+ = \max\{u, 0\}$.

Tarantello [8] studied problem (1.2) when $c < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - c)$. She showed that (1.2) has at least two solutions, one of which is a negative solution. She obtained this result by the degree theory. Micheletti and Pistoia [6] also proved that if $c < \lambda_1$ and $b \geq \lambda_2(\lambda_2 - c)$, then (1.2) has at least three solutions by the Leray-Schauder degree theory. Choi and Jung [2] showed that the problem

$$\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \tag{1.3}$$
$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega$$

has at least two nontrivial solutions when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and, $s < 0$ or when $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. The authors obtained these results by using the variational reduction method. The authors [5] also proved that when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$, (1.2) has at least three nontrivial solutions by using degree theory.

The eigenvalue problem $\Delta^2 u + c\Delta u = \mu u$ in Ω with $u = 0, \quad \Delta u = 0$ on $\partial\Omega$ has also infinitely many eigenvalues $\mu_k = \lambda_k(\lambda_k - c)$, $k \geq 1$ and corresponding eigenfunctions ϕ_k, $k \geq 1$. We note that $\lambda_1(\lambda_1 - c) < \lambda_2(\lambda_2 - c) \leq \lambda_3(\lambda_3 - c) < \cdots$.

We suppose that $\lambda_1 < \lambda_2 < \lambda_3 \ldots \to +\infty$, and that $\lambda_2 < c < \lambda_3$. Then

$$\lambda_1(\lambda_1 - c) < \lambda_2(\lambda_2 - c) < 0 < \lambda_3(\lambda_3 - c) < \cdots .$$

Jung and Choi [4] showed that: (i) Let $\lambda_k < c < \lambda_{k+1}$ and $\lambda_k(\lambda_k - c) < 0$, $b < \lambda_{k+1}(\lambda_{k+1} - c)$. Then (1.2) has a unique solution.
(ii) Let $\lambda_k < c < \lambda_{k+1}$ and $\lambda_k(\lambda_k - c) < 0 < \lambda_{k+1}(\lambda_{k+1} - c) < \cdots < \lambda_{k+n}(\lambda_{k+n} - c) < b < \lambda_{k+n+1}(\lambda_{k+n+1} - c)$, $k \geq 1$, $n \geq 1$. Then (1.2) has at least two nontrivial solutions.

In section 2, we introduce the Hilbert space and prove $(P.S.)^*_\gamma$- condition for the energy functional. In section 3, we state the existence result for two solutions and prove it by using the critical point theory and variation of linking method. In section 4, we state the existence result for three solutions and prove it by using the category theory.

2. Preliminaries

We assume that $\lambda_k < c < \lambda_{k+1}$. Let H be a subspace of $L^2(\Omega)$ defined by

$$H_c(\Omega) = \{u \in L^2(\Omega) | \sum |\lambda_k(\lambda_k - c)| h_k^2 < \infty\},$$

where $u = \sum h_k \phi_k \in L^2(\Omega)$ with $\sum h_k^2 < \infty$. Then this is a complete normed space with a norm

$$\|u\| = [\sum |\lambda_k(\lambda_k - c)| h_k^2]^{\frac{1}{2}}.$$

Here after we set $H_c(\Omega) = H$. Since $\lambda_k(\lambda_k - c) \to +\infty$ and c is fixed, we have
(i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.
(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$.
For the proof of the above results we refer [1].

Lemma 2.1. *Assume that c is not an eigenvalue of $-\Delta$, $b \neq \lambda_k(\lambda_k - c)$. If $u \in L^2(\Omega)$ and $(u^+)^{p-1} - (u^-)^{q-1} \in L^2(\Omega)$, then all solutions of*

$$\Delta^2 u + c\Delta u + bu^+ = (u^+)^{p-1} - (u^-)^{q-1} \qquad \text{in} \quad L^2(\Omega)$$

belong to H, where $p, q > 2$ and $p \neq q$.

Proof. Let $u \in L^2(\Omega)$ and $(u^+)^{p-1} - (u^-)^{q-1} \in L^2(\Omega)$. Then $bu^+ \in L^2(\Omega)$ and we put $-bu^+ + (u^+)^{p-1} - (u^-)^{q-1} = \sum h_k \phi_k \in L^2(\Omega)$.

$$u = (\Delta^2 + c\Delta)^{-1}(-bu^+ + (u^+)^{p-1} - (u^-)^{q-1}) = \sum \frac{1}{\lambda_k(\lambda_k - c)} h_k \phi_k \in L^2(\Omega).$$

$$\|u\| = \sum |\lambda_k(\lambda_k - c)| \frac{1}{(\lambda_k(\lambda_k - c))^2} h_k^2 \leq C \sum h_k^2 = C\|u\|^2_{L^2(\omega)} < \infty$$

for some $C > 0$. Thus $u \in H$. □

With the aid of Lemma 2.1 it is enough that we investigate the existence of the solutions of (1.1) in the subspace H of $L^2(\Omega)$.

