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SELF-DUAL CODES OVER $\mathbb{Z}_{p^2}$ OF SMALL LENGTHS

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Abstract. Self-dual codes of lengths less than 5 over $\mathbb{Z}_p$ are completely classified by the second author [The classification of self-dual modular codes, Finite Fields Appl. **17** (2011), 442-460]. The number of such self-dual codes are also determined. In this article we will extend the results to classify self-dual codes over $\mathbb{Z}_{p^2}$ of length less than 5 and give the number of codes in each class. Explicit and complete classifications for small p's are also given.

1. Introduction

A code over $\mathbb{Z}_{p^e}$ of length n is a $\mathbb{Z}_{p^e}$-submodule of $\mathbb{Z}_{p^e}^n$. Codes of length n over $\mathbb{Z}_{p^e}$ have generator matrices permutation equivalent to the *standard form*

$$(1)\qquad G = \begin{pmatrix} I_{k_0} & A_{01} & A_{02} & A_{03} & \dots & A_{0,e-1} & A_{0e} \\ 0 & pI_{k_1} & pA_{12} & pA_{13} & \dots & pA_{1,e-1} & pA_{1e} \\ 0 & 0 & p^2I_{k_2} & p^2A_{23} & \dots & p^2A_{2,e-1} & p^2A_{2e} \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & p^{e-1}I_{k_{e-1}} & p^{e-1}A_{e-1,e} \end{pmatrix},$$

where the columns are grouped into blocks of sizes $k_0, k_1, \cdots, k_e$, and the k_i are nonnegative integers adding to n [4]. A matrix with this standard

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form is said to be of *type*

$$(1)^{k_0}(p)^{k_1}(p^2)^{k_2}\cdots(p^{e-1})^{k_{e-1}}. \tag{2}$$

The number of nonzero rows is called the *rank* of M and denoted by $\operatorname{rank} M$. k_0 is called the *free rank*.

The ambient space $\mathbb{Z}_{p^e}^n$ is endowed with the standard inner product

$$(v_1,\cdots,v_n)\cdot(w_1,\cdots,w_n)=v_1w_1+\cdots+v_nw_n.$$

For a code C of length n over $\mathbb{Z}_{p^e}$, the dual code $C^\perp$ of C is defined by

$$C^\perp=\{\mathbf{v}\in\mathbb{Z}_{p^e}^n \mid \mathbf{v}\cdot\mathbf{w}=0 \text{ for all } \mathbf{w}\in C\}.$$

If C is a code of length n over $\mathbb{Z}_{p^e}$ with generator matrix of the form (1) then $C^\perp$ has generator matrix of the form

$$G^\perp=\begin{pmatrix} B_{0e} & B_{0,e-1} & \cdots & B_{03} & B_{02} & B_{01} & I_{k_e}\\ pB_{1e} & pB_{1,e-1} & \cdots & pB_{13} & pB_{12} & pI_{k_{e-1}} & 0\\ p^2B_{2e} & p^2B_{2,e-1} & \cdots & p^2B_{23} & p^eI_{k_{e-2}} & 0 & 0\\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot\\ p^{e-1}B_{e-1,e} & p^{e-1}I_{k_1} & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the column blocks have the same size as in G [4]. If C has type $1^{k_0}(p)^{k_1}\cdots(p^{e-1})^{k_{e-1}}$ then the dual code has type $1^{k_e}p^{k_{e-1}}(p^2)^{k_{e-2}}\cdots(p^{e-1})^{k_1}$, where $k_e=n-\sum_{i=0}^{e-1}k_i$.

C is *self-orthogonal* if $C\subset C^\perp$. C is *self-dual* if $C=C^\perp$. If C is self-dual with type $1^{k_0}(p)^{k_1}\cdots(p^{e-1})^{k_{e-1}}$, then $k_i=k_{e-i}$ for all i. For any code C of length n over $\mathbb{Z}_{p^e}$ $|C||C^\perp|=p^{en}$. If C is a self-orthogonal code of length n and $|C|=p^{en/2}$, then C is self-dual.

Next we discuss the equivalence of self-dual codes. Let

$$\mathbb{D}=\mathbb{D}_m^n=\{\operatorname{diag}(\gamma_1,\gamma_2,\cdots,\gamma_n)\mid \gamma_i\in\mathbb{Z}_m,\ \gamma_i^2=1\}.$$

and let $\mathbb{T}_m=\mathbb{T}_m^n$ be the group of all *monomial transformations* on $\mathbb{Z}_m^n$ defined by

$$\mathbb{T}_m=\{\gamma\sigma\mid\gamma\in\mathbb{D},\sigma\in S_n\}$$

as in [8]. We will use the same notations and terminology as in [8]. The group $\mathbb{T}_m$ acts on the set of all self-dual codes of length n over $\mathbb{Z}_m$ by $Ct=\{ct\mid c\in C\}$. Two self-dual codes C and C' are *equivalent* (denoted $C\sim C'$) if there exists an element $t\in\mathbb{T}_m^n$ such that $Ct=C'$. The group of all automorphisms of C will be denoted by $\operatorname{Aut}(C)$.

Self-dual codes of lengths less than 5 over $\mathbb{Z}_p$ are completely classified in [8]. The number of such self-dual codes are also determined. In this article we will classify self-dual codes over $\mathbb{Z}_{p^2}$ of length less than 5.

2. Self-dual codes over $\mathbb{Z}_{p^2}$

For codes over $\mathbb{Z}_{p^2}$, every code C over $\mathbb{Z}_{p^2}$ is permutation equivalent to a code with generator matrix in standard form:

$$G = \begin{pmatrix} I_{k_1} & A_1 & B_1 + pB_2 \\ 0 & pI_{k_2} & pC_1 \end{pmatrix}$$

where A_1, B_1, B_2, C_1 are matrices with entries from $\{0, 1, \cdots, p-1\}$. Associated with C there are two codes over $\mathbb{Z}_p$, the *residue code*

$$R(C) = \{x \in \mathbb{Z}_p^n \ : \exists y \in \mathbb{Z}_p^n \text{ such that } x + py \in C\}$$

and the *torsion code* $\mathrm{Tor}(C) = \{y \in \mathbb{Z}_p^n \ : py \in C\}$ which have generator matrices

$$R(C) = \begin{pmatrix} I_{k_1} & A_1 & B_1 \end{pmatrix}, \quad \mathrm{Tor}(C) = \begin{pmatrix} I_{k_1} & A_1 & B_1 \\ 0 & I_{k_2} & C_1 \end{pmatrix}$$

respectively. If C is self-dual, then $R(C)$ is self-orthogonal.

THEOREM 2.1. *Let p be an odd prime. There is a one-one correspondence between self-dual codes C of free rank 1 over $\mathbb{Z}_{p^2}$*

$$C : \begin{pmatrix} 1 & a_2 & a_3 & \cdots & a_{n-1} & a_n+pb_1 \\ & p & & & & pb_2 \\ & & p & & & pb_3 \\ & & & \ddots & & \vdots \\ & & & & p & pb_{n-1} \end{pmatrix}$$

where n is the length of the code, $0 \leq a_i, b_j < p$, and self-orthogonal codes $R(C) = (1 \ a_2 \ \cdots \ a_{n-1} \ a_n)$ over $\mathbb{Z}_p$.

THEOREM 2.2. *If C is a self-dual code of free rank 1 over $\mathbb{Z}_{p^2}$, then* $\mathrm{Aut}(C) = \mathrm{Aut}(R(C))$.

THEOREM 2.3. *[9] Let $\sigma_p(n.k)$ be the number of self-orthogonal codes of length n and dimension k over $\mathbb{Z}_p$, where p is odd prime. Then*

1. If n is odd,

$$\sigma_p(n,k) = \frac{\prod_{i=0}^{k-1}(p^{(n-1-2i)} - 1)}{\prod_{i=1}^{k}(p^i - 1)}.$$

2. If n is even and $k \geq 2$,

$$\sigma_p(n,k) = \frac{(p^{n-k} + \eta((-1)^{\frac{n}{2}})(p^k - 1)p^{\frac{n}{2}-k}) \prod_{i=1}^{k-1}(p^{n-2i} - 1)}{\prod_{i=1}^{k}(p^i - 1)}.$$

Here η is the quadratic character of $\mathbb{Z}_p$.

THEOREM 2.4. *[1] Let p be an odd prime. Given a self-orthogonal code C_p of dimension k over $\mathbb{Z}_p$, there are $p^{k(k-1)/2}$ self-dual codes over $\mathbb{Z}_{p^2}$ whose residue code is C_p. Therefore, the number of self-dual codes of length n over $\mathbb{Z}_{p^2}$ is $N_{p^2}(n) = \sum_{0 \leq k \leq [n/2]} \sigma_p(n,k) p^{k(k-1)/2}$.*

THEOREM 2.5. *If n is even, $\sigma_p(n,1) = \frac{p^{n-1} + \eta((-1)^{\frac{n}{2}})(p-1)p^{\frac{n}{2}-1} - 1}{p-1}$.*

Proof. The number of solutions of $x_1^2 + \cdots + x_n^2 = 0$ in $\mathbb{Z}_p$ is given by $p^{n-1} + \eta((-1)^{n/2})(p-1)p^{\frac{n}{2}-1}$ [5]. □

3. Classification

There is a unique self-dual codes (p) of length 1 over $\mathbb{Z}_{p^2}$ and there is a (unique) inequivalent self-dual code $(1 \;\; a)$ over $\mathbb{Z}_{p^2}$ of length 2 if and only if $p \equiv 1 \pmod 4$. It is clear that $\begin{pmatrix} p & \\ & p \end{pmatrix}$ is a self-dual code over $\mathbb{Z}_{p^2}$.

The types of self-dual codes of length 3 are $1^{e_0} p^{e_1}$, where $2e_0 + e_1 = 3$. Thus any self-dual code C of length 3 over $\mathbb{Z}_{p^2}$ is equivalent to

$$\begin{pmatrix} p & & \\ & p & \\ & & p \end{pmatrix} \text{ or } C_{a,b} : \begin{pmatrix} 1 & a & b+pb_1 \\ & p & pc \end{pmatrix}$$

where $0 \leq a, b, b_1 < p$ and $b \neq 0$.

For binary case, $(2) \oplus (2) \oplus (2)$ is the only self-dual code over $\mathbb{Z}_4$ of length 3, and for ternary case there are two classes of self-dual codes over $\mathbb{Z}_9$ of length 3:

$$(3) \oplus (3) \oplus (3), \quad \begin{pmatrix} 1 & 2 & 2 \\ & 3 & 6 \end{pmatrix}.$$

THEOREM 3.1. *Let $p \neq 2, 3$. Then the non-trivial self-dual code over $\mathbb{Z}_{p^2}$ of length 3 is equivalent to one of the following classes of inequivalent codes:*

Class	$C_{a,b}$	$\mathrm{Aut}(C_{a,b})$
(i)	$a = 0$	$4.\{(1),(13)\}$
(ii)	$a^6 = 1,\ a \neq \pm 1$	$2.\langle(123)\rangle$
(iii)	$a^2 = 1,\ b^2 + 2 = 0$	$2.\{(12)\}$
(iv)	*else*	$2.(1)$

THEOREM 3.2. *For $p \neq 2, 3$, let N_1, N_2, N_3, N_4 be the number of class (i), (ii), (iii), (iv) self-dual codes over $\mathbb{Z}_{p^2}$ of length 3, respectively. These numbers are determined as follows.*

$p \pmod{24}$	N_1	N_2	N_3	N_4
1	1	1	1	$\frac{p-25}{24}$
5	1	0	0	$\frac{p-5}{24}$
7	0	1	0	$\frac{p-7}{24}$
11	0	0	1	$\frac{p-11}{24}$
13	1	1	0	$\frac{p-13}{24}$
17	1	0	1	$\frac{p-17}{24}$
19	0	1	1	$\frac{p-19}{24}$
23	0	0	0	$\frac{p+1}{24}$

Proof. We have the one-to-one correspondence between the set of self-dual codes over $\mathbb{Z}_p$, the set of self-orthogonal codes over $\mathbb{Z}_{p^2}$ and the set of self-dual codes over $\mathbb{Z}_{p^2}$ as follows:

$$\begin{pmatrix} 1 & & a & b \\ & 1 & -b & a \end{pmatrix} \leftrightarrow (1\ a\ b) \leftrightarrow \begin{pmatrix} 1 & a & b+pb_1 \\ & p & pc \end{pmatrix}$$

where $1 + a^2 + b^2 = 0 \pmod{p}$. □

For $5 \leq p \leq 67$, we give the classification in the following table. Here (a, b) denotes the code $C_{a,b}$.

p^2	(i)	(ii)	(iii)	(iv)
5^2	(0,7)			
7^2		(2,32)		
11^2			(1,19)	
13^2	(0,70)	(3,126)		
17^2	(0,38)		(1,24)	
19^2		(7,315)	(1,63)	
23^2				(2,169)
29^2	(0,41)			(2,71)
31^2		(5,800)		(4,142)
37^2	(0,117)	(10,248)		(3,510)
41^2	(0,378)		(1,71)	(2,703)
43^2		(36,49)	(1,801)	(2,826)
47^2				(2,1052), (3,361)
53^2	(0,500)			(3,231), (4,1172)
59^2			(1,1275)	(3,1246), (6,776)
61^2	(0,682)	(13,1328)		(2,774), (8,1259)
67^2		(29,1645)	(1,2030)	(2,2091), (12,1626)

Next, we consider the codes of length 4. The types of self-dual codes of length 4 are $1^{e_0}p^{e_1}$, where $2e_0 + e_1 = 4$. Thus any self-dual code C of length 4 over $\mathbb{Z}_{p^2}$ is equivalent to one of

1. $(p)^4$,
2. $C^2_{a,b} : \left(\begin{smallmatrix} 1 & & a & b \\ & 1 & -b & a \end{smallmatrix}\right)$
3. $C^1_{a,b,c} : \left(\begin{smallmatrix} 1 & a & b & c+pc_1 \\ & p & & pc_2 \\ & & p & pc_3 \end{smallmatrix}\right)$ where $0 \le a, b, c < p$ and $c \ne 0$.

There are two classes of self-dual codes over $\mathbb{Z}_4$ of length 4:

$$(2) \oplus (2) \oplus (2) \oplus (2), \quad \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ & 2 & & 2 \\ & & 2 & 2 \end{smallmatrix}\right)$$

and there are three classes of self-dual codes over $\mathbb{Z}_9$ of length 4:

$$(3) \oplus (3) \oplus (3), \quad \left(\begin{smallmatrix} 1 & & 1 & 4 \\ & 1 & 5 & 1 \end{smallmatrix}\right), \quad \left(\begin{smallmatrix} 1 & & 1 & 4 \\ & 3 & & \\ & & 3 & 6 \end{smallmatrix}\right)$$

THEOREM 3.3. *Let $p \ne 2, 3$. Then the self-dual code*

$$C^2_{a,b} : \left(\begin{smallmatrix} 1 & & a & b \\ & 1 & -b & a \end{smallmatrix}\right)$$

over $\mathbb{Z}_{p^2}$ is one of the following four classes of inequivalent codes:

Class	$C^2_{a,b}$	$\mathrm{Aut}(C^2_{a,b})$
(i)	$a^2 + 1 = 0,\ b = 0$	$4.B_8$
(ii)	$a^6 = 1,\ a \ne \pm 1$	$2.A_4$
(iii)	$a^2 = 1,\ b^2 + 2 = 0$	$2.B_8$
(iv)	*else*	$2.B_4$

Codes from classes (i),(ii),(iii) are unique, if exist, up to equivalence.

THEOREM 3.4. *For $p \ne 2, 3$, let N_1, N_2, N_3, N_4 be the number of class* (i)), (iii), (iv) *self-dual codes over $\mathbb{Z}_{p^2}$ of length* 4 *and free rank* 2*, respectively. These numbers are determined as follows.*

$p \pmod{24}$	N_1	N_2	N_3	N_4
1	1	1	1	$\frac{p^2+p-26}{24}$
5	1	0	0	$\frac{p^2+p-6}{24}$
7	0	1	0	$\frac{p^2+p-8}{24}$
11	0	0	1	$\frac{p^2+p-12}{24}$
13	1	1	0	$\frac{p^2+p-14}{24}$
17	1	0	1	$\frac{p^2+p-18}{24}$
19	0	1	1	$\frac{p^2+p-20}{24}$
23	0	0	0	$\frac{p^2+p}{24}$

Proof. The number of self-dual codes over $\mathbb{Z}_{p^2}$ of length 4 and free rank 2 is given by $\sigma_p(4,2)p = 2(p+1)p$. By the mass formula

$$N_4 = \frac{1}{48}(2(p+1)p - 12N_1 - 16N_2 - 24N_3).$$

Here $48 = \frac{2^4 \cdot 4!}{|2.B_4|}$, $12 = \frac{2^4 \cdot 4!}{|4.B_8|}$, etc. □

THEOREM 3.5. *Let $p \neq 2, 3$. Then any self-dual code $C^1_{a,b,c}$ of rank 3 is equivalent to one of the following inequivalent codes:*

Class	$C^1_{a,b,c}$	$\mathrm{Aut}(C^1_{a,b,c})$
(i)	$a = b = 0$	$8.\langle(14),(23)\rangle$
(ii)	$b = 0,\ a^6 = 1,\ a^2 \neq 1,\ c^2 \neq 1$	$4.\langle(124)\rangle$
(iii)	$b = 0,\ a^2 = 1$	$4.S_2$
(iv)	$b = 0,\ a \neq 0,\ a^6 \neq 1,\ c^6 \neq 1,\ a^2 \neq c^2$	$4.(1)$
(v)	$a^2 = 1 \neq b^2 = c^2$	$2.\langle(1324),(12)\rangle$
(vi)	$a^2 = b^2 = 1$	$2.S_3$
(vii)	$1 = a^2, b^2, c^2$ *distinct*	$2.S_2$
(viii)	$a^2 = -1,\ b^2 \neq \pm 1,\ b^4 \neq -1$	$2.\{(1),(14)(23)\}$
(ix)	$a^2 = -1,\ b^2 \neq \pm 1,\ b^4 = -1$	$2.\langle(1243)\rangle$
(x)	$1, a^2, b^2, c^2$ *are all distinct,* $a^2, b^2, c^2 \neq -1$	$2.(1)$

Proof. It is enough to classify $R(C) = \langle(1,a,b,c)\rangle$ over $\mathbb{Z}_p$. When $b = 0$, the classification goes back to the case of $C^2_{a,c}$. Suppose $b \neq 0$. For $t = \gamma\sigma \in \mathbb{T}$, $\sigma \in S_4$, $k \in \mathbb{Z}_p$, we have that

$$(1,a,b,c)\gamma\sigma = k(1,a,b,c) \iff (1,a^2,b^2,c^2)\sigma = k^2(1,a^2,b^2,c^2).$$

Thus $k^2 = 1, a^2, b^2, c^2$ and σ can be determined once we know the equalities among $1, a^2, b^2, c^2$. For example, suppose that $1 = a^2, b^2, c^2$ are distinct. Now $(1,1,b^2,c^2)\sigma = (k^2,k^2,k^2b^2,k^2c^2)$ implies that $k^2 = 1$, $\sigma(1) = 1, 2$ and $\sigma(3) = 3$, $\sigma(4) = 4$. Next, for $\gamma \in \mathbb{D}$, $(1,1,b,c)\gamma = k(1,1,b,c)$ implies $\gamma = \pm(1,1,1,1)$. □

THEOREM 3.6. *For $p \neq 2, 3$, let $N_1, N_2, \cdots, N_{10}$ be the number of class* (i), (ii), $\cdots$, (x) *self-dual codes over $\mathbb{Z}_{p^2}$ of length 4 and free rank 1, respectively. These numbers are determined as follows.*

p (24)	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}
1	1	1	1	$\frac{p-25}{24}$	1	1	$\frac{p-17}{8}$	$\frac{p-9}{8}$	1	$\frac{(p+1)^2-28p+216}{192}$
5	1	0	0	$\frac{p-5}{24}$	1	0	$\frac{p-5}{8}$	$\frac{p-5}{8}$	0	$\frac{(p+1)^2-28p+104}{192}$
7	0	1	0	$\frac{p-7}{24}$	0	1	$\frac{p-7}{8}$	0	0	$\frac{(p+1)^2-16p+48}{192}$
11	0	0	1	$\frac{p-11}{24}$	0	0	$\frac{p-3}{8}$	0	0	$\frac{(p+1)^2-16p+32}{192}$
13	1	1	0	$\frac{p-13}{24}$	1	1	$\frac{p-13}{8}$	$\frac{p-5}{8}$	0	$\frac{(p+1)^2-28p+168}{192}$
17	1	0	1	$\frac{p-17}{24}$	1	0	$\frac{p-9}{8}$	$\frac{p-9}{8}$	1	$\frac{(p+1)^2-28p+152}{192}$
19	0	1	1	$\frac{p-19}{24}$	0	1	$\frac{p-11}{8}$	0	0	$\frac{(p+1)^2-16p+96}{192}$
23	0	0	0	$\frac{p+1}{24}$	0	0	$\frac{p+1}{8}$	0	0	$\frac{(p+1)^2-16p-16}{192}$

Proof. We consider the classes (viii) and (ix). In these cases $\{1, a^2, b^2, c^2\} = \{1, -1, b^2, -b^2\}$, where $b^2 \neq 0, \pm 1$, $p \equiv 1 \pmod 4$. There exists b with $b^4 = -1$ if and only if $p \equiv 1 \pmod 8$, and in that case, $(1, a, b, c) = (1, i, \pm b, \pm ib)$ or $(1, i, \pm bi, \pm b)$ with $i^2 = -1$, and hence $N_9 = 1$.

Now $(1, a^2, b^2, c^2) \sim (1, -1, \pm b^2, \mp b^2) \sim (1, -1, \pm 1/b^2, \mp 1/b^2)$. These four are distinct iff $b^4 \neq -1$. Thus $4N_8 + 2N_9 = \frac{(p-1)}{2} - 2$.

Once $N_1, \cdots, N_9$ is determined, N_{10} can be computed by the mass formula:

$$\sum_i \frac{2^4 \cdot 4!}{|\operatorname{Aut}(C_i)|} = 3p^2 + 4p + 2,$$

where C_i runs through the representatives of inequivalent self-dual codes. □

Finally we give the complete classification for small p's in the following table. Here (a, b, c) denotes the codes $C^1_{a,b,c}$.

p^2	i	ii	iii	iv	v
5^2	$(0,0,7)$				$(1,2,12)$
7^2		$(2,0,17)$			
11^2			$(1,0,19)$		
13^2	$(0,0,70)$	$(3,0,43)$			$(1,5,34)$
17^2	$(0,0,38)$		$(1,0,24)$		$(1,4,72)$
19^2		$(7,0,46)$	$(1,0,63)$		
23^2				$(2,0,169)$	
29^2	$(0,0,41)$			$(2,0,71)$	$(1,12,70)$
31^2		$(5,0,161)$		$(4,0,142)$	
37^2	$(0,0,117)$	$(10,0,248)$		$(3,0,510)$	$(1,6,228)$

p^2	vi	vii	viii	ix	x
5^2					
7^2	$(1,1,12)$				
11^2		$(1,2,29)$			
13^2	$(1,1,45)$		$(5,6,48)$		
17^2		$(1,6,110)$	$(4,5,139)$	$(4,8,53)$	
19^2	$(1,1,137)$	$(1,5,50)$			$(2,3,104)$
23^2		$(1,3,239)$ $(1,6,56)$ $(1,7,100)$			$(2,4,212)$
29^2		$(1,2,136)$ $(1,6,181)$ $(1,11,333)$	$(12,13,47)$ $(12,14,325)$ $(12,19,149)$		$(3,5,96)$
31^2	$(1,1,82)$	$(1,2,98)$ $(1,3,446)$ $(1,9,107)$			$(2,44,234)$ $(2,9,289)$ $(3,8,53)$
37^2	$(1,1,206)$	$(1,3,64)$ $(1,5,618)$ $(1,9,425)$	$(6,7,143)$ $(6,8,248)$ $(6,9,609)$ $(6,12,298)$		$(2,5,231)$ $(2,13,97)$ $(3,4,495)$

Remark. Many of the results in this article reappear in [3] with more details.

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