Korean J. Math. **25** (2017), No. 3, pp. 437–453
https://doi.org/10.11568/kjm.2017.25.3.437

GENERALIZED (Λ, b)-CLOSED SETS IN TOPOLOGICAL SPACES

Chawalit Boonpok

Abstract. A new class of sets, called generalized (Λ, b)-closed sets, has been introduced and studied. Also, several properties of generalized (Λ, b)-closed sets are investigated. Some characterizations of Λ_b-regular spaces and Λ_b-normal spaces are discussed.

1. Introduction

In 1970, Levine [5] introduced the notion of generalized closed sets in topological spaces. Since then, many variations of generalized closed sets are introduced and investigated. As an application of these sets, many low separation axioms are introduced. Andrijević [1] introduced a new class of generalized open sets called b-open sets into the field of topology. This class is a subset of the class of semi-preopen sets [2], i.e. a subset of a topological space which is contained in the closure of the interior of its closure. Also the class of b-open sets in a superset of the class of semi-open sets [5], i.e. a set which is contained in the closure of its interior and the class of locally dense set [4] or preopen sets [6], i.e. a set which is contained in the interior of its closure. Andrijević studied several fundamental and interesting properties of b-open sets. Caldas et al. [3] introduced the concept of Λ_b-sets which is the intersection of b-open sets and studied the fundamental properties of b-open sets. The concepts of

Received July 1, 2017. Revised September 3, 2017. Accepted September 4, 2017.
2010 Mathematics Subject Classification: 54A05, 54D10.
Key words and phrases: (Λ, b)-open set, generalized (Λ, b)-closed set, Λ_b-normal space, Λ_b-regular space.

the (Λ, b)-closure and (Λ, b)-open sets were introduced by using b-open sets and b-closure operators due to Andrijević [1]. In the present paper, we introduce the concept of generalized (Λ, b)-closed sets and investigate some of their fundamental properties. Finally, some characterizations of Λ_b-normal spaces and Λ_b-normal spaces have been given.

2. Preliminaries

Throughout the paper (X, τ) (or simply X) will always denote a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of X, the closure, interior and complement of A in (X, τ) are denoted by $Cl(A), Int(A)$ and $X - A$, respectively. By $BO(X, \tau)$ and $BC(X, \tau)$ we denote the collection of all b-open sets and the collection of all b-closed sets of (X, τ), respectively. A subset A of a topological space (X, τ) is said to be b-open [1] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$. The complement $X - A$ of a b-open set A is called b-closed and the b-closure of a set A, denoted by $bCl(A)$, is the intersection of all b-closed sets containing A. The b-interior of a set A denoted by $bInt(A)$, is the union of all b-open sets contained in A.

Proposition 2.1. [1] *For a topological space (X, τ), the following properties hold:*

(1) The union of any family of b-open sets is b-open.
(2) The intersection of an open and a b-open set is a b-open set.

Definition 2.2. Let A be a subset of a topological space (X, τ). The subset A^{Λ_b} [3] is defined to be the set $\cap\{U \in BO(X, \tau) \mid A \subseteq U\}$.

Lemma 2.3. [3] *For a subsets A, B and $A_i (i \in I)$ of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq A^{\Lambda_b}$.
(2) If $A \subseteq B$, *then* $A^{\Lambda_b} \subseteq B^{\Lambda_b}$.
(3) $(A^{\Lambda_b})^{\Lambda_b} = A^{\Lambda_b}$.
(4) $(\cap\{A_i \mid i \in I\})^{\Lambda_b} \subseteq \cap\{A_i^{\Lambda_b} \mid i \in I\}$.
(5) $(\cup\{A_i \mid i \in I\})^{\Lambda_b} = \cup\{A_i^{\Lambda_b} \mid i \in I\}$.

Definition 2.4. [3] A subset A of a topological space (X, τ) is called a Λ_b-*set* if $A = A^{\Lambda_b}$.

Remark 2.5. [3] If $G \in BO(X, \tau)$, then G is a Λ_b-set.

LEMMA 2.6. [3] *For a topological space (X, τ), the following properties hold:*

(1) The subsets $\emptyset$ and X are Λ_b-sets.
(2) Every union of Λ_b-sets is a Λ_b-set.
(3) Every intersection of Λ_b-sets is a Λ_b-set.

3. Generalized (Λ, b)-closed sets

In this section, we introduce the notion of generalized (Λ, b)-closed sets and obtain several properties of generalized (Λ, b)-closed sets.

DEFINITION 3.1. A subset A of a topological space (X, τ) is called (Λ, b)-*closed* if $A = T \cap C$, where T is a Λ_b-set and C is a b-closed set. The collection of all (Λ, b)-closed sets in a topological space (X, τ) is denoted by $\Lambda_b C(X, \tau)$.

THEOREM 3.2. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is (Λ, b)-closed.
(2) $A = T \cap bCl(A)$, where T is a Λ_b-set.
(3) $A = A^{\Lambda_b} \cap bCl(A)$.

Proof. (1) $\Rightarrow$ (2): Let $A = T \cap C$,where T is a Λ_b-set and C is a b-closed set. Since $A \subseteq C$, we have $bCl(A) \subseteq C$ and

$$A = T \cap C \supseteq T \cap bCl(A) \supseteq A.$$

Therefore, we obtain $A = T \cap bCl(A)$.

(2) $\Rightarrow$ (3): Let $A = T \cap bCl(A)$, where T is a Λ_b-set. Since $A \subseteq T$, we have $A^{\Lambda_b} \subseteq T^{\Lambda_b} = T$ and hence,

$$A \subseteq A^{\Lambda_b} \cap bCl(A) \subseteq T \cap bCl(A) = A.$$

Therefore, we obtain $A = A^{\Lambda_b} \cap bCl(A)$.

(3) $\Rightarrow$ (1): Since A^{Λ_b} is a Λ_b-set, $bCl(A)$ is b-closed and

$$A = A^{\Lambda_b} \cap bCl(A).$$

This shows that A is (Λ, b)-closed. □

LEMMA 3.3. *Every Λ_b-set (resp. b-closed set) is (Λ, b)-closed.*

Definition 3.4. A subset A of a topological space (X,τ) is said to be (Λ,b)-*open* if the complement of A is (Λ,b)-closed. The collection of all (Λ,b)-open sets in a topological space (X,τ) is denoted by $\Lambda_bO(X,\tau)$.

Theorem 3.5. *Let $A_\alpha(\alpha\in\nabla)$ be a subset of a topological space (X,τ). Then the following properties hold:*

(1) If A_α is (Λ,b)-closed for each $\alpha\in\nabla$, then $\cap\{A_\alpha\mid\alpha\in\nabla\}$ is (Λ,b)-closed.