Assume that $k \geq 1$ and $\lambda_k < c < \lambda_{k+1}$. We denote by $(\Lambda_i^-)_{i\geq 1}$ the sequence of the negative eigenvalues of $\Delta^2 + c\Delta$, by $(\Lambda_i^+)_{i\geq 1}$ the sequence of the positive ones, so that

$$\Lambda_k^- = \lambda_1(\lambda_1 - c) < \cdots < \Lambda_1^- = \lambda_k(\lambda_k - c) < 0$$

$$< \Lambda_1^+ = \lambda_{k+1}(\lambda_{k+1} - c) < \Lambda_2^+ = \lambda_{k+1}(\lambda_{k+1} - c) < \cdots .$$

We consider an orthonormal system of eigenfunctions $\{e_i^-, e_i^+, i \geq 1\}$ associated with the eigenvalues $\{\Lambda_i^-, \Lambda_i^+, i \geq 1\}$. We set

$$H^+ = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \geq 0\},$$

$$H^- = \text{closure of span}\{\text{eigenfunctions with eigenvalue} \leq 0\}.$$

We define the linear projections $P^- : H \to H^-, P^+ : H \to H^+$.

We also introduce two linear operators $R : H \to H^+, S : H \to H^-$ by

$$S(u) = \sum_{i=1}^{\infty} \frac{a_i^- e_i^-}{\sqrt{-\Lambda_i^-}}, R(u) = \sum_{i=1}^{\infty} \frac{a_i^+ e_i^+}{\sqrt{\Lambda_i^+}}$$

if

$$u = \sum_{i=1}^{\infty} a_i^- e_i^- + \sum_{i=1}^{\infty} a_i^+ e_i^+.$$

It is clear that S and R are compact and self adjoint on H.

Definition 2.1. Let $I_b : H \to R$ be defined by

$$I_b(u) = \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 + \frac{b}{2}\|[Au]^+\|^2 - \int_\Omega F(Au)dx$$

where $A = R + S$ and $F(s) = \int_0^s f(x,\tau)d\tau$, $f(x,\tau) = (\tau^+)^2 - (\tau^-)^3$.

It is straightforward that

$$\nabla I_b(u) = P^+u - P^-u + bA(Au)^+ - Af(Au).$$

Following the idea of Hofer [3] one can show that

Proposition 2.2. *$I_b \in C^{1,1}(H, R)$. Moreover $\nabla I_b(u) = 0$ if and only if $w = (R + S)(u)$ is a weak solution of (1.1), that is,*

$$\int_\Omega (w(v_{tt} + v_{xxxx}) + b[w]^+ v)dxdt = \int_\Omega f(w)v dxdt \text{ for all smooth } v \in H.$$

In this section, we suppose $b > 0$. Under this assumption, we have a concern with multiplicity of solutions of equation (1.1). Here we suppose that f is defined by equation $f(x,\tau) = (\tau^+)^{p-1} - (\tau^-)^{q-1}$.

In the following, we consider the following sequence of subspaces of $L^2(R^N)$:

$$H_n = (\oplus_{i=1}^n H_{\Lambda_i^-}) \oplus (\oplus_{i=1}^n H_{\Lambda_i^+})$$

where H_Λ is the eigenspace associated to Λ and $H_{\Lambda_i^-} = \phi$ if $i > k$.

LEMMA 2.5. *The functional I_b satisfies $(P.S.)^*_\gamma$ condition, with respect to (H_n), for all γ.*

Proof. Let (k_n) be any sequence in N with $k_n \to \infty$. And let (u_n) be any sequence in H such that $u_n \in H_n$ for all n, $I_b(u_n) \to \gamma$ and $\nabla(I_b)\mid_{H_{k_n}}(u_n) \to 0$.

