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ON CHARACTERIZATIONS OF THE CONTINUOUS DISTRIBUTIONS BY INDEPENDENCE PROPERTY OF RECORD VALUES

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ABSTRACT. A sequence $\{X_n, n \geq 1\}$ of independent and identically distributed random variables with absolutely continuous (with respect to Lebesque measure) cumulative distribution function $F(x)$ is considered. We obtain two characterizations of a family of continuous probability distribution by independence property of record values.

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1. Introduction

Suppose that $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Let $Y_n = \max(\min)\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper(lower) record value of $\{X_n, n \geq 1\}$ if $Y_j > (<) Y_{j-1}$ for $j > 1$. By definition, X_1 is an upper as well as a lower record value. One can transform the upper records to lower records by replacing the original sequence of $\{X_j\}$ by $\{-X_j, j \geq 1\}$ or (if $P(X_j > 0) = 1$ for all j) by $\{1/X_j, j \geq 1\}$. We defined the upper record times $\{U(n), n \geq 2\}$ where $U(1) = 1$, and $U(n) = \min\{j \mid j > U(n-1), X_j > X_{U(n-1)}\}$. Similarly, the lower record times $\{L(n), n \geq 2\}$ where $L(1) = 1$, and $L(n) = \min\{j \mid j > L(n-1), X_j > X_{L(n-1)}\}$.

In [3] Lee and Chang showed that X has a Pareto random variable with parameter θ if and only if $\frac{X_{U(n+1)}}{X_{U(n)}}$ and $X_{U(n)}$ for $n \geq 1$ are independent. In [4] Yanev and Ahsanullah studied characterizations based on the regression of linear combinations of record values. Recently, in [2] Juhas and Skrivankova presented

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characterization of general classes of distributions with the independent property that the random variables $g(L_n)$ and $g(L_{n+1}) - g(L_n)$ are independent for $n \geq 1$ if and only if X has general classes of distributions.

The current investigation was induced by characterizations based on independent property in [2]. Namely, one can ask whether the independence $\frac{g(X_{U(n+1)})}{g(X_{U(n)})}$ and $g(X_{U(n)})$ or $\frac{g(X_{L(n+1)})}{g(X_{L(n)})}$ and $g(X_{L(n)})$ guarantee the characterization of general classes of distributions.

In this paper we investigate characterizations of continuous distributions by independence property of record values.

2. Main results

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed nonnegative random variables with cdf $F(x)$ which is absolutely continuous with pdf $f(x)$ and $F(x) < 1$. Let $g(x)$ is an increasing and differentiable function with $g(x) \to 1$ as $x \to a^+$ and $g(x) \to \infty$ as $x \to b^-$ for all $x \in (a, b)$. Then $F(x) = 1 - \big(g(x)\big)^{-\alpha}$, for $\alpha > 0$, if and only if $\frac{g(X_{U(n+1)})}{g(X_{U(n)})}$ and $g(X_{U(n)})$ are independent for $n \geq 1$.*

Proof. The joint pdf $f_{n+1,n}(x, y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ can be written as

$$f_{n+1,n}(x, y) = \frac{\{R(x)\}^{n-1}}{\Gamma(n)} r(x) f(y)$$

where $R(x) = -\ln \bar{F}(x)$, $\bar{F}(x) = 1 - F(x)$ and $r(x) = \frac{d}{dx} R(x) = \frac{f(x)}{\bar{F}(x)}$, for $n \geq 1$.

Consider the functions $U = g(X_{U(n)})$ and $V = \frac{g(X_{U(n+1)})}{g(X_{U(n)})}$. It follows that $x_{U(n)} = g^{-1}(u)$, $x_{U(n+1)} = g^{-1}(uv)$ and $J = \frac{\partial}{\partial u}\big(g^{-1}(u)\big) \frac{\partial}{\partial v}\big(g^{-1}(uv)\big)$. Since $g(x)$ is an increasing and differentiable function, both $\frac{\partial}{\partial u}\big(g^{-1}(u)\big)$ and $\frac{\partial}{\partial v}\big(g^{-1}(uv)\big)$ are positive.

Thus we can write the joint pdf $f_{U,V}(u, v)$ of u and v as

$$f_{U,V}(u, v) = \frac{R\big(g^{-1}(u)\big)^{n-1}}{\Gamma(n)} r\big(g^{-1}(u)\big) f\big(g^{-1}(uv)\big) \frac{\partial}{\partial u}\big(g^{-1}(u)\big) \frac{\partial}{\partial v}\big(g^{-1}(uv)\big)$$

for $u > 1$ and $v > 1$.

If $F(x) = 1 - \big(g(x)\big)^{-\alpha}$ for all $g(x) > 1$ and $\alpha > 0$, then we get

$$\begin{aligned} f_{U,V}(u, v) =& \frac{(\alpha)^{n-1}\big(ln(g(g^{-1}(u)))\big)^{n-1} \alpha(g(g^{-1}(u)))^{-\alpha-1} g'\big(g^{-1}(u)\big)}{\Gamma(n)} \\ &\times \frac{\alpha(g(g^{-1}(uv)))^{-\alpha-1} g'\big(g^{-1}(uv)\big) \frac{\partial}{\partial u}\big(g^{-1}(u)\big) \frac{\partial}{\partial v}\big(g^{-1}(uv)\big)}{\big(g(g^{-1}(u))\big)^{-\alpha}} \end{aligned} \tag{1}$$

$$= \frac{(\alpha)^{n-1}\big(ln(g(g^{-1}(u)))\big)^{n-1}\alpha(g(g^{-1}(u)))^{-\alpha-1}\alpha(g(g^{-1}(uv)))^{-\alpha-1}}{\Gamma(n)}$$
$$\times \frac{\frac{\partial}{\partial u}\big(g(g^{-1}(u))\big)\frac{\partial}{\partial v}\big(g(g^{-1}(uv))\big)}{\big(g(g^{-1}(u))\big)^{-\alpha}} = \frac{(\alpha)^{n-1}}{\Gamma(n)}\alpha^2\big(ln(u)\big)^{n-1}(uv)^{-\alpha-1},$$

for all $u > 1$, $v > 1$ and $\alpha > 0$.

We can get the pdf $f_V(v)$ of v by integration of $f_{U,V}(u,v)$ as

$$f_V(v) = v^{\alpha-1}\int_1^\infty f_{U,V}(u,v)du = \alpha v^{-\alpha-1}, v > 1, \alpha > 0. \tag{2}$$

Also, the pdf $f_U(u)$ of u is given by

$$\begin{aligned} f_U(u) =& \frac{\alpha^{n-1}\big(\ln(g(g^{-1}(u)))\big)^{n-1}}{\Gamma(n)}\alpha\big(g(g^{-1}(u))\big)^{-\alpha-1} \\ =& \frac{\alpha^n}{\Gamma(n)}\big(\ln(u)\big)^{n-1}u^{-\alpha-1},\ u > 1,\ \alpha > 0. \end{aligned} \tag{3}$$

From (1), (2) and (3) we obtain $f_U(u)f_V(v) = f_{U,V}(u,v)$. Hence $U = g(X_{U(n)})$ and $V = \frac{g(X_{U(n+1)})}{g(X_{U(n)})}$ are independent.

Now we will prove the sufficient condition. Let us use the transformation $U = g(X_{U(n)})$ and $V = \frac{g(X_{U(n+1)})}{g(X_{U(n)})}$. Jacobian of the transformation is $J = \frac{\partial}{\partial u}\big(g^{-1}(u)\big)\times \frac{\partial}{\partial v}\big(g^{-1}(uv)\big)$. Since $g(x)$ is an increasing and differentiable function, both $\frac{\partial}{\partial u}\big(g^{-1}(u)\ \big)$ and $\frac{\partial}{\partial v}\big(g^{-1}(uv)\big)$ are positive. Thus we can write the joint pdf $f_{U,V}(u,v)$ of U and V as

$$\begin{aligned} &f_{U,V}(u,v) \\ &= \frac{R\big(g^{-1}(u)\big)^{n-1}}{\Gamma(n)}r\big(g^{-1}(u)\big)f\big(g^{-1}(uv)\big)\frac{\partial}{\partial u}\big(g^{-1}(u)\big)\frac{\partial}{\partial v}\big(g^{-1}(uv)\big) \end{aligned} \tag{4}$$

for all $u > 1$ and $v > 1$.

The pdf $f_U(u)$ of U is given by

$$f_U(u) = \frac{R\big(g^{-1}(u)\big)^{n-1}}{\Gamma(n)}f\big(g^{-1}(u)\big)\frac{\partial}{\partial u}\big(g^{-1}(u)\big) \tag{5}$$

for all $u > 1$ and $n \geq 1$.

Since u and v are independent, we get the pdf $f_V(u)$ of v from (4) and (5) as

$$f_V(v) = \frac{f\big(g^{-1}(uv)\big)}{\bar{F}\big(g^{-1}(u)\big)}\frac{\partial}{\partial v}\big(g^{-1}(uv)\big),\ u > 1,\quad v > 1. \tag{6}$$

Integrating of (6) with respect to v from v_1 to ∞ and simplifying, we get

$$\int_{v_1}^\infty f_V(v)dv = \bar{F}_V(v_1) = \frac{F\big(g^{-1}(\infty)\big) - F\big(g^{-1}(uv_1)\big)}{\bar{F}\big(g^{-1}(u)\big)} \tag{7}$$

for any fixed v_1, $v_1 > 1$.

Now taking $u \to 1$, $g^{-1}(u) \to a^+$ and $F\big(g^{-1}(1)\big) \to 0$. Since $F\big(g^{-1}(\infty)\big) = 1$, it holds $\bar{F}_V(v_1) = \bar{F}\big(g^{-1}(v_1)\big)$ for any fixed v_1, $v_1 > 1$.

Let $\bar{F}\big(g^{-1}(y)\big) = \bar{F}(y)$, for $y > 1$. Then we obtain the following equation from (7)

$$\bar{F}(u)\bar{F}(v_1) = \bar{F}(uv_1). \tag{8}$$

for all u, $u > 1$ and any fixed v_1, $v_1 > 1$.

By the theory of functional equation see [1], the only continuous solution of (8) with the boundary conditions $\bar{F}(a) = 1$ and $\bar{F}(b) = 0$ is

$$\bar{F}(x) = (g(x))^{-\alpha}$$

for all $g(x) > 1$ and $\alpha > 0$.

This completes the proof. □

Remark 2.1. If we set $g(x) = \big(\frac{1}{1-\exp[-e^{-\lambda x}]}\big)^{\frac{1}{\alpha}}$, we can obtain characterization by independence property concerning the Generalized Gumbel distribution. For $\lambda = 1$, we have the Gumbel distribution in [2].

Remark 2.2. If we set $g(x) = \big(\frac{1}{1-\exp[-(-x)^{-\beta}]}\big)^{\frac{1}{\alpha}}$, we can obtain characterization by independence property concerning the Weibull for extream value distribution in [2].

Remark 2.3. A list of continuous distributions with cdf and the corresponding forms of $g(x)$ are given in Table 1.

TABLE 1. Examples based on the distribution function $\bar{F}(x) = (g(x))^{-\alpha}$

Distribution	$g(x)$	$\bar{F}(x)$
Pareto	x	$x^{-\alpha},\ 1 < x < \infty$
Weibull	e^{x^p}	$e^{-\alpha x^p},\ 0 < x < \infty$
Beta 2nd kind	$(1+x)$	$(1+x)^{-\alpha},\ 0 < x < \infty$
Lomax	$(1+\frac{x}{\lambda})^{\frac{1}{\alpha}}$	$(1+\frac{x}{\lambda})^{-1},\ 0 < x < \infty$
Singh-Maddala	$(1+\theta x^p)$	$(1+\theta x^p)^{-\alpha},\ 0 < x < \infty$
Gompertz	$\big(\exp[\frac{\lambda}{\mu}(e^{\mu x}-1)]\big)^{\frac{1}{\alpha}}$	$\exp[-\frac{\lambda}{\mu}(e^{\mu x}-1)], 0 < x < \infty$
Rayleigh	$\big(\exp[2^{-1}\theta^{-2}x^2]\big)^{\frac{1}{\alpha}}$	$\exp[-2^{-1}\theta^{-2}x^2], 0 < x < \infty$
Inverse Weibull	$\big(\frac{1}{1-e^{-\theta x^{-p}}}\big)^{\frac{1}{\alpha}}$	$1-e^{-\theta x^{-p}}, 0 < x < \infty$
MW	$\exp[x^\lambda e^{\beta x}]$	$\exp[-\alpha x^\lambda e^{\beta x}],\ 0 < x < \infty$
EP	$\big(e^{-1}\exp(e^{x^\beta})\big)^{\frac{1}{\alpha}}$	$\exp[1-e^{x^\beta}],\ 0 < x < \infty$
Extream value I	$\big(\exp[e^x]\big)^{\frac{1}{\alpha}}$	$\exp[-e^x],\ -\infty < x < \infty$
Lognormal	$\big(\frac{1}{1-\Phi(\frac{\ln x-\mu}{\sigma})}\big)^{\frac{1}{\alpha}}$	$1-\Phi(\frac{\ln x-\mu}{\sigma}),\ 0 < x < \infty$

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed nonnegative random variables with cdf $F(x)$ which is absolutely continuous with pdf $f(x)$ and $F(x) < 1$. Let $g(x)$ is an increasing and differentiable function with $g(x) \to 0$ as $x \to a^+$ and $g(x) \to 1$ as $x \to b^-$ for all $x \in (a,b)$. Then $F(x) = \big(g(x)\big)^\alpha$, for $\alpha > 0$, if and only if $\frac{g(X_{L(n+1)})}{g(X_{L(n)})}$ and $g(X_{L(n)})$ are independent for $n \geq 1$.*

Proof. If $F(x) = \big(g(x)\big)^\alpha$, then it is easy to see that $\frac{g(X_{L(n+1)})}{g(X_{L(n)})}$ and $g(X_{L(n)})$ are independent.

Let us use the transformation $U = g(X_{L(n)})$ and $V = \frac{g(X_{L(n+1)})}{g(X_{L(n)})}$. Jacobian of the transformation is $J = \frac{\partial}{\partial u}\big(g^{-1}(u)\big)\frac{\partial}{\partial v}\big(g^{-1}(uv)\big)$. Since $g(x)$ is an increasing and differentiable function, both $\frac{\partial}{\partial u}\big(g^{-1}(u)\big)$ and $\frac{\partial}{\partial v}\big(g^{-1}(uv)\big)$ are positive.

Thus we can write the joint pdf $f_{U,V}(u,v)$ of U and V as

$$f_{U,V}(u,v) = \frac{H\big(g^{-1}(u)\big)^{n-1}}{\Gamma(n)} h\big(g^{-1}(u)\big) f\big(g^{-1}(uv)\big) \frac{\partial}{\partial u}\big(g^{-1}(u)\big)\frac{\partial}{\partial v}\big(g^{-1}(uv)\big) \tag{9}$$

for all $0 < u < 1$ and $0 < v < 1$.

The pdf $f_U(u)$ of U is given by

$$f_U(u) = \frac{H\big(g^{-1}(u)\big)^{n-1}}{\Gamma(n)} f\big(g^{-1}(u)\big)\frac{\partial}{\partial u}\big(g^{-1}(u)\big) \tag{10}$$

for $0 < u < 1$.

Since u and v are independent, we get the pdf $f_V(u)$ of v from (9) and (10) as

$$f_V(v) = \frac{f\big(g^{-1}(uv)\big)}{F\big(g^{-1}(u)\big)}\frac{\partial}{\partial v}\big(g^{-1}(uv)\big) \tag{11}$$

for all $0 < u < 1$ and $0 < v < 1$.

Integrating of (11) with respect to v from 0 to v_1 and simplifying, we get

$$\int_0^{v_1} f_V(v)dv = F_V(v) = \frac{F\big(g^{-1}(uv_1)\big) - F\big(g^{-1}(0)\big)}{F\big(g^{-1}(u)\big)}$$

for any fixed v_1, $0 < v_1 < 1$.

Now taking $u \to 1$, $g^{-1}(u) \to b^-$ and $F\big(g^{-1}(1)\big) \to 1$. Since $F\big(g^{-1}(0)\big) = 0$, it holds $F_V(v_1) = F\big(g^{-1}(v_1)\big)$ for any fixed v_1, $0 < v_1 < 1$.

Let $F\big(g^{-1}(y)\big) = F(y)$, for $0 < y < 1$. Then we obtain the following equation

$$F(u)F(v_1) = F(uv_1), \tag{12}$$

for all u, $0 < u < 1$ and any fixed v_1, $0 < v_1 < 1$.

By the theory of functional equation, see [1], the only continuous solution of (12) with the boundary conditions $F(a) = 0$ and $F(b) = 1$ is

$$F(x) = (g(x))^\alpha$$

for all $0 < g(x) < 1$ and $\alpha > 0$.

This completes the proof. □

Remark 2.4. If we set $g(x) = \big(\exp[-e^{-\lambda x}]\big)^{1/\alpha}$, we obtain characterization by independence property concerning the Generalized Gumbel distribution. For $\lambda = 1$, we have the Gumbel distribution in [2].

Remark 2.5. If we set $g(x) = \big(e^{-(-x)^{-\beta}}\big)^{\frac{1}{\alpha}}$, we obtain characterization by independence property concerning the Weibull for extream value distribution in [2].

Remark 2.6. A list of continuous distributions with cdf and the corresponding forms of $g(x)$ are given in Table 2.

TABLE 2. Examples based on the distribution function $F(x) = \big(g(x)\big)^{\alpha}$

Distribution	$g(x)$	$F(x)$
Power	x	$x^{\alpha},\ 0 < x < 1$
Weibull	$\big(1 - e^{-(x/\lambda)^{\beta}}\big)^{\frac{1}{\alpha}}$	$1 - e^{-(x/\lambda)^{\beta}},\ 0 < x < \infty$
EP	$1 - (1+x)^{-\lambda}$	$(1 - (1+x)^{-\lambda})^{\alpha},\ 0 < x < \infty$
GR	$1 - e^{-(\lambda x)^2}$	$(1 - e^{-(\lambda x)^2})^{\alpha},\ 0 < x < \infty$
Inverse Weibull	$\big(e^{-\theta x^{-p}}\big)^{1/\alpha}$	$e^{-\theta x^{-p}}, 0 < x < \infty$
Burr Type II	$(1 + e^{-x})^{-1}$	$(1 + e^{-x})^{-\alpha},\ -\infty < x < \infty$
Lognormal	$\big(\Phi(\frac{\ln x - \mu}{\sigma})\big)^{\frac{1}{\alpha}}$	$\Phi(\frac{\ln x - \mu}{\sigma}),\ 0 < x < \infty$
Cauchy	$\big(\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)\big)^{1/\alpha}$	$\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x),\ -\infty < x < \infty$
Kappa	$(\frac{x^p}{\lambda + x^p})^{1/\alpha}$	$\frac{x^p}{\lambda + x^p},\ 0 < x < \infty$

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