(2) If A_α is (Λ,b)-open for each $\alpha\in\nabla$, then $\cup\{A_\alpha\mid\alpha\in\nabla\}$ is (Λ,b)-open.

Proof. (1) Suppose that A_α is (Λ,b)-closed for each $\alpha\in\nabla$. Then, for each α, there exist a Λ_b-set T_α and a b-closed set C_α such that $A_\alpha=T_\alpha\cap C_\alpha$. We have $\cap_{\alpha\in\nabla}A_\alpha=\cap_{\alpha\in\nabla}(T_\alpha\cap C_\alpha)=(\cap_{\alpha\in\nabla}T_\alpha)\cap(\cap_{\alpha\in\nabla}C_\alpha)$. By Lemma 2.6, $\cap_{\alpha\in\nabla}T_\alpha$ is a Λ_b-set and $\cap_{\alpha\in\nabla}C_\alpha$ is a b-closed. This shows that $\cap_{\alpha\in\nabla}A_\alpha$ is (Λ,b)-closed.

(2) Let A_α is (Λ,b)-open for each $\alpha\in\nabla$. Then $X-A_\alpha$ is (Λ,b)-closed and $X-\cup_{\alpha\in\nabla}A_\alpha=\cap_{\alpha\in\nabla}(X-A_\alpha)$. Therefore, by (1) $X-\cup_{\alpha\in\nabla}A_\alpha$ is (Λ,b)-closed and hence, $\cup_{\alpha\in\nabla}A_\alpha$ is (Λ,b)-open. □

Definition 3.6. Let A be a subsets of a topological space (X,τ). A point $x\in X$ is called a (Λ,b)-*cluster point* of A if for every (Λ,b)-open set U of X containing x we have $A\cap U\neq\emptyset$. The set of all (Λ,b)-cluster points is called the (Λ,b)-*closure* of A and is denoted by $A^{(\Lambda,b)}$.

Lemma 3.7. *Let A and B be subsets of a topological space (X,τ). For the (Λ,b)-closure, the following properties hold:*

(1) $A\subseteq A^{(\Lambda,b)}$ and $[A^{(\Lambda,b)}]^{(\Lambda,p)}=A^{(\Lambda,b)}$.

(2) If $A\subseteq B$, then $A^{(\Lambda,b)}\subseteq B^{(\Lambda,b)}$.

(3) $A^{(\Lambda,b)}=\cap\{F|A\subseteq F$ and F is (Λ,b)-closed$\}$.

(4) $A^{(\Lambda,b)}$ is (Λ,b)-closed.

(5) A is (Λ,b)-closed if and only if $A=A^{(\Lambda,b)}$.

Lemma 3.8. *Let A be a subset of a topological space (X,τ). Then $x\in A^{(\Lambda,b)}$ if and only if $U\cap A\neq\emptyset$ for every $U\in\Lambda_bO(X,\tau)$ containing x.*

Lemma 3.9. *For a subset A of a topological space (X,τ), the following properties hold:*

(1) If A is (Λ,b)-closed, then $A=A^{\Lambda_b}\cap A^{(\Lambda,b)}$.

(2) If A is b-closed, then A is (Λ,b)-closed.

Proof. (1) Let A be (Λ, b)-closed, then there exist a Λ_b-set T and a b-closed set C such that $A = T \cap C$. By $A \subseteq T$, we have $A \subseteq A^{\Lambda_b} \subseteq T^{\Lambda_b} = T$, and also by $A \subseteq C$, $A \subset A^{(\Lambda,b)} \subseteq C^{(\Lambda,b)} = C$. Now $A \subseteq A^{\Lambda_b} \cap A^{(\Lambda,b)} \subseteq T \cap C = A$. Hence, $A = A^{\Lambda_b} \cap A^{(\Lambda,b)}$.

(2) It is sufficient to observe that $A = X \cap A$, where the whole set X is a Λ_b-set. □

A subset N of a topological space (X, τ) is said to be (Λ, b)-*neighbourhood* of a point $x \in X$ if there exists a (Λ, b)-open set U such that $x \in U \subseteq N$.

Lemma 3.10. *A subset of a topological space (X, τ) is (Λ, b)-open in (X, τ) if and only if it is a (Λ, b)-neighbourhood of each of its points.*

Definition 3.11. Let A be a subset of a topological space (X, τ). The union of all (Λ, b)-open sets contained in A is called the (Λ, b)-*interior* of A and is denoted by $A_{(\Lambda,b)}$.

Lemma 3.12. *Let A and B be subsets of a topological space (X, τ). For the (Λ, b)-interior, the following properties hold:*

(1) $A_{(\Lambda,b)} \subseteq A$ and $[A_{(\Lambda,b)}]_{(\Lambda,b)} = A_{(\Lambda,b)}$.
(2) If $A \subseteq B$, then $A_{(\Lambda,b)} \subseteq B_{(\Lambda,b)}$.
(3) $A_{(\Lambda,b)} = \cup\{G \mid G \subseteq A$ and G is (Λ, b)-open$\}$.
(4) $A_{(\Lambda,b)}$ is (Λ, b)-open.
(5) A is (Λ, b)-open if and only if $A_{(\Lambda,b)} = A$.

Definition 3.13. A subset A of a topological space (X, τ) is said to be *generalized* (Λ, b)-*closed* (briefly g-(Λ, b)-closed) set if $A^{(\Lambda,b)} \subseteq U$ whenever $A \subseteq U$ and $U \in \Lambda_b O(X, \tau)$.

Definition 3.14. A topological space (X, τ) is said to be Λ_b-*symmetric* if for x and y in X, $x \in \{y\}^{(\Lambda,b)}$ implies $y \in \{x\}^{(\Lambda,b)}$.

Theorem 3.15. *A topological space (X, τ) is Λ_b-symmetric if and only if $\{x\}$ is g-(Λ, b)-closed for each $x \in X$.*

Proof. Assume that $x \in \{y\}^{(\Lambda,b)}$ but $y \notin \{x\}^{(\Lambda,b)}$. This implies that the complement of $\{x\}^{(\Lambda,b)}$ contains y. Therefore the set $\{y\}$ is a subset of the complement of $\{x\}^{(\Lambda,b)}$. This implies that $\{y\}^{(\Lambda,b)}$ is a subset of the complement of $\{x\}^{(\Lambda,b)}$. Now the complement of $\{x\}^{(\Lambda,b)}$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subseteq V \in \Lambda_b O(X, \tau)$, but $\{x\}^{(\Lambda,b)}$ is not a subset of V. This means that $\{x\}^{(\Lambda,b)}$ and the complement of V are not disjoint. Let y belongs to their intersection. Now we have $x \in \{y\}^{(\Lambda,b)}$ which is a subset of the complement of V and $x \notin V$. But this is a contradiction. □

Theorem 3.16. *A subset A of a topological space (X,τ) is g-(Λ,b)-closed if and only if $A^{(\Lambda,b)} - A$ contains no nonempty (Λ,b)-closed set.*

Proof. Let F be a (Λ,b)-closed subset of $A^{(\Lambda,b)} - A$. Now $A \subseteq X - F$ and since A is g-(Λ,b)-closed, we have $A^{(\Lambda,b)} \subseteq X - F$ or $F \subseteq X - A^{(\Lambda,b)}$. Thus $F \subseteq A^{(\Lambda,b)} \cap [X - A^{(\Lambda,b)}] = \emptyset$ and F is empty.

Conversely, suppose that $A \subseteq U$ and that U is (Λ,b)-open. If $A^{(\Lambda,b)} \nsubseteq U$, then $A^{(\Lambda,b)} \cap (X - U)$ is a nonempty (Λ,b)-closed subset of $A^{(\Lambda,b)} - A$. □

Corollary 3.17. *Let A be a g-(Λ,b)-closed subset of a topological space (X,τ). Then A is (Λ,b)-closed if and only if $A^{(\Lambda,b)} - A$ is (Λ,b)-closed.*

Proof. If A is (Λ,b)-closed, then $A^{(\Lambda,b)} - A = \emptyset$.

Conversely, suppose that $A^{(\Lambda,b)} - A$ is (Λ,b)-closed. But A is g-(Λ,b)-closed and $A^{(\Lambda,b)} - A$ is a (Λ,b)-closed subset of itself. By Theorem 3.16, $A^{(\Lambda,b)} - A = \emptyset$ and hence $A^{(\Lambda,b)} = A$. □

Proposition 3.18. *For a subset A of a topological space (X,τ), the following properties hold:*

(1) If A is (Λ,b)-closed, then A is g-(Λ,b)-closed.
(2) If A is g-(Λ,b)-closed and (Λ,b)-open, then A is (Λ,b)-closed.
(3) If A is g-(Λ,b)-closed and $A \subseteq B \subseteq A^{(\Lambda,b)}$, then B is g-(Λ,b)-closed.

Proof. (1) Let A be (Λ,b)-closed and $A \subseteq U \in \Lambda_b O(X,\tau)$. Then, by Lemma 3.7 $A = A^{(\Lambda,b)} \subseteq U$ and hence, A is g-(Λ,b)-closed.

(2) Let A be g-(Λ,b)-closed and (Λ,b)-open. Then $A^{(\Lambda,b)} \subseteq A$ and hence, A is (Λ,b)-closed.

(3) Let $B \subseteq U$ and $U \in \Lambda_b O(X,\tau)$. Then $A \subseteq U$ and A is g-(Λ,b)-closed. Hence, $A^{(\Lambda,b)} \subseteq U$. By Lemma 3.7, $A^{(\Lambda,b)} = B^{(\Lambda,b)}$ and hence, $B^{(\Lambda,b)} \subseteq U$. Therefore, B is g-(Λ,b)-closed. □

Definition 3.19. Let A be a subset of a topological space (X,τ). The (Λ,b)-frontier of A, $\Lambda_b Fr(A)$, is defined as follows:

$$\Lambda_b Fr(A) = A^{(\Lambda,b)} \cap [X - A]^{(\Lambda,b)}.$$

Proposition 3.20. *Let A be a subset of a topological space (X, τ). If A is g-(Λ, b)-closed and $A \subseteq V \in \Lambda_b O(X, \tau)$, then*

$$\Lambda_b Fr(V) \subseteq [X - A]_{(\Lambda,b)}.$$

Proof. Let A be g-(Λ, b)-closed and $A \subseteq V \in \Lambda_b O(X, \tau)$. Then $A^{(\Lambda,b)} \subseteq V$. Suppose that $x \in \Lambda_b Fr(V)$. Since $V \in \Lambda_b O(X, \tau)$, $\Lambda_b Fr(V) = V^{(\Lambda,b)} - V$. Therefore, $x \notin V$ and $x \notin A^{(\Lambda,b)}$. This shows that $x \in [X - A]_{(\Lambda,b)}$ and hence, $\Lambda_b Fr(V) \subseteq [X - A]_{(\Lambda,b)}$. □

Proposition 3.21. *Let (X, τ) be a topological space. For each $x \in X$, either $\{x\}$ is (Λ, b)-closed or $\{x\}$ is g-(Λ, b)-open.*

Proof. Suppose that $\{x\}$ is not (Λ, b)-closed. Then $X - \{x\}$ is not (Λ, b)-open and the only (Λ, b)-open set containing $X - \{x\}$ is X itself. Therefore, $[X - \{x\}]^{(\Lambda,b)} \subseteq X$ and hence, $X - \{x\}$ is g-(Λ, b)-closed. Thus, $\{x\}$ is g-(Λ, b)-open. □

Theorem 3.22. *A subset A of a topological space (X, τ) is g-(Λ, b)-open if and only if $F \subseteq A_{(\Lambda,b)}$ whenever $F \subseteq A$ and F is (Λ, b)-closed.*

Proof. Suppose that A is g-(Λ, b)-open. Let $F \subseteq A$ and F is (Λ, b)-closed. Then $X - A \subseteq X - F \in \Lambda_b O(X, \tau)$ and $X - A$ is g-(Λ, b)-closed. Therefore, we obtain $X - A_{(\Lambda,b)} = [X - A]^{(\Lambda,b)} \subseteq X - F$ and hence $F \subseteq A_{(\Lambda,b)}$.

Conversely, let $X - A \subseteq U$ and $U \in \Lambda_b O(X, \tau)$. Then $X - U \subseteq A$ and $X - U$ is (Λ, b)-closed. By the hypothesis, we have $X - U \subseteq A_{(\Lambda,b)}$ and hence, $[X - A]^{(\Lambda,b)} = X - A_{(\Lambda,b)} \subseteq U$. Therefore, $X - A$ is g-(Λ, b)-closed and A is g-(Λ, b)-open. □

Corollary 3.23. *For a subsets A, B of a topological space (X, τ), the following properties hold:*

(1) If A is (Λ, b)-open, then A is g-(Λ, b)-open.
(2) If A is g-(Λ, b)-open and (Λ, b)-closed, then A is (Λ, b)-open.
(3) If A is g-(Λ, b)-open and $A_{(\Lambda,b)} \subseteq B \subseteq A$, then B is g-(Λ, b)-open.

Proof. This follows from Proposition 3.18. □

Lemma 3.24. *Let A be a subset of a topological space (X, τ). If $A \cap G = \emptyset$, then $A^{(\Lambda,b)} \cap G = \emptyset$ for each $G \in \Lambda_b O(X, \tau)$.*

Theorem 3.25. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) *A is* g-(Λ, b)*-closed.*
(2) $A^{(\Lambda,b)} - A$ *contains no nonempty* (Λ, b)*-closed set.*
(3) $A^{(\Lambda,b)} - A$ *is* g-(Λ, b)*-open.*

Proof. (1) $\Rightarrow$ (2): This follows from Theorem 3.16.

(2) $\Rightarrow$ (3): Let $F \subseteq A^{(\Lambda,b)} - A$ and F be (Λ, b)-closed. By (2), we have $F = \emptyset$ and $F \subseteq [A^{(\Lambda,b)} - A]_{(\Lambda,b)}$. It follows from Theorem 3.22 that $A^{(\Lambda,b)} - A$ is g-(Λ, b)-open.

(3) $\Rightarrow$ (1): Suppose that $A \subseteq U$ and $U \in \Lambda_b O(X, \tau)$. Then,

$$A^{(\Lambda,b)} - U \subseteq A^{(\Lambda,b)} - A.$$

By (3), $A^{(\Lambda,b)} - A$ is g-(Λ, b)-open. Since $A^{(\Lambda,b)} - U$ is (Λ, b)-closed and by Theorem 3.22, we have $A^{(\Lambda,b)} - U \subseteq [A^{(\Lambda,b)} - A]_{(\Lambda,b)} = \emptyset$. Therefore, we have $A^{(\Lambda,b)} \subseteq U$ and hence, A is g-(Λ, b)-closed. Now, the proof of $[A^{(\Lambda,b)} - A]_{(\Lambda,b)} = \emptyset$ is given as follows. Suppose that $[A^{(\Lambda,b)} - A]_{(\Lambda,b)} \neq \emptyset$. There exists $x \in [A^{(\Lambda,b)} - A]_{(\Lambda,b)}$. Then, there exists $G \in \Lambda_b O(X, \tau)$ such that $x \in G \subseteq A^{(\Lambda,b)} - A$. Since $G \subseteq X - A$, we have $G \cap A = \emptyset$ and $G \in \Lambda_b O(X, \tau)$. By Lemma 3.24, $G \cap A^{(\Lambda,b)} = \emptyset$ and hence, $G \subseteq X - A^{(\Lambda,b)}$. Therefore, we obtain $G \subseteq [X - A^{(\Lambda,b)}] \cap A^{(\Lambda,b)} = \emptyset$. This is a contradiction. □

THEOREM 3.26. *A subset A of a topological space* (X, τ) *is* g-(Λ, b)*-closed if and only if* $F \cap A^{(\Lambda,b)} = \emptyset$ *whenever* $A \cap F = \emptyset$ *and F is* (Λ, b)*-closed.*

Proof. Suppose that A is (Λ, b)-closed. Let $A \cap F = \emptyset$ and F be (Λ, b)-closed. Then $A \subseteq X - F \in \Lambda_b O(X, \tau)$ and $A^{(\Lambda,b)} \subseteq X - F$. Therefore, we have $F \cap A^{(\Lambda,b)} = \emptyset$.

Conversely, let $A \subseteq U$ and $U \in \Lambda_b O(X, \tau)$. Then $A \cap (X - U) = \emptyset$ and $X - U$ is (Λ, b)-closed. By the hypothesis, $(X - U) \cap A^{(\Lambda,b)} = \emptyset$ and hence $A^{(\Lambda,b)} \subseteq U$. Therefore, A is (Λ, b)-closed. □

THEOREM 3.27. *A subset A of a topological space* (X, τ) *is* g-(Λ, b)*-closed if and only if* $A \cap \{x\}^{(\Lambda,b)} \neq \emptyset$ *for every* $x \in A^{(\Lambda,b)}$.

Proof. Suppose that A is g-(Λ, b)-closed. Let $A \cap \{x\}^{(\Lambda,b)} = \emptyset$ for some $x \in A^{(\Lambda,b)}$. By Lemma 3.7, $\{x\}^{(\Lambda,b)}$ is (Λ, b)-closed and hence, $A \subseteq X - \{x\}^{(\Lambda,b)} \in \Lambda_b O(X, \tau)$. Since A is g-(Λ, b)-closed,

$$A^{(\Lambda,b)} \subseteq X - \{x\}^{(\Lambda,b)} \subseteq X - \{x\}.$$

This contradicts that $x \in A^{(\Lambda,b)}$.

Conversely, suppose that A is not g-(Λ, b)-closed. Then $\emptyset \neq A^{(\Lambda,b)} - U$ for some $U \in \Lambda_b O(X, \tau)$ containing A. There exists $x \in A^{(\Lambda,b)} - U$. Since $x \notin U$, by Lemma 3.8 $U \cap \{x\}^{(\Lambda,b)} = \emptyset$ and hence,

$$A \cap \{x\}^{(\Lambda,b)} \subseteq U \cap \{x\}^{(\Lambda,b)} = \emptyset.$$

This shows that $A \cap \{x\}^{(\Lambda,b)} = \emptyset$ for some $x \in A^{(\Lambda,b)}$. □

COROLLARY 3.28. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is g-(Λ, b)-open.
(2) $A - A_{(\Lambda,b)}$ contains no nonempty (Λ, b)-closed set.
(3) $A - A_{(\Lambda,b)}$ is g-(Λ, b)-open.
(4) $(X - A) \cap \{x\}^{(\Lambda,b)} \neq \emptyset$ for every $x \in A - A_{(\Lambda,b)}$.

Proof. This follows from Theorem 3.22, 3.25 and 3.27. □

THEOREM 3.29. *For a topological space (X, τ), the following properties are equivalent:*

(1) For every (Λ, b)-open set U, $U^{(\Lambda,b)} \subseteq U$.
(2) Every subset of X is g-(Λ, b)-closed.

Proof. (1) ⇒ (2): Let A be any subset of X and $A \subseteq U \in \Lambda_b O(X, \tau)$. By (1), $U^{(\Lambda,b)} \subseteq U$ and hence $A^{(\Lambda,b)} \subseteq U^{(\Lambda,b)} \subseteq U$. Therefore, A is g-(Λ, b)-closed.

(2) ⇒ (1): Let $U \in \Lambda_b O(X, \tau)$. By (2), U is g-(Λ, b)-closed and hence, $U^{(\Lambda,b)} \subseteq U$. □

THEOREM 3.30. *A subset A of a topological space (X, τ) is g-(Λ, b)-open if and only if $U = X$ whenever U is (Λ, b)-open and*

$$(X - A) \cup A_{(\Lambda,b)} \subseteq U.$$

Proof. Suppose that A is g-(Λ, b)-open and $U \in \Lambda_b O(X, \tau)$ such that $(X - A) \cup A_{(\Lambda,b)} \subseteq U$. Then $X - U \subseteq [X - A]^{(\Lambda,b)} - (X - A)$. Since $X - A$ is g-(Λ, b)-closed and $X - U$ is (Λ, b)-closed, by Theorem 3.16 $X - U = \emptyset$ and hence, $X = U$.

Conversely, suppose that $F \subseteq A$ and F is (Λ, b)-closed. By Lemma 3.12, we have $(X - A) \cup A_{(\Lambda,b)} \subseteq (X - F) \cup A_{(\Lambda,b)} \in \Lambda_b O(X, \tau)$. By the hypothesis, $X = (X - F) \cup A_{(\Lambda,b)}$. Hence,

$$F = F \cap [(X - F) \cup A_{(\Lambda,b)}] = F \cap A_{(\Lambda,b)} \subseteq A_{(\Lambda,b)}.$$

It follows from Theorem 3.22 that A is g-(Λ, b)-open. □

Proposition 3.31. *Let A be a subset of a topological space (X, τ). If A is g-(Λ, b)-open and $A_{(\Lambda,b)} \subseteq B \subseteq A$, then B is g-(Λ, b)-open.*

Proof. We have $X - A \subseteq X - B \subseteq X - A_{(\Lambda,b)} = [X - A]^{(\Lambda,b)}$. Since $X - A$ is g-(Λ, b)-closed, it follows from Proposition 3.18(3) that $X - B$ is g-(Λ, b)-closed and hence, B is g-(Λ, b)-open. □

Definition 3.32. A subset A of a topological space (X, τ) is said to be *locally* (Λ, b)*-closed* if $A = U \cap F$, where $U \in \Lambda_b O(X, \tau)$ and F is (Λ, b)-closed.

Theorem 3.33. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is locally (Λ, b)-closed;
(2) $A = U \cap A^{(\Lambda,b)}$ for some $U \in \Lambda_b O(X, \tau)$;
(3) $A^{(\Lambda,b)} - A$ is (Λ, b)-closed;
(4) $A \cup [X - A^{(\Lambda,b)}] \in \Lambda_b O(X, \tau)$;
(5) $A \subseteq [A \cup [X - A^{(\Lambda,b)}]]_{(\Lambda,b)}$.

Proof. (1) ⇒ (2): Suppose that $A = U \cap F$, where $U \in \Lambda_b O(X, \tau)$ and F is (Λ, b)-closed. Since $A \subseteq F$, we have $A^{(\Lambda,b)} \subseteq F^{(\Lambda,b)} = F$. Since $A \subseteq U$, $A \subseteq U \cap A^{(\Lambda,b)} \subseteq U \cap F = A$. Therefore, we obtain $A = U \cap A^{(\Lambda,b)}$ for some $U \in \Lambda_b O(X, \tau)$.

(2) ⇒ (3): Suppose that $A = U \cap A^{(\Lambda,b)}$ for some $U \in \Lambda_b O(X, \tau)$. Then, $A^{(\Lambda,b)} - A = (X - [U \cap A^{(\Lambda,b)}]) \cap A^{(\Lambda,b)} = (X - U) \cap A^{(\Lambda,b)}$. Since $(X - U) \cap A^{(\Lambda,b)}$ is (Λ, b)-closed and hence, $A^{(\Lambda,b)} - A$ is (Λ, b)-closed.

(3) ⇒ (4): We have $X - [A^{(\Lambda,b)} - A] = [X - A^{(\Lambda,b)}] \cup A$ and hence, by (3) we obtain $A \cup [X - A^{(\Lambda,b)}] \in \Lambda_b O(X, \tau)$.

(4) ⇒ (5): By (4), $A \subseteq A \cup [X - A^{(\Lambda,b)}] = [A \cup [X - A^{(\Lambda,b)}]]_{(\Lambda,b)}$.

(5) ⇒ (1): We put $U = [A \cup [X - A^{(\Lambda,b)}]]_{(\Lambda,b)}$. Then $U \in \Lambda_b O(X, \tau)$ and $A = A \cap U \subseteq U \cap A^{(\Lambda,b)} \subseteq [A \cup [X - A^{(\Lambda,b)}]] \cap A^{(\Lambda,b)} = A \cap A^{(\Lambda,b)} = A$. Therefore, we obtain $A = U \cap A^{(\Lambda,b)}$, where $U \in \Lambda_b O(X, \tau)$ and $A^{(\Lambda,b)}$ is (Λ, b)-closed. This shows that A is locally (Λ, b)-closed. □

Theorem 3.34. *A subset A of a topological space (X, τ) is (Λ, b)-closed if and only if A is locally (Λ, b)-closed and g-(Λ, b)-closed.*

Proof. Let A be (Λ, b)-closed. By Proposition 3.18(1), A is g-(Λ, b)-closed. Since $X \in \Lambda_b O(X, \tau)$ and $A = X \cap A$, A is locally (Λ, b)-closed.

Conversely, suppose that A is locally (Λ, b)-closed and g-(Λ, b)-closed. Since A is locally (Λ, b)-closed, by Theorem 3.33 we have

$$A \subseteq [A \cup [X - A^{(\Lambda,b)}]]_{(\Lambda,b)}.$$

By Lemma 3.12, $[A \cup [X - A^{(\Lambda,b)}]]_{(\Lambda,b)} \in \Lambda_b O(X, \tau)$ and A is g-(Λ, b)-closed. Therefore, we obtain

$$A^{(\Lambda,b)} \subseteq [A \cup [X - A^{(\Lambda,b)}]]_{(\Lambda,b)} \subseteq A \cup [X - A^{(\Lambda,b)}]$$

and hence, $A^{(\Lambda,b)} = A$ and by Lemma 3.7 A is (Λ, b)-closed. □

Definition 3.35. Let A be a subset of a topological space (X, τ). The subset $\Lambda_{(\Lambda,b)}$ is defined as follows:

$$\Lambda_{(\Lambda,b)}(A) = \cap\{U \in \Lambda_b O(X, \tau) \mid A \subseteq U\}.$$

Lemma 3.36. *For subsets A, B of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq \Lambda_{(\Lambda,b)}(A)$.
(2) If $A \subseteq B$, *then* $\Lambda_{(\Lambda,b)}(A) \subseteq \Lambda_{(\Lambda,b)}(B)$.
(3) $\Lambda_{(\Lambda,b)}[\Lambda_{(\Lambda,b)}(A)] = \Lambda_{(\Lambda,b)}(A)$.
(4) If A is (Λ, b)-open, $\Lambda_{(\Lambda,b)}(A) = A$.

Definition 3.37. A subset A of a topological space (X, τ) is called a $\Lambda_{(\Lambda,b)}$*-set* if $A = \Lambda_{(\Lambda,b)}(A)$. The family of all $\Lambda_{(\Lambda,b)}$-sets of (X, τ) is denoted by $\Lambda_{(\Lambda,b)}(X, \tau)$ (or simply $\Lambda_{(\Lambda,b)}$).

Definition 3.38. A subset A of a topological space (X, τ) is called a *generalized* $\Lambda_{(\Lambda,b)}$*-set* (briefly g-$\Lambda_{(\Lambda,b)}$-set) if $\Lambda_{(\Lambda,b)}(A) \subseteq F$ whenever $A \subseteq F$ and F is (Λ, b)-closed.

Definition 3.39. A topological space (X, τ) is called a Λ_b-$T_{\frac{1}{2}}$*-space* if every g-(Λ, b)-closed set of (X, τ) is (Λ, b)-closed.

Lemma 3.40. *For a topological space (X, τ), the following properties hold:*

(1) For each $x \in X$, the singleton $\{x\}$ is (Λ, b)-closed or $X - \{x\}$ is g-(Λ, b)- closed.
(2) For each $x \in X$, the singleton $\{x\}$ is (Λ, b)-open or $X - \{x\}$ is g-$\Lambda_{(\Lambda,b)}$-set.

Proof. (1) Let $x \in X$ and the singleton $\{x\}$ be not (Λ, b)-closed. Then $X - \{x\}$ is not (Λ, b)-open and X is the only (Λ, b)-open set which contains $X - \{x\}$ and hence, $X - \{x\}$ is g-(Λ, b)-closed.

(2) Let $x \in X$ and the singleton $\{x\}$ be not (Λ, b)-open. Then $X - \{x\}$ is not (Λ, b)-closed and the only (Λ, b)-closed set which contains $X - \{x\}$ is X and hence, $X - \{x\}$ is a g-$\Lambda_{(\Lambda,b)}$-set. □

THEOREM 3.41. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is a Λ_b-$T_{\frac{1}{2}}$-space.
(2) For each $x \in X$, the singleton $\{x\}$ is (Λ, b)-open or (Λ, b)-closed.
(3) Every g-$\Lambda_{(\Lambda,b)}$-set is a $\Lambda_{(\Lambda,b)}$-set.

Proof. (1) ⇒ (2): By Lemma 3.40, for each $x \in X$, the singleton $\{x\}$ is (Λ, b)-closed or $X - \{x\}$ is g-(Λ, b)-closed. Since (X, τ) is a Λ_b-$T_{\frac{1}{2}}$-space, $X - \{x\}$ is (Λ, b)-closed and hence, $\{x\}$ is (Λ, b)-open in the latter case. Therefore, the singleton $\{x\}$ is (Λ, b)-open or (Λ, b)-closed.

(2) ⇒ (3): Suppose that there exists a g-$\Lambda_{(\Lambda,b)}$-set A which is not a $\Lambda_{(\Lambda,b)}$-set. There exists $x \in \Lambda_{(\Lambda,b)}(A)$ such that $x \notin A$. In case the singleton $\{x\}$ is (Λ, b)-open, $A \subseteq X - \{x\}$ and $X - \{x\}$ is (Λ, b)-closed. Since A is a g-$\Lambda_{(\Lambda,b)}$-set, $\Lambda_{(\Lambda,b)}(A) \subseteq X - \{x\}$. This is a contradiction. In case the singleton $\{x\}$ is (Λ, b)-closed, $A \subseteq X - \{x\}$ and $X - \{x\}$ is (Λ, b)-open. By Lemma 3.36, $\Lambda_{(\Lambda,b)}(A) \subseteq \Lambda_{(\Lambda,b)}(X - \{x\}) = X - \{x\}$. This is a contradiction. Therefore, every g-$\Lambda_{(\Lambda,b)}$-set is a $\Lambda_{(\Lambda,b)}$-set.

(3) ⇒ (1): Suppose that (X, τ) is not a Λ_b-$T_{\frac{1}{2}}$-space. Then, there exists a g-(Λ, b)-closed set A which is not (Λ, b)-closed. Since A is not (Λ, b)-closed, there exists a point $x \in A^{(\Lambda,b)}$ such that $x \notin A$. By Lemma 3.40, the singleton $\{x\}$ is (Λ, b)-open or $X - \{x\}$ is a $\Lambda_{(\Lambda,b)}$-set. (a) In case $\{x\}$ is (Λ, b)-open, since $x \in A^{(\Lambda,b)}$, $\{x\} \cap A \neq \emptyset$ and $x \in A$. This is a contradiction. (b) In case $X - \{x\}$ is a $\Lambda_{(\Lambda,b)}$-set, if $\{x\}$ is not (Λ, b)-closed, $X - \{x\}$ is not (Λ, b)-open and $\Lambda_{(\Lambda,b)}(X - \{x\}) = X$. Hence, $X - \{x\}$ is not a $\Lambda_{(\Lambda,b)}$-set. This contradicts (3). If $\{x\}$ is (Λ, b)-closed, $A \subseteq X - \{x\} \in \Lambda_b O(X, \tau)$ and A is g-(Λ, b)-closed. Hence we have $A^{(\Lambda,b)} \subseteq X - \{x\}$. This contradicts that $x \in A^{(\Lambda,b)}$. Therefore, (X, τ) is a Λ_b-$T_{\frac{1}{2}}$-space. □

4. Λ_b-regular spaces and Λ_b-normal spaces

In this section, we introduce the notions of Λ_b-regular spaces and Λ_b-normal spaces. Some characterizations of Λ_b-regular spaces and Λ_b-normal spaces are obtained.

DEFINITION 4.1. A topological space (X, τ) is said to be Λ_b-*regular* if for each (Λ, b)-closed sets F not containing x, there exist disjoint (Λ, b)-open sets U and V such that $x \in U$ and $F \subseteq V$.

THEOREM 4.2. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_b-regular.

(2) For each $x \in X$ and each $U \in \Lambda_b O(X, \tau)$ such that $x \in U$, there exists $V \in \Lambda_b O(X, \tau)$ such that $x \in V \subseteq V^{(\Lambda,b)} \subseteq U$.

(3) For each (Λ, b)-closed set F, $\cap\{V^{(\Lambda,b)} \mid F \subseteq V \in \Lambda_b O(X, \tau)\} = F$.

(4) For each subset A of X and each $U \in \Lambda_b O(X, \tau)$ such that $A \cap U \neq \emptyset$, there exists $V \in \Lambda_b O(X, \tau)$ such that $A \cap V \neq \emptyset$ and $V^{(\Lambda,b)} \subseteq U$.

(5) For each nonempty subset A of X and each (Λ, b)-closed set F of X such that $A \cap F = \emptyset$, there exist $V, W \in \Lambda_b O(X, \tau)$ such that $A \cap V \neq \emptyset$, $F \subseteq W$ and $V \cap W = \emptyset$.

(6) For each (Λ, b)-closed set F and $x \notin F$, there exist $U \in \Lambda_b O(X, \tau)$ and a g-(Λ, b)-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$.

(7) For each subset A of X and each (Λ, b)-closed set F such that $A \cap F = \emptyset$, there exist $U \in \Lambda_b O(X, \tau)$ and a g-(Λ, b)-open set V such that $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let $G \in \Lambda_b O(X, \tau)$ and $x \in G$. Then $x \notin X - G$, there exist disjoint $U, V \in \Lambda_b O(X, \tau)$ such that $X - G \subseteq U$ and $x \in V$. Thus $V \subseteq X - U$ and so $x \in V \subseteq V^{(\Lambda,b)} \subseteq X - U \subseteq G$.

(2) $\Rightarrow$ (3): For any $F \in \Lambda_b C(X, \tau)$, we always have

$$F \subseteq \cap\{V^{(\Lambda,b)} \mid F \subseteq V \in \Lambda_b O(X, \tau)\}.$$

On the other hand, let $F \in \Lambda_b C(X, \tau)$ such that $x \in X - F$. Then by (2), there exists $U \in \Lambda_b O(X, \tau)$ such that $x \in U \subseteq U^{(\Lambda,b)} \subseteq X - F$. So $F \subseteq X - U^{(\Lambda,b)} = V \in \Lambda_b O(X, \tau)$ and $U \cap V = \emptyset$. Then $x \notin V^{(\Lambda,b)}$. Thus $F \supseteq \cap\{V^{(\Lambda,b)} \mid F \subseteq V \in \Lambda_b O(X, \tau)\}$.

(3) $\Rightarrow$ (4): Let A be a subset of X and $U \in \Lambda_b O(X, \tau)$ such that $A \cap U \neq \emptyset$. Let $x \in A \cap U$. Then $x \notin X - U$. Hence by (3), there exists $W \in \Lambda_b O(X, \tau)$ such that $X - U \subseteq W$ and $x \notin W^{(\Lambda,b)}$. Put $V =$

$X - W^{(\Lambda,b)}$ which is a (Λ, b)-open set containing x and hence $A \cap V \neq \emptyset$. Now $V \subseteq X - W$ and so $V^{(\Lambda,b)} \subseteq X - W \subseteq U$.

(4) $\Rightarrow$ (5): Let A be a nonempty subset of X and F be a (Λ, b)-closed set such that $A \cap F = \emptyset$. Then $X - F \in \Lambda_b O(X, \tau)$ such that $A \cap (X - F) \neq \emptyset$ and hence by (4), there exists $V \in \Lambda_b O(X, \tau)$ such that $A \cap V \neq \emptyset$ and $V^{(\Lambda,b)} \subseteq X - F$. Put $W = X - V^{(\Lambda,b)}$, then W is a (Λ, b)-open set such that $F \subseteq W$ and $W \cap V = \emptyset$.

(5) $\Rightarrow$ (1): Let F be a (Λ, b)-closed set not containing x. Then $F \cap \{x\} = \emptyset$. Thus by (5), there exist $V, W \in \Lambda_b O(X, \tau)$ such that $x \in V, F \subseteq W$ and $V \cap W = \emptyset$.

(1) $\Rightarrow$ (6): Obvious.

(6) $\Rightarrow$ (7): Let A be a subset of X and F be a (Λ, b)-closed set such that $A \cap F = \emptyset$. Then for $x \in A, x \notin F$, and hence by (6), there exist $U \in \Lambda_b O(X, \tau)$ and a g-(Λ, b)-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. So $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.

(7) $\Rightarrow$ (1): Let F be a (Λ, b)-closed set such that $x \notin F$. Since $\{x\} \cap F = \emptyset$, by (7) there exist $U \in \Lambda_b O(X, \tau)$ and a g-(Λ, b)-open set W such that $x \in U$, $F \subseteq W$ and $U \cap W = \emptyset$. Since W is g-(Λ, b)-open, by Theorem 3.22 we have $F \subseteq W_{(\Lambda,b)} = V \in \Lambda_b O(X, \tau)$ and hence $U \cap V = \emptyset$. □

Definition 4.3. A topological space (X, τ) is said to be Λ_b-*normal* if for any pair of disjoint (Λ, b)-closed sets F and H, there exist disjoint (Λ, b)-open sets U and V such that $F \subseteq U$ and $H \subseteq V$.

Theorem 4.4. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_b-normal.

(2) For every pair of (Λ, b)-open sets U and V whose union is X, there exist (Λ, b)-closed sets F and H such that $F \subseteq U$, $H \subseteq V$ and $F \cup H = X$.

(3) For every (Λ, b)-closed set F and every (Λ, b)-open set G containing F, there exists a (Λ, b)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,b)} \subseteq G$.

Proof. (1) $\Rightarrow$ (2): Let U and V be a pair of (Λ, b)-open sets in a Λ_b-normal space (X, τ) such that $X = U \cup V$. Then $X - U$ and $X - V$ are disjoint (Λ, b)-closed sets. Since (X, τ) is Λ_b-normal, there exist disjoint (Λ, b)-open sets G and W such that $X - U \subseteq G$ and $X - V \subseteq W$. Put $F = X - G$ and $H = X - W$. Then F and H are (Λ, b)-closed sets such that $F \subseteq U$, $H \subseteq V$ and $F \cup H = X$.