First, we prove that (u_n) is bounded. By contradiction let $t_n = \|u_n\| \to \infty$ and set $\hat{u_n} = u_n/t_n$. Up to a subsequence $\hat{u_n} \rightharpoonup \hat{u}$ in H for some $\hat{u}$ in H. Moreover

$$\begin{aligned} 0 &\leftarrow < \nabla(I_b)_{H_{k_n}}(u_n), \hat{u_n} > -\frac{2}{t_n} I_b(u_n) \\ &= \frac{2}{t_n}\int_\Omega F(Au_n)dx - \frac{1}{t_n}\int_\Omega f(Au_n)Au_n dx \\ &= \int_\Omega -\frac{p-2}{p}(t_n)^{p-1}[(A\hat{u_n})^+]^p + \frac{q+2}{q}(t_n)^{q-1}[(A\hat{u_n})^-]^q dx. \end{aligned}$$

Since $t_n \to \infty$, $(A\hat{u_n})^+ \to 0$ and $(A\hat{u_n})^- \to 0$. This implies $A\hat{u} = 0$ and $\hat{u} = 0$, a contradiction.

So (u_n) is bounded and we can suppose $u_n \rightharpoonup u$ for some $u \in H$. We know that

$$\nabla(I_b)_{H_{k_n}}(u_n) = P^+u_n - P^-u_n + bA(Au_n)^+ - Af(Au_n).$$

Since A is the compact operator, $P^+u_n - P^-u_n$ converges strongly, hence $u_n \to u$ strongly and $\nabla I_b(u) = 0$. □

3. An Application of Linking Theory

Fixed Λ_i^- and $\Lambda_i^- < -b < \Lambda_{i-1}^-$. We prove the Theorem via a linking argument.

First of all, we introduce a suitable splitting of the space H. Let

$$Z_1 = \oplus_{j=i+1}^\infty H_{\Lambda_j^-}, Z_2 = H_{\Lambda_i^-}, Z_3 = \oplus_{j=1}^{i-1} H_{\Lambda_j^-} \oplus H^+,$$

where $H_{\Lambda_j^-} = \phi$ if $j > k$.

LEMMA 3.1. *There exists R such that* $\sup_{v \in Z_1 \oplus Z_2, \|v\|=R} I_b(v) \le 0$.

Proof. If $v \in Z_1 \oplus Z_2$ then

$$I_b(v) = -\frac{1}{2}\|v\|^2 + \frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega F(Sv)dx.$$

Since

$$\frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega F(Sv)dx = \int_\Omega \frac{b}{2}([Sv]^+)^2 - \frac{1}{p}([Sv]^+)^p - \frac{1}{q}([Sv]^-)^q dx,$$

there exists R such that $\frac{b}{2}\|[Sv]^+\|^2 - \int_\Omega F(Sv)dx \leq 0$ for all $\|v\| = R$. Hence

$$I_b(v) \leq -\frac{1}{2}\|v\|^2 \leq 0$$

□

Lemma 3.2. *There exists ρ such that* $\inf_{u \in Z_2 \oplus Z_3, \|u\|=\rho} I_b(u) > 0$.

Proof. Let $\sigma \in [0,1]$. We consider the functional $I_{b,\sigma} : Z_2 \oplus Z_3 \to R$ defined by

$$I_{b,\sigma}(u) = \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 + \frac{b}{2}\|[Au]^+\|^2 - \sigma\int_\Omega F(Au)dx.$$

We claim that there exists a ball $B_\rho = \{u \in Z_2 \oplus Z_3 | \|u\| < \rho\}$ such that

1. $I_{b,\sigma}$ are continuous with respect to σ,
2. $I_{b,\sigma}$ satisfies $(P.S)$ condition,
3. 0 is a minimum for $I_{b,0}$ in B_ρ,
4. 0 is the unique critical point of $I_{b,\sigma}$ in B_ρ.

Then by a continuation argument of Li-Szulkin's [5], it can be shown that 0 is a local minimum for $I_b|_{Z_2 \oplus Z_3} = I_{b,1}$ and Lemma is proved.

The continuity in σ and the fact that 0 is a local minimum for $I_{b,0}$ are straightforward. To prove $(P.S.)$ condition one can argue as in the previous Lemma, when dealing with I_b.

