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SOME PROPERTIES OF DEGENERATED EULER POLYNOMIALS OF THE SECOND KIND USING DEGENERATED ALTERNATIVE POWER SUM

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ABSTRACT. We construct degenerated Euