

TWO NEW RELATIONS BETWEEN Q-PRODUCT IDENTITIES, THETA FUNCTION IDENTITIES AND COMBINATORIAL PARTITION IDENTITIES

M. P. CHAUDHARY*, AHMED BUSERI ASHINE, AND FEYISSA KABA WAKENE

Abstract. The objective of this research article is to establish two relationships between q-product identities, theta function identities and combinatorial partition identities, using elementary results.

1. Introduction

Throughout this paper, \mathbb{N} , \mathbb{Z} , and \mathbb{C} denote the sets of positive integers, integers, and complex numbers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The following q-notations are recalled (see, e.g., [4, Chapter 6]): The q-shifted factorial $(a; q)_n$ is defined by

$$(1.1) \quad (a; q)_n := \begin{cases} 1 & (n=0) \\ \prod_{k=0}^{n-1} (1 - aq^k) & (n \in \mathbb{N}), \end{cases}$$

where $a, q \in \mathbb{C}$ and it is assumed that $a \neq q^{-m}$ ($m \in \mathbb{N}_0$). We also write

$$(a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k)$$

$$(1.2) \quad = \prod_{k=1}^{\infty} (1 - aq^{k-1}) \quad (a, q \in \mathbb{C}; |q| < 1).$$

It is noted that, when $a \neq 0$ and $|q| \geq 1$, the infinite product in (1.2) diverges. So, whenever $(a; q)_\infty$ is involved in a given formula, the constraint $|q| < 1$ will be tacitly assumed. The following notations are also

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*Corresponding author

frequently used:

$$(1.3) \quad (a_1, a_2, \dots, a_m; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$$

and

$$(1.4) \quad (a_1, a_2, \dots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

Ramanujan defined the general theta function $f(a, b)$ as follows (see, for details, [3, p. 31, Eq.(18.1)] and [5]; see also [1]):

$$(1.5) \quad f(a, b) = 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} (a^n + b^n)$$

$$= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = f(b, a) \quad (|ab| < 1).$$

We find from (1.5) that

$$(1.6) \quad f(a, b) = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f(a(ab)^n, b(ab)^{-n}) = f(b, a) \quad (n \in \mathbb{Z}).$$

Ramanujan also rediscovered the Jacobi's famous triple-product identity (see [3, p. 35, Entry 19]):

$$(1.7) \quad f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty,$$

which was first proved by Gauss.

Several q-series identities emerging from Jacobi's triple-product identity (1.7) are worthy of note here (see [3, pp. 36-37, Entry 22]):

$$(1.8) \quad \phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2}$$

$$= \{(-q; q^2)_\infty\}^2 (q^2; q^2)_\infty = \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (-q^2; q^2)_\infty};$$

$$(1.9) \quad \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty};$$

$$(1.10) \quad f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = (q; q)_\infty$$

Equation (1.10) is known as *Euler's Pentagonal Number Theorem*. The following q-series identity:

$$(1.11) \quad (-q; q)_\infty = \frac{1}{(q; q^2)_\infty} = \frac{1}{\chi(-q)}$$

provides the analytic equivalence of Euler's famous theorem: The number of partitions of a positive integer n into distinct parts is equal to the

number of partitions of n into odd parts.

We also recall the Rogers-Ramanujan continued fraction of $R(q)$:

$$(1.12) \quad R(q) := q^{\frac{1}{5}} \frac{H(q)}{G(q)} = q^{\frac{1}{5}} \frac{f(-q, -q^4)}{f(-q^2, -q^3)} = q^{\frac{1}{5}} \frac{(q; q^5)_\infty (q^4; q^5)_\infty}{(q^2; q^5)_\infty (q^3; q^5)_\infty}$$

$$= \frac{q^{\frac{1}{5}}}{1+} \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \dots \quad (|q| < 1).$$

Here $G(q)$ and $H(q)$ are widely investigated Roger-Ramanujan identities defined by

$$(1.13) \quad G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{f(-q^5)}{f(-q, -q^4)}$$

$$= \frac{1}{(q; q^5)_\infty (q4; q^5)_\infty} = \frac{(q^2; q^5)_\infty (q^3; q^5)_\infty (q^5; q^5)_\infty}{(q; q)_\infty};$$

$$(1.14) \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n} = \frac{f(-q^5)}{f(-q^2, -q^3)}$$

$$= \frac{1}{(q^2; q^5)_\infty (q^3; q^5)_\infty} = \frac{(q; q^5)_\infty (q^4; q^5)_\infty (q^5; q^5)_\infty}{(q; q)_\infty};$$

and the functions $f(a, b)$ and $f(-q)$ are given in (1.5) and (1.10), respectively. For a detailed historical account of (and for various investigated developments stemming from) the Rogers-Ramanujan continued fraction (1.12) and identities (1.13) and (1.14), the interested reader may refer to the monumental work [3, p. 77 et seq.] (see also [1, 4]). The following continued fraction was recalled in [6, p. 5, Eq. (2.8)] from an earlier work cited therein: For $|q| < 1$,

$$(1.15) \quad (q^2; q^2)_\infty (-q; q)_\infty = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}$$

$$= \frac{1}{1-} \frac{q}{1+} \frac{q(1-q)}{1-} \frac{q^3}{1+} \frac{q^2(1-q^2)}{1-} \frac{q^5}{1+} \frac{q^3(1-q^3)}{1-} \dots;$$

$$(1.16) \quad \frac{(q; q^5)_\infty (q^4; q^5)_\infty}{(q^2; q^5)_\infty (q^3; q^5)_\infty} = \frac{1}{1+} \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \frac{q^4}{1+} \frac{q^5}{1+} \frac{q^6}{1+} \dots$$

$$(1.17) \quad C(q) := \frac{(q^2; q^5)_\infty (q^3; q^5)_\infty}{(q; q^5)_\infty (q^4; q^5)_\infty} = 1 + \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \frac{q^4}{1+} \frac{q^5}{1+} \frac{q^6}{1+} \dots$$

Andrews et al. [2] investigated new double summation hypergeometric q-series representations for several families of partitions and further explored the role of double series in combinatorial partition identities by introducing the following general family:

$$(1.18) \quad R(s, t, l, u, v, w) := \sum_{n=0}^{\infty} q^{s(\frac{n}{2}) + tn} r(l, u, v, w; n),$$

where

$$(1.19) \quad r(l, u, v, w : n) := \sum_{j=0}^{[n]} (-1)^j \frac{q^{uv\binom{j}{2}} + (w-ul)j}{(q;q)_{n-u_j} (q^{uv};q^{uv})_j}.$$

The following interesting special cases of (1.18) are recalled (see [2, p. 106, Theorem 3]; see also [1]):

$$(1.20) \quad R(2, 1, 1, 1, 2, 2) = (-q; q^2)_\infty;$$

$$(1.21) \quad R(2, 2, 1, 1, 2, 2) = (-q^2; q^2)_\infty;$$

$$(1.22) \quad R(m, m, 1, 1, 1, 2) = \frac{(q^{2m}; q^{2m})_\infty}{(q^m; q^{2m})_\infty}.$$

Here, in this paper, we aim to present certain interrelations between q-product identities, theta function identities and combinatorial partition identities associated with the identities in (1.8)-(1.10) and (1.20)-(1.22).

2. The Main Results

Here we state and prove certain interesting interrelations among q-product identities and combinatorial partition identities.

Theorem. Each of the following relations holds true:

$$\begin{aligned} (3.1) \quad & \frac{3162}{q^{26}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \times \\ & \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{18} + \\ & + \frac{9691326}{q^{16}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \times \\ & \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} + \\ & + \frac{56044062}{q^{11}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \times \\ & \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 + \\ & + \frac{349029}{q^{21}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \times \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{14} + \\
& + \frac{256122}{q^{19}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{18} \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^2 + \\
& + \frac{784997406}{q^9} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^2 + \\
& + \frac{4539569022}{q^4} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^2 + \\
& + 87528332700 q^6 \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^2 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^2 + \\
& + \frac{28271349}{q^{14}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{14} \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^2 + \\
& + \frac{45675}{q^{22}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{12} + \\
& + \frac{2815884}{q^{17}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 + \\
& + \frac{7735419}{q^{12}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 \times
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 + \\
& + \frac{1303}{q^{27}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{16} + \\
& + \frac{299673675}{q^8} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{12} \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^4 + \\
& + \frac{18475014924}{q^3} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^4 \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 + \\
& + 50752084059 q^2 \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^4 + \\
& + \frac{8548983}{q^{13}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{16} \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^4 + \\
& + \frac{252}{q^{28}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{14} + \\
& + \frac{33462}{q^{23}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} + \\
& + \frac{424305}{q^{18}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 \times
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 + \\
& + \frac{99144}{q^{13}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 + \\
& + \frac{133923132}{q^7} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{14} \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^6 + \\
& + \frac{17783078742}{q^2} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^6 + \\
& + \frac{225493073505}{q^3} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^6 + \\
& + 52689186504q^8 \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^6 + \\
& + \frac{24}{q^{29}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{12} + \\
& + \frac{5640}{q^{24}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 + \\
& + \frac{31131}{q^{19}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 \times
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 + \\
& + \frac{1033121304}{q} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{12} \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^8 + \\
& + 242783506440 q^4 \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^8 + \\
& + 1340087471451 q^9 \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^8 + \\
& + q^{30} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} + \\
& + \frac{198}{q^{25}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 + \\
& + \frac{819}{q^{20}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 + \\
& + 3486784401 q^5 \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{10} + \\
& + 690383311398 q^{10} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 \times
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{10} + \\
& + 2855676424419 q^{15} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{10} + \\
& + \frac{36}{q^{26}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{12} \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 + \\
& + \frac{7}{q^{21}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{12} + \\
& + 10167463313316 q^{16} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 \times \\
& \quad \times \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{12} + \\
& + 1977006755367 q^{21} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{12} + \\
& + \frac{1}{q^{27}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{14} \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 + \\
& + 22876792454961 q^{22} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{14} \times \\
& \quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 + \\
& + \frac{79184709}{q^5} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 + \\
& + \frac{2322}{q^{25}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{20} + \\
& + \frac{367497}{q^{20}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{16}
\end{aligned}$$

$$\begin{aligned}
&= 215233605q^9 \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \times \\
&\quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^{10} + \\
&+ 1312127829q^4 \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \times \\
&\quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^6 + \\
&+ \frac{140772816}{q^6} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \times \\
&\quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 + \\
&+ \frac{1080596700}{q} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \times \\
&\quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^2 + \\
&+ 17433922005q^{16} \left\{ \frac{(q^6, q^{12}q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^2 \times \\
&\quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^{10} + \\
&+ 106282354149q^{11} \left\{ \frac{(q^6, q^{12}q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^2 \times \\
&\quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^6 + \\
&+ 11402598096q^6 \left\{ \frac{(q^6, q^{12}q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^2 \times \\
&\quad \times \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 + \\
&+ 14348907q^8 \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 \times \\
&\quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^{12} +
\end{aligned}$$

$$\begin{aligned}
& + 140831865q^3 \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^8 + \\
& + \frac{185827203}{q^2} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^4 + \\
& + \frac{46438758}{q^7} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 + \\
& + 94143178827q^{\frac{19}{2}} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)} \right\}^4 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^2 + \\
& + 923997866265q^{17} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)} \right\}^4 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^8 + \\
& + 1219212278883q^{12} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)} \right\}^4 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^4 + \\
& + 304684691238q^7 \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)} \right\}^4 + \\
& + 7971615q^2 \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^{10} + \\
& + \frac{17301357}{q^3} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^6 +
\end{aligned}$$

$$\begin{aligned}
& + \frac{8597097}{q^8} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^2 + \\
& + 4236443047215q^{23} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^6 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^{10} + \\
& + 9194650465437q^{18} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^6 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^6 + \\
& + 4568849826777q^{13} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^6 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^2 + \\
& + \frac{793881}{q^4} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^8 + \\
& + \frac{739206}{q^9} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^4 + \\
& + \frac{121338}{q^{14}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 + \\
& + 34173973914201q^{24} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^8 \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^8 + \\
& + 31820394443526q^{19} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^8 \times
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ \frac{(q, q^2, q^2; q^2)_{\infty} (q^{18}, q^{36}, q^{36}; q^{36})_{\infty}}{(q^3, q^6, q^6; q^6)_{\infty} (q^6, q^{12}, q^{12}; q^{12})_{\infty}} \right\}^4 + \\
& + 5223203032698q^{14} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_{\infty} (q^{18}, q^{36}q^{36}; q^{36})_{\infty}}{(q, q^2, q^2; q^2)_{\infty} R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^8 + \\
& + \frac{6561}{q^5} \left\{ \frac{(q, q^2, q^2; q^2)_{\infty} R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_{\infty} (q^{18}, q^{36}, q^{36}; q^{36})_{\infty}} \right\}^{10} \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_{\infty} (q^{18}, q^{36}, q^{36}; q^{36})_{\infty}}{(q^3, q^6, q^6; q^6)_{\infty} (q^6, q^{12}, q^{12}; q^{12})_{\infty}} \right\}^{10} + \\
& + \frac{11259}{q^{10}} \left\{ \frac{(q, q^2, q^2; q^2)_{\infty} R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_{\infty} (q^{18}, q^{36}, q^{36}; q^{36})_{\infty}} \right\}^{10} \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_{\infty} (q^{18}, q^{36}, q^{36}; q^{36})_{\infty}}{(q^3, q^6, q^6; q^6)_{\infty} (q^6, q^{12}, q^{12}; q^{12})_{\infty}} \right\}^6 + \\
& + \frac{14688}{q^{15}} \left\{ \frac{(q, q^2, q^2; q^2)_{\infty} R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_{\infty} (q^{18}, q^{36}, q^{36}; q^{36})_{\infty}} \right\}^{10} \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_{\infty} (q^{18}, q^{36}, q^{36}; q^{36})_{\infty}}{(q^3, q^6, q^6; q^6)_{\infty} (q^6, q^{12}, q^{12}; q^{12})_{\infty}} \right\}^2 + \\
& + 22876792454961q^{30} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_{\infty} (q^{18}, q^{36}q^{36}; q^{36})_{\infty}}{(q, q^2, q^2; q^2)_{\infty} R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{10} \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_{\infty} (q^{18}, q^{36}, q^{36}; q^{36})_{\infty}}{(q^3, q^6, q^6; q^6)_{\infty} (q^6, q^{12}, q^{12}; q^{12})_{\infty}} \right\}^{10} + \\
& + 39257705570859q^{25} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_{\infty} (q^{18}, q^{36}q^{36}; q^{36})_{\infty}}{(q, q^2, q^2; q^2)_{\infty} R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{10} \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_{\infty} (q^{18}, q^{36}, q^{36}; q^{36})_{\infty}}{(q^3, q^6, q^6; q^6)_{\infty} (q^6, q^{12}, q^{12}; q^{12})_{\infty}} \right\}^6 + \\
& + 51213889281888q^{20} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_{\infty} (q^{18}, q^{36}q^{36}; q^{36})_{\infty}}{(q, q^2, q^2; q^2)_{\infty} R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{10} \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_{\infty} (q^{18}, q^{36}, q^{36}; q^{36})_{\infty}}{(q^3, q^6, q^6; q^6)_{\infty} (q^6, q^{12}, q^{12}; q^{12})_{\infty}} \right\}^2 + \\
& + \frac{1080}{q^{16}} \left\{ \frac{(q, q^2, q^2; q^2)_{\infty} R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_{\infty} (q^{18}, q^{36}, q^{36}; q^{36})_{\infty}} \right\}^{12} \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_{\infty} (q^{18}, q^{36}, q^{36}; q^{36})_{\infty}}{(q^3, q^6, q^6; q^6)_{\infty} (q^6, q^{12}, q^{12}; q^{12})_{\infty}} \right\}^4 +
\end{aligned}$$

$$\begin{aligned}
& + \frac{567}{q^{11}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{12} \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^8 + \\
& + 305023899399480 q^{36} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36} q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{12} \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^4 + \\
& + 160137547184727 q^{31} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36} q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{12} \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^8 + \\
& + \frac{48}{q^{17}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{14} \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^6 + \\
& + 1098086037838128 q^{32} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36} q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{14} \times \\
& \quad \times \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^6 + \\
& + 10130328342 q^5 \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^4 + \\
& + \frac{1}{q^{30}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^{24} + 5311752795
\end{aligned}$$

and

$$\begin{aligned}
(3.2) \quad & \left\{ \frac{R(6, 6, 1, 1, 1, 2)}{R(18, 18, 1, 1, 1, 2)} \right\}^3 \left\{ \frac{(q^6, q^{12}; q^{12})_\infty (-q^{18}, -q^{36}, q^{36})}{(-q^6, q^{12}; q^{12})_\infty (q^{18}, q^{36}; q^{36})_\infty} \right\}^6 \\
& = \frac{27(q, q^2; q^2)_\infty (-q^3, -q^6; q^6)_\infty}{R(2, 1, 1, 1, 2, 2) R(2, 2, 1, 1, 2, 2) (q^3, q^6; q^6)_\infty} \left\{ \frac{R(1, 1, 1, 1, 1, 2)}{R(3, 3, 1, 1, 1, 2)} \right\}^2 + \\
& + \frac{9R(6, 6, 1, 1, 1, 2)}{R(3, 3, 1, 1, 1, 2)} \left\{ \frac{(q, q^2; q^2)_\infty (-q^3, q^6; q^6_\infty)}{R(2, 1, 1, 1, 2, 2) R(2, 2, 1, 1, 2, 2) (q^3, q^6; q^6)_\infty} \right\}^2 \times \\
& \quad \times \left\{ \frac{R(1, 1, 1, 1, 1, 2)}{R(3, 3, 1, 1, 1, 2)} \right\}^4 \left\{ \frac{(q^6; q^6)_\infty (-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, q^{12}; q^{12})_\infty} \right\}^2 +
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \frac{(6, 6, 1, 1, 1, 2)(q; q)_\infty (-q^3, -q^6; q^6)_\infty}{R(2, 1, 1, 1, 2, 2)R(2, 2, 1, 1, 2, 2)} \right\}^3 \left\{ \frac{\{R(1, 1, 1, 1, 1, 2)\}^2}{R(3, 3, 3, 1, 1, 1, 2)} \right\}^3 \times \\
& \times \frac{1}{\{(q^6; q^6)_\infty R(3, 3, 1, 1, 1, 2)\}^3} \left\{ \frac{(-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty (-q^6, q^{12}; q^{12})_\infty} \right\}^4 \frac{(q^6; q^{12})_\infty}{(q^{12}; q^{12})_\infty}
\end{aligned}$$

Proofs: Applying the identity (1.10), and further using (1.20)-(1.22), we obtain the following identities :

$$\begin{aligned}
3 \left(P^2 + \frac{9^2}{P^2} \right) &= \frac{3}{q^{\frac{7}{2}}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \\
(3.3) \quad &+ 234q^{\frac{7}{2}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)} \right\}^2
\end{aligned}$$

$$(3.4) \quad 1054Q^{18} = \frac{1054}{q^{\frac{45}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{18}$$

$$\begin{aligned}
3^5 \left(13294Q^{10} - \frac{295245}{Q^{10}} \right) &= \frac{3^5 \times 13294}{q^{\frac{25}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} \\
(3.5) \quad &- 3^5 \times 295245q^{\frac{25}{2}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^{10}
\end{aligned}$$

$$\begin{aligned}
3^7 \left(8542Q^6 - \frac{199989}{Q^6} \right) &= \frac{3^7 \cdot 8542}{q^{\frac{15}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 \\
(3.6) \quad &- 3^7 \times 199989q^{\frac{15}{2}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^6
\end{aligned}$$

$$\begin{aligned}
-78732 \left(596Q^2 + \frac{4575}{Q^2} \right) &= -78732 \times \frac{596}{q^{\frac{5}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \\
(3.7) \quad &- 78732 \times 4575 \cdot q^{\frac{5}{2}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^2
\end{aligned}$$

$$(3.8) \quad 116343Q^{14} = \frac{116343}{q^{\frac{35}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{14}$$

$$(3.9) \quad \left(P^4 + \frac{9^4}{P^4} \right) = \frac{1}{q^7} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 \\ + 9^4 q^7 \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^4$$

$$(3.10) \quad 3^2 \left(5075 Q^{12} - \frac{3^{13}}{Q^{12}} \right) = \frac{3^2 \times 5075}{q^{15}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{12} - \\ - 3^{15} \times 5075 q^{15} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^{12}$$

$$(3.11) \quad 3^5 \left(11588 Q^8 - \frac{579555}{Q^8} \right) = \frac{3^5 \times 11588}{q^{10}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 \\ - 3^5 \times 579555 q^{10} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^8$$

$$(3.12) \quad 3^{10} \left(131 Q^4 - \frac{3147}{Q^4} \right) = 3^{10} \times \frac{131}{q^5} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 - \\ - 3^{10} \times 3147 q^5 \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^3, q^6, q^6; q^6)_\infty} \right\}^4$$

$$(3.13) \quad 1303 Q^{16} = \frac{1303}{q^{20}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{16}$$

$$(3.14) \quad 9 \left(P^6 + \frac{9^6}{P^6} \right) = \frac{9}{q^{\frac{21}{2}}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 + \\ + 9^7 q^{\frac{21}{2}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^6$$

$$(3.15) \quad 28 Q^{14} = \frac{28}{q^{\frac{35}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{14}$$

$$(3.16) \quad 3718 Q^{10} - \frac{885735}{Q^{10}} = \frac{3718}{q^{\frac{25}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} \\ - 885735 q^{\frac{25}{2}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^{10}$$

$$(3.17) \quad 3\left(15715Q^6 - \frac{640791}{Q^6}\right) = \frac{47145}{q^{\frac{15}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 \\ - 1922373q^{\frac{15}{2}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^6$$

$$(3.18) \quad 3^4 \left(136Q^2 - \frac{11793}{Q^2}\right) = \frac{3^4 \times 136}{q^{\frac{5}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 \\ - 3^4 \times 11793q^{\frac{5}{2}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^2$$

$$(3.19) \quad 3\left(P^8 + \frac{9^8}{P^8}\right) = \frac{3}{q^{14}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 \\ + 3 \times 8^9 q^{14} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)} \right\}^8$$

$$(3.20) \quad 8Q^{12} = \frac{8}{q^{15}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{12}$$

$$(3.21) \quad 1880Q^8 - \frac{24627}{Q^8} = \frac{1880}{q^{10}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 \\ - 264627q^{10} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^8$$

$$(3.22) \quad 9\left(1153Q^4 - \frac{27378}{Q^4}\right) = \frac{10377}{q^5} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 \\ - 246402q^5 \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^4$$

$$(3.23) \quad \left(P^{10} + \frac{9^{10}}{P^{10}}\right) = \frac{1}{q^{\frac{25}{2}}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} \\ + 9^{10} q^{\frac{25}{2}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)} \right\}^{10}$$

$$(3.24) \quad Q^{10} - \frac{3^8}{Q^{10}} = \frac{1}{q^{\frac{25}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{10} - 3^8 q^{\frac{25}{2}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^{10}$$

$$(3.25) \quad 3^2 \left(22Q^6 - \frac{1251}{Q^6} \right) = \frac{198}{q^{\frac{15}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^6 - 11259 q^{\frac{15}{2}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^6$$

$$(3.26) \quad 3^2 \left(91Q^2 - \frac{1632}{Q^2} \right) = \frac{819}{q^{\frac{5}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 - 14688 q^{\frac{5}{2}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^2$$

$$(3.27) \quad P^{12} + \frac{9^{12}}{P^{12}} = \frac{1}{q^{21}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{12} + 9^{12} q^{21} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{12}$$

$$(3.28) \quad 6^2 \left(Q^4 - \frac{30}{Q^4} \right) = \frac{36}{q^5} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 - 1080 q^5 \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^4$$

$$(3.29) \quad - \frac{567}{Q^8} = -567 q^{10} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^8$$

$$(3.30) \quad P^{14} + \frac{9^{14}}{P^{14}} = \frac{1}{q^{\frac{49}{2}}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{14} + 9^{14} q^{\frac{49}{2}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2) R(6, 6, 1, 1, 1, 2)} \right\}^{14}$$

$$(3.31) \quad Q^2 - \frac{48}{Q^6} = \frac{1}{q^{\frac{5}{2}}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^2 - \\ - 48q^{\frac{15}{2}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^6$$

$$(3.32) \quad - \frac{1}{Q^8} = -q^{10} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^8 \\ \left(P^{16} + \frac{9^{16}}{P^{16}} \right) = \frac{1}{q^{28}} \left\{ \frac{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)}{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{16} + \\ + 9^{16} q^{28} \left\{ \frac{(q^6, q^{12}, q^{12}; q^{12})_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q, q^2, q^2; q^2)_\infty R(3, 3, 1, 1, 1, 2)R(6, 6, 1, 1, 1, 2)} \right\}^{16}$$

$$(3.34) \quad 3^8 \left(2284Q^{12} + \frac{3^{12}}{Q^{12}} \right) = \frac{3^8 \times 2284}{q^{15}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{12} + \\ + 3^{20} q^{15} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^{12}$$

$$(3.35) \quad 3^{10} \left(3896Q^8 - \frac{61965}{Q^8} \right) = \frac{3^{10} \times 3896}{q^{10}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^8 - \\ - 3^{10} \times 61965 q^{10} \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^8$$

$$(3.36) \quad 3^{12} \left(149Q^4 - \frac{19062}{Q^4} \right) = \frac{3^{12} \times 149}{q^5} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^4 - \\ - 3^{12} \times 19062 q^5 \left\{ \frac{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty}{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty} \right\}^4$$

$$(3.37) \quad -Q^{24} + 2322Q^{20} + 367497Q^{16} = \frac{1}{q^{30}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{24} + \\ + \frac{2322}{q^{25}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{20} + \\ + \frac{367497}{q^{20}} \left\{ \frac{(q^3, q^6, q^6; q^6)_\infty (q^6, q^{12}, q^{12}; q^{12})_\infty}{(q, q^2, q^2; q^2)_\infty (q^{18}, q^{36}, q^{36}; q^{36})_\infty} \right\}^{16}$$

(3.3)- (3.37) into (2.1), by little algebra and moving negative terms to the right hand side we obtain (3.1).

Further, we attempt to prove our second identity (3.2), to establish it

applying the identity(1.8)-(1.9), and further using (1.20) - (1.22), we obtain the following identities:

(3.38)

$$27PR^2 = \frac{27(q, q^2; q^2)_\infty(-q^3, -q^6; q^6)_\infty}{R(2, 1, 1, 1, 2, 2)R(2, 2, 1, 1, 2, 2)(q^3, q^6; q^6)_\infty} \left\{ \frac{R(1, 1, 1, 1, 1, 2)}{R(3, 3, 1, 1, 1, 2)} \right\}^2$$

$$\begin{aligned} 9P^2R^4Q^2T &= \frac{9R(6, 6, 1, 1, 1, 2)}{R(3, 3, 1, 1, 1, 2)} \left\{ \frac{(q, q^2; q^2)_\infty(-q^3, q^6; q^6)_\infty}{R(2, 1, 1, 1, 2, 2)R(2, 2, 1, 1, 2, 2)(q^3, q^6; q^6)_\infty} \right\}^2 \times \\ (3.39) \quad &\times \left\{ \frac{R(1, 1, 1, 1, 1, 2)}{R(3, 3, 1, 1, 1, 2)} \right\}^4 \left\{ \frac{(q^6; q^6)_\infty(-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty(-q^6, q^{12}; q^{12})_\infty} \right\}^2 \end{aligned}$$

$$\begin{aligned} P^3R^6T^2Q^4 &= \left\{ \frac{(6, 6, 1, 1, 1, 2)(q; q)_\infty(-q^3, -q^6; q^6)_\infty}{R(2, 1, 1, 1, 2, 2)R(2, 2, 1, 1, 2, 2)} \right\}^3 \left\{ \frac{\{R(1, 1, 1, 1, 1, 2)\}^2}{R(3, 3, 3, 1, 1, 1, 2)} \right\}^3 \times \\ (3.40) \quad &\times \frac{1}{\{(q^6; q^6)_\infty R(3, 3, 1, 1, 1, 2)\}^3} \times \left\{ \frac{(-q^{18}, -q^{36}; q^{36})_\infty}{(q^{18}; q^{18})_\infty(-q^6, q^{12}; q^{12})_\infty} \right\}^4 \times \frac{(q^6; q^{12})_\infty}{(q^{12}; q^{12})_\infty} \end{aligned}$$

(3.41)

$$Q^6T^3 = \left\{ \frac{R(6, 6, 1, 1, 1, 2)}{R(18, 18, 1, 1, 1, 2)} \right\}^3 \left\{ \frac{(q^6, q^{12}; q^{12})_\infty(-q^{18}, -q^{36}; q^{36})_\infty}{(-q^6, q^{12}; q^{12})_\infty(q^{18}, q^{36}; q^{36})_\infty} \right\}^6$$

using (3.38)-(3.41) into (2.2), arranging the terms, we obtain (3.2). This completes the proof of our theorem.

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M.P. Chaudhary

International Scientific Research and Welfare Organization
New Delhi, India.

E-mail:dr.m.p.chaudhary@gmail.com

Ahemed Buseri Ashine

Department of Mathematics, College of Natural and Computational Sciences
Madda Walabu University, Bale Robe, Ethiopia.

Feyissa Kaba Wakene

Department of Mathematics, College of Natural and Computational Sciences
Madda Walabu University, Bale Robe, Ethiopia.