To prove that 0 is isolated we argue by contradiction and suppose that there exists a sequence (σ_n) in $[0,1]$ and sequence (u_n) in $Z_2 \oplus Z_3$ such that $\nabla I_{b,\sigma_n}(u_n) = 0$ for all $n, u_n \neq 0, \text{and} u_n \to 0$. Set $t_n = \|u_n\|$ and $\hat{u_n} = u_n/t_n$ then $t_n \to 0$. Let $\hat{v_n} = P^-\hat{u_n}$ and $\hat{w_n} = P^+\hat{u_n}$. Since $\hat{v_n}$ varies in a finite dimensional space, we can suppose that $\hat{v_n} \to \hat{v}$ for some $\hat{v}$. We get

$$(1) \qquad \frac{1}{t_n}\nabla I_{b,\sigma}(u_n) = \hat{w_n} - \hat{v_n} + \frac{b}{t_n}A(Au_n)^+ - \frac{\sigma_n}{t_n}Af(Au_n) = 0.$$

Multiplying by $\hat{w_n}$ yields

$$\|\hat{w_n}\|^2 = \frac{\sigma_n}{t_n}\int_\Omega f(Au_n)A\hat{w_n}dx - \frac{b}{t_n}\int_\Omega (Au_n)^+ A\hat{w_n}dx.$$

We know that

$$\begin{aligned}\int_\Omega (Au_n)^+ A\hat{w_n}dx &= \int_\Omega P^+(Au_n)^+ A\hat{u_n}dx \\ &= \int_\Omega P^+(Au_n)^+(A\hat{u_n})^+dx.\end{aligned}$$

Since $b > 0$, there exists a sequence (ϵ_n) such that $\epsilon_n \to 0$ and $0 < \epsilon_n < b$ for all n. That is

$$\frac{b}{t_n}\int_\Omega (Au_n)^+ A\hat{w_n}dx \geq \frac{\epsilon_n}{t_n}\int_\Omega P^+(Au_n)^+(A\hat{u_n})^+dx.$$

Then

$$\begin{aligned}\|\hat{w_n}\|^2 &\leq \frac{1}{t_n}\int_\Omega f(Au_n)A\hat{w_n}dx - \frac{\epsilon_n}{t_n}\int_\Omega P^+(Au_n)^+(A\hat{u_n})^+dx \\ &\leq \int_\Omega \frac{|f(Au_n)|}{t_n}|A\hat{w_n}|dx + \epsilon_n\int_\Omega |P^+(A\hat{u_n})^+||(A\hat{u_n})^+|dx.\end{aligned}$$

Since A is a compact operator

$$\begin{aligned}|f(Au_n)| &= |\{([t_nA\hat{u_n}]^+)^{p-1} - ([t_nA\hat{u_n}]^-)^{q-1}\}| \\ &\leq {t_n}^{p-1}|[A\hat{u_n}]^+|^{p-1} + {t_n}^{q-1}|[A\hat{u_n}]^-|^{q-1} \\ &\leq {t_n}^m(M_1 + {t_n}^{M-m}M_2)\end{aligned}$$

for some M_1 and M_2 where $m = \min\{p-1, q-1\}$ and $M = \max\{p-1, q-1\}$. We get that

$$\int_\Omega \frac{|f(Au_n)|}{t_n}|A\hat{w_n}|dx \leq {t_n}^m(M_1 + {t_n}^{M-m}M_2)\int_\Omega |A\hat{w_n}|dx \leq o(1).$$

Hence

$$\|\hat{w_n}\|^2 \leq o(1) + \epsilon_n\int_\Omega |P^+(A\hat{u_n})^+||(A\hat{u_n})^+|dx. \tag{2}$$

Since $\int_\Omega |P^+(A\hat{u_n})^+||(A\hat{u_n})^+|dx$ is bounded and equation (7) holds for every ϵ_n, $\hat{w_n} \to 0$ and so $(\hat{u_n})$ converges. Since $|f(Au_n)| \leq {t_n}^m(M_1 + {t_n}^{M-m}M_2)$, we get

$$\frac{\sigma_n}{t_n}|f(Au_n)| \leq \frac{1}{t_n}|f(Au_n)| \leq {t_n}^{m-1}(|M_1 + {t_n}^{M-m}M_2) \leq o(1).$$

Then $\frac{\sigma_n}{t_n}Af(Au_n) \to 0$. From equation (6), $(\hat{v_n})$ converges to zero, but this is impossible if $\|\hat{(}u_n)\| = 1$. $\square$

We give the definitions for the next step:

Definition 3.3. Let H be an Hilbert space, $Y \subset H$, $\rho > 0$ and $e \in H \setminus Y$, $e \neq 0$. Set:

$$\begin{aligned} B_\rho(Y) &= \{x \in Y \mid \|x\| \le \rho\}, \\ S_\rho(Y) &= \{x \in Y \mid \|x\| = \rho\}, \\ \triangle_\rho(e, Y) &= \{\sigma e + v \mid \sigma \ge 0, v \in Y, \|\sigma e + v\| \le \rho\}, \\ \Sigma_\rho(e, Y) &= \{\sigma e + v \mid \sigma \ge 0, v \in Y, \|\sigma e + v\| = \rho\} \cup \{v \mid v \in Y, \|v\| \le \rho\}. \end{aligned}$$