(2) $\Rightarrow$ (3): Let F be a (Λ, b)-closed set and G a (Λ, b)-open set containing F. Then $X - F$ and G are (Λ, b)-open sets whose union is X. Then by (2), there exist (Λ, b)-closed sets M and N such that $M \subseteq X - F$, $N \subseteq G$ and $M \cup N = X$. Then $F \subseteq X - M$, $X - G \subseteq X - N$ and $(X - M) \cap (X - N) = \emptyset$. Put $U = X - M$ and $V = X - N$. Then U and V are disjoint (Λ, b)-open sets such that $F \subseteq U \subseteq X - V \subseteq G$. As $X - V$ is (Λ, b)-closed set, we have $U^{(\Lambda,b)} \subseteq X - V$ and $F \subseteq U \subseteq U^{(\Lambda,b)} \subseteq G$.

(3) $\Rightarrow$ (1): Let F and H be two disjoint (Λ, b)-closed sets. Then $F \subseteq X - H$ and $X - H$ is (Λ, b)-open and hence there exists a (Λ, b)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,b)} \subseteq X - H$. Put $V = X - U^{(\Lambda,b)}$. Then U and V are disjoint (Λ, b)-open sets such that $F \subseteq U$ and $H \subseteq V$. This shows that (X, τ) is Λ_b-normal. □

THEOREM 4.5. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is Λ_b-normal.
(2) For every pair of disjoint (Λ, b)-closed sets F and H, there exist disjoint g-(Λ, b)-open sets U and V such that $F \subseteq U$ and $H \subseteq V$.
(3) For each (Λ, b)-closed set F and each (Λ, b)-open set G containing F, there exists a g-(Λ, b)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,b)} \subseteq G$.
(4) For each (Λ, b)-closed set F and each g-(Λ, b)-open set G containing F, there exists a (Λ, b)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,b)} \subseteq G_{(\Lambda,b)}$.
(5) For each (Λ, b)-closed set F and each g-(Λ, b)-open set G containing F, there exists a g-(Λ, b)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,b)} \subseteq G_{(\Lambda,b)}$.
(6) For each g-(Λ, b)-closed set F and each (Λ, b)-open set G containing F, there exists a (Λ, b)-open set U such that $F^{(\Lambda,b)} \subseteq U \subseteq U^{(\Lambda,b)} \subseteq G$.
(7) For each g-(Λ, b)-closed set F and each (Λ, b)-open set G containing F, there exists a g-(Λ, b)-open set U such that $F^{(\Lambda,b)} \subseteq U \subseteq U^{(\Lambda,b)} \subseteq G$.

Proof. (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (3): Let F be a (Λ, b)-closed set and G be a (Λ, b)-open set containing F. Then F and $X - G$ are two disjoint (Λ, b)-closed sets. Hence by (2) there exist disjoint g-(Λ, b)-open sets U and V such that

$F \subseteq U$ and $X - G \subseteq V$. Since V is g-(Λ, b)-open and $X - G$ is (Λ, b)-closed, by Theorem 3.22, $X - G \subseteq V_{(\Lambda,b)}$. Hence,

$$[X - V]^{(\Lambda,b)} = X - V_{(\Lambda,b)} \subseteq G.$$

Thus, $F \subseteq U \subseteq U^{(\Lambda,b)} \subseteq G$.

(3) $\Rightarrow$ (1): Let F and H be two disjoint (Λ, b)-closed sets. Then F is a (Λ, b)-closed set and $X - H$ is a (Λ, b)-open set containing F. Thus by (3), there exists a g-(Λ, b)-open set U such that

$$F \subseteq U \subseteq U^{(\Lambda,b)} \subseteq X - H.$$

Thus by Theorem 3.22, $F \subseteq U_{(\Lambda,b)}$, $H \subseteq X - U^{(\Lambda,b)}$, where $U_{(\Lambda,b)}$ and $X - U^{(\Lambda,b)}$ are two disjoint (Λ, b)-open sets. This shows that (X, τ) is Λ_b-normal.

(4) $\Rightarrow$ (5) $\Rightarrow$ (2): Obvious.

(6) $\Rightarrow$ (7) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (5): Let F be a (Λ, b)-closed set and G be a g-(Λ, b)-open set containing F. Since G is g-(Λ, b)-open and F is (Λ, b)-closed, by Theorem 3.22 $F \subseteq G_{(\Lambda,b)}$. Thus by (3), there exists a g-(Λ, b)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,b)} \subseteq G_{(\Lambda,b)}$.

(5) $\Rightarrow$ (6): Let F be a g-(Λ, b)-closed set and G be a (Λ, b)-open set containing F. Then $F^{(\Lambda,b)} \subseteq G$. Since G is g-(Λ, b)-open, there exists a g-(Λ, b)-open set U such that $F^{(\Lambda,b)} \subseteq U \subseteq U^{(\Lambda,b)} \subseteq G$. Since U is g-(Λ, b)-open and $F^{(\Lambda,b)}$ is (Λ, b)-closed, by Theorem 3.22 we have $F^{(\Lambda,b)} \subseteq U_{(\Lambda,b)}$. Put $V = U_{(\Lambda,b)}$. Then V is (Λ, b)-open and hence,

$$F^{(\Lambda,b)} \subseteq V \subseteq V^{(\Lambda,b)} = [U_{(\Lambda,b)}]^{(\Lambda,b)} \subseteq U^{(\Lambda,b)} \subseteq G.$$

(6) $\Rightarrow$ (4): Let F be a (Λ, b)-closed set and G be a g-(Λ, b)-open set containing F. Then by Theorem 3.22, we obtain $F \subseteq G_{(\Lambda,b)}$. Since F is g-(Λ, b)-closed and $G_{(\Lambda,b)}$ is (Λ, b)-open, by (6) there exists a (Λ, b)-open set U such that $F = F^{(\Lambda,b)} \subseteq U \subseteq U^{(\Lambda,b)} \subseteq G_{(\Lambda,b)}$. $\square$

References

[1] D. Andrijević, *On b-open sets*, Mat. Vesnik, **48** (1996), 59–64.

[2] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik, **38** (1986), 24–32.

[3] M. Caldas, S. Jafari and T. Noiri, *On Λ_b-sets and the associated topology τ^{Λ_b}*, Acta Math. Hungar., **110** (2006), 337–345.

[4] H.H. Corson and E. Michael, *Metrizability of certain countable unions*, Illinois J. Math., **8** (1964), 351–360.

[5] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41.
[6] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Pro. Math. Phys. Soc. Egypt, **53** (1982), 47–53.

Chawalit Boonpok
Mathematics and Applied Mathematics Research Unit
Department of Mathematics
Faculty of Science
Mahasarakham University
Mahasarakham 44150, Thailand
E-mail: `chawalit.b@msu.ac.th`