Theorem 3.4. *If $\Lambda_i^- \le -b (i = 1, 2, \cdots, k)$, then problem (1.1) has at least one nontrivial solution.*

Proof. Let $e \in Z_2$. By Lemma 3.1 and Lemma 3.2, for a suitable large R and a suitable small ρ, we have the linking inequality

$$\sup I_b(\Sigma_R(e, Z_1)) < \inf I_b(S_\rho(Z_2 \oplus Z_3)). \tag{3}$$

Moreover $(P.S.)_\gamma^*$ holds. By standard linking arguments, it follows that there exists a critical point u for I_b with $\alpha \le I_b(u) \le \beta$, where $\alpha = \inf I_b(S_\rho(Z_2 \oplus Z_3))$ and $\beta = \sup I_b(\Delta_R(e, Z_1))$. Since $\alpha > 0$, then $u \neq 0$. $\square$

We assume in this section that $i \ge 2$ and we set

$$W_1 = \oplus_{j=i}^{\infty} H_{\Lambda_j^-}, W_2 = \oplus_{j=1}^{i-1} H_{\Lambda_j^-}, W_3 = H^+.$$

Notice that $W_1 = Z_1 \oplus Z_2$ and $W_2 \oplus W_3 = Z_3$.

Lemma 3.5. $\liminf_{\|u\| \to +\infty, u \in W_1 \oplus W_2} I_b(u) \le 0$.

Proof. Let $(u_n)_n$ be a sequence in $W_1 \oplus W_2$ such that $\|u_n\| \to \infty$. We set $t_n = \|u_n\|$ and $\hat{u_n} = u_n / t_n$. Since S is a compact operator,

$$\begin{aligned} \frac{b}{2}\frac{\|[Su_n]^+\|^2}{t_n^2} &- \int_\Omega \frac{F(Su_n)}{t_n^2} dx \\ &= \int_\Omega \frac{b}{2}([S\hat{u_n}]^+)^2 - \frac{{t_n}^{p-2}}{p}([S\hat{u_n}]^+)^p - \frac{{t_n}^{q-2}}{q}([S\hat{u_n}]^-)^q dx \\ &\to -\infty. \end{aligned}$$

Then

$$\frac{I_b(u_n)}{\|u_n\|^2} = -\frac{1}{2} + \frac{b}{2}\frac{\|[Su_n]^+\|^2}{t_n^2} - \int_\Omega \frac{F(Su_n)}{t_n^2}dx \to -\infty.$$

Hence

$$\liminf_{\|u\|\to+\infty, u\in W_1\oplus W_2} I_b(u) \leq 0.$$

□

Lemma 3.6. *There exists $\hat{\rho}$ such that* $\inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)) > 0$.

Proof. Repeating the same arguments used in Lemma 3.2, we get the conclusion. □

Theorem 3.7. *Assume that $\lambda_k < c < \lambda_{k+1}$. Let $k \geq i \geq 2$. If $\Lambda_i^- \leq -b$, then problem (1.1) has at least two nontrivial solution.*

Proof. Using the conclusion of Theorem 3.4, we have that there exist a nontrivial critical point u with

$$I_b(u) \leq \sup I_b(\Delta_R(e, Z_1))$$

where e, R were given in Lemma 3.1 and 3.2. We can choose that $\hat{R} \geq R$. Take any $\hat{e}$ in W_2, then we have a second linking inequality,

$$\sup I_b(\Sigma_{\hat{R}}(\hat{e}, W_1)) \leq \inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)).$$

Since $(P.S.)^*_\gamma$ holds, there exists a critical point $\hat{u}$ such that

$$\inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)) \leq I_b(\hat{u}) \leq \sup I_b(\Delta_{\hat{R}}(\hat{e}, W_1)).$$

Since $\hat{R} \geq R$ and $Z_1 \oplus Z_2 = W_1$,

$$\Delta_R(e, Z_1) \subset B_{\hat{R}}(W_1) \subset \Sigma_{\hat{R}}(\hat{e}, W_1).$$

Then

$$\begin{aligned} I_b(u) &\leq \sup I_b(\Delta_R(e, Z_1)) \\ &\leq \sup I_b(\Sigma_{\hat{R}}(\hat{e}, W_1)) < \inf I_b(S_{\hat{\rho}}(W_2 \oplus W_3)) \leq I_b(\hat{u}). \end{aligned}$$

Hence $u \neq \hat{u}$. □

4. An Application of Category Theory

We define the map $\Psi : H\backslash(Z_1 \oplus Z_3) \to H$ by

$$\Psi(u) = u - \frac{P_{Z_2}u}{\|P_{Z_2}u\|} = P_{Z_1\oplus Z_3}u + (1 - \frac{1}{\|P_{z_2}u\|})P_{z_2}u.$$

We have

$$\begin{aligned}\Psi'(u)(v) &= v - \frac{P_{Z_2}v}{\|P_{Z_2}u\|} + < \frac{P_{Z_2}u}{\|P_{Z_2}u\|}, P_{Z_2}v > \frac{P_{Z_2}u}{\|P_{Z_2}u\|}\frac{1}{\|P_{Z_2}u\|}\\ &= v - \frac{1}{\|P_{z_2}u\|}(P_{z_2}v - < \frac{P_{Z_2}u}{\|P_{Z_2}u\|}, P_{Z_2}v > \frac{P_{Z_2}u}{\|P_{Z_2}u\|}).\end{aligned}$$

Moreover, we introduce the smooth manifold with boundary

$$C = \{u \in H | \|P_{Z_2}u\| \geq 1\}$$

and the constrained functional $\tilde{I}_b : C \to R$ defined by $\tilde{I}_b = I_b \circ \Psi$ which is of class $C^{1,1}_{loc}$.

In particular the lower gradient of $\tilde{I}_b$ at a point $\tilde{u}$ is

(4)

$$grad_{\bar{C}}\tilde{I}_b(\tilde{u}) = \begin{cases} P_{Z_1\oplus Z_3}(\nabla I_b)(u) + (1 - \frac{1}{\|P_{Z_2}\tilde{u}\|})P_{Z_2}(\nabla I_b)(u) & \text{if } \tilde{u} \in int(C)\\ P_{Z_1\oplus Z_3}(\nabla I_b)(u) - [< \nabla I_b(u), P_{Z_2}\tilde{u} >]^+ P_{Z_2}\tilde{u} & \text{if } \tilde{u} \in \partial C,\end{cases}$$

where $u = \Psi(\tilde{u})$.

We can prove the following result.

LEMMA 4.1. *We set $C_n = C \cap H_n$ for all n. Then the functional $\tilde{I}_b$ satisfies $(P.S.)^*_\gamma$ condition, with respect to (C_n), for all γ.*

Proof. Let $(k_n)_n$ and $(u_n)_n$ and γ be such that $k_n \to \infty$, $\tilde{u_n} \in C_{k_n}$ for all n, $\tilde{I}_b(\tilde{u_n}) \to \gamma$ and $grad_{\bar{C_{k_n}}}\tilde{I}_b\tilde{u_n} \to 0$. Apply the Definition of the lower gradient of $\tilde{I}_b$,

$$grad_{\bar{C_{k_n}}}\tilde{I}_b\tilde{u_n} = P_{H_{k_n}}grad_{\bar{C}}\tilde{I}_b\tilde{u_n} \to 0. \tag{5}$$

We set $u_n = \Psi(\tilde{u_n})$ and $u_{n,1} = P_{Z_1}u_n$, $u_{n,2} = P_{Z_2}u_n$, $u_{n,3} = P_{Z_3}u_n$.

Case 1. $\inf\|P_{Z_2}\tilde{u_n}\| > 1$.

In this case, $P_{H_{k_n}}\nabla I_b(u_n) \to 0$, so by the $(P.S.)^*_\gamma$ condition for I_b, (u_n) has converging subsequence (u_{n_j}) which converges to a point u which is a critical point for I_b and $u \notin Z_1 \oplus Z_3$. Since Ψ is a diffeomorphism in a neighborhood of u, $(\tilde{u_{n_j}})$ converges to $\tilde{u} = \Psi^{-1}(u)$ and $\tilde{u}$ is critical for $\tilde{I}_b$.

Case 2. $\inf\|P_{Z_2}\tilde{u_n}\| = 1$.

We can suppose that $P_{Z_2}u_n \to 0$. We claim that (u_n) is bounded. If not we can suppose $t_n = \|u_n\| \to \infty$. We take $\hat{u_n} = u_n/\|u_n\|$, $\hat{u_{n,i}} = u_{n,1}/\|u_n\|$ for $i = 1, 2, 3$.

Applying $P_{Z_1 \oplus Z_3}$ to equation (10) and using equation (9), we get

$$P^+\hat{u_{n,3}} - \hat{u_{n,1}} - P^-\hat{u_{n,3}} + P_{H_{k_n}}P_{Z_1 \oplus Z_3}\frac{bA(Au_n)^+ - Af(Au_n)}{t_n} \to 0.$$

Multiplying by $\hat{u_{n,3}}$ and integrating over Ω yields

$$\begin{aligned}\|P^+\hat{u_{n,3}}\|^2 \quad &- \quad \|P^-\hat{u_{n,3}}\|^2 \\ &+ \quad P_{H_{k_n}}[\frac{b}{t_n}\int_\Omega (Au_n)^+A\hat{u_{n,3}}dx - \frac{1}{t_n}\int_\Omega f(Au_n)A\hat{u_{n,3}}dx] \to 0.\end{aligned}$$

We know that there exists a sequence (ϵ_n) such that $\epsilon_n \to 0$ and $0 < \epsilon_n < b$ for all n, that is,

$$\begin{aligned}-\frac{b}{t_n}\int_\Omega (Au_n)^+A\hat{u_{n,3}}dx \quad &\le \quad -\frac{\epsilon_n}{t_n}\int_\Omega P_{Z_3}(Au_n)^+(A\hat{u_n})^+dx \\ &\le \quad \epsilon_n\int_\Omega |P_{Z_3}(Au_n)^+||(A\hat{u_n})^+|dx.\end{aligned}$$

And we know that

$$\begin{aligned}\frac{1}{t_n}\int_\Omega f(Au_n)A\hat{u_{n,3}}dx \quad &\le \quad \int_\Omega \frac{|f(Au_n)|}{t_n}|A\hat{u_{n,3}}|dx \\ &\le \quad {t_n}^{m-1}(M_1 + {t_n}^{M-m}M_2)\int_\Omega |A\hat{u_{n,3}}|dx \le o(1).\end{aligned}$$

Hence

$$-\frac{b}{t_n}\int_\Omega (Au_n)^+A\hat{u_{n,3}}dx + \frac{1}{t_n}\int_\Omega f(Au_n)A\hat{u_{n,3}}dx \to 0$$

and $\hat{u_{n,3}} \to 0$.

Similarly, $\hat{u_{n,1}} \to 0$. Since $\hat{u_{n,2}} \to 0$, $\hat{u_n} \to 0$ which is impossible.

Since $(u_n)_n$ is bounded, we can suppose $u_{n,1} \rightharpoonup u_1$, $u_{n,2} \to 0$ and $u_{n,3} \rightharpoonup u_3$ for suitable u_i in Z_i, $i = 1, 2, 3$.

Let $z_n = P^-u_n$ and $v_n = P^+u_n$. Applying P^+ to equation (9)

$$v_n + P^+P_{H_{k_n}}(bA(Au_n)^+ - Af(Au_n)) \to 0. \tag{6}$$

Since A is compact and $(u_n)_n$ is bounded, $Au_n \to Au$. Hence $bA(Au_n)^+ - Af(Au_n) \to bA(Au)^+ - Af(Au)$ strongly and by equation (11), v_n converges strongly to v. Similarly Z_n converges strongly to z. Since $P_{Z_2}u_n \to 0$, $u_n \to u = v + z$ where $v = P^+u$ and $z = P^-u$. Since $u_{n,1} \to$

u_1 and $u_{n,3} \to u_3$, $P_{Z_1}\tilde{u_n} \to \Psi^{-1}(u_1) = u_1$ and $P_{Z_3}\tilde{u_n} \to \Psi^{-1}(u_3) = u_3$. Since $P_{Z_2}\tilde{u_n}$ is in a finite dimensional space, $P_{Z_2}\tilde{u_n}$ converges to a point $\tilde{u_2}$.

Hence $\tilde{u_n}$ converges to $\tilde{u} = u_1 + \tilde{u_2} + u_3$ and $\tilde{u}$ is critical for $\tilde{I}_b$. □

LEMMA 4.2. *The functional $I_b\,|_{Z_1 \oplus Z_3}$ has no critical points u such that $I_b(u) < 0$.*

Proof. By contradiction, let (u_n) be a sequence such that $u_n \in Z_1 \oplus Z_3$, $I_b(u_n) < 0$ for all n, $I_b(u_n) \to 0$, and $P_{Z_1\oplus Z_3}\nabla I_b(u_n) = 0$.

Arguing as in the proof of Lemma 2.5, up to a subsequence, (u_n) converges to some u such that $I_b(u) = 0$ and $P_{Z_1 \oplus Z_3}\nabla I_b(u) = 0$. Then

$$\begin{aligned} 0 &= \; < P_{Z_1\oplus Z_3}\nabla I_b(u), u > -2I_b(u) \\ &= \int_\Omega [2F(Au) - f(Au)Au]dx \\ &= -\int_\Omega \frac{p-2}{p}[(Au)^+]^p + \frac{q-2}{q}[(Au)^-]^q dx. \end{aligned}$$

Hence $Au = 0$ and $u = 0$.

Let $\hat{u_n} = u_n / \|u_n\|$ and $t_n = \|u_n\|$. We have

$$t_n P^+ \hat{u_n} - t_n P^- \hat{u_n} + bA(Au_n)^+ - Af(Au_n) = 0. \tag{7}$$

Multiplying equation (12) by P^+, we get

$$\hat{v_n} + bA(Au_n)^+ - \frac{1}{t_n} Af(Au_n) = 0. \tag{8}$$

Multiplying equation (13) by $\hat{v_n}$ and integrating over Ω,

$$\|\hat{v_n}\|^2 = \frac{1}{t_n}\int_\Omega f(Au_n)A\hat{v_n}dx - b\int_\Omega (A\hat{u_n})^+ A\hat{v_n}dx.$$

Arguing as in the proof of Lemma 3.2, $\hat{v_n} \to 0$.

Similarly, $\hat{z_n} \to 0$ and then $\hat{u_n}$, which gives a contradiction. □

THEOREM 4.1. *Assume that $\lambda_k < c < \lambda_{k+1}$. Let $k \geq i \geq 2$. Then problem (1.1) has at least three nontrivial solutions.*

Proof. We claim that there exists two critical points u_i for I_b such that, for $i = 1, 2$

$$\inf I_b(S_\rho(Z_2 \oplus Z_3)) \leq I_b(u_i) \leq \sup I_b(\Delta_R(S_1(Z_2), Z_1)) \tag{9}$$

where ρ and R are as Theorem 3.4. By specify which theorem, we know that the critical point $\hat{u}$ is distinguished from u_1 and u_2.

To prove the claim, we consider the functional $\tilde{I}_b$. If we set $\tilde{S} = \Psi^{-1}(S_\rho(Z_2 \oplus Z_3)), \tilde{\Sigma} = \Psi^{-1}(\Sigma_R(S_1(Z_2), Z_1)), \tilde{\Delta} = \Psi^{-1}(\Delta_R(S_1(Z_2), Z_1))$. By equation (9) and the definition of Ψ,

$$\sup \tilde{I}_b(\tilde{\Sigma}) < \inf \tilde{I}_b(\tilde{S}).$$

Due to Lemma 3.1,

$$\inf \tilde{I}_b(\tilde{S}) < \sup \tilde{I}_b(\tilde{\Delta}) \leq 0.$$

Since the $(P.S.)^*_\gamma$ condition holds for $\tilde{I}_b$ using the Theorem 3.7 in Section 3.2, there exists two critical points $\tilde{u_i}$, $i = 1, 2$ for $\tilde{I}_b$ such that

$$\inf \tilde{I}_b(\tilde{S}) \leq \tilde{I}_b(\tilde{u_i}) \leq \sup \tilde{I}_b(\tilde{\Delta}). \tag{10}$$

We claim that $\tilde{u_i} \notin \partial C$. Suppose that $\tilde{u_i} \in \partial C$. Since

$$0 = grad_{\tilde{C}}\tilde{I}_b(\tilde{u_i}) = P_{Z_1 \oplus Z_3}(\nabla I_b)(u_i) - [< \nabla I_b(u_i), P_{Z_2}\tilde{u_i} >]^+ P_{Z_2}\tilde{u_i},$$

$P_{Z_1 \oplus Z_3}(\nabla I_b)(u_i) = 0$ where $u_i = \Psi(\tilde{u_i})$. Then u_i are critical for $I_b \mid_{Z_1 \oplus Z_3}$. By equation (14) and equation (15), $I_b(u_i) < 0$, but this contradicts Lemma 4.2.

So $\tilde{u_i} \notin \partial C$, since Ψ is a diffeomorphism in a neighborhood of $\tilde{u_n}$, then $\nabla I_b(u_i) = 0$ where $u_i = \Psi(\tilde{u_i}), i = 1, 2$. □

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Tacksun Jung
Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
E-mail: tsjung@kunsan.ac.kr

Q-Heung Choi
Department of Mathematics Education
Inha University
Incheon 402-751, Korea
E-mail: qheung@inha.ac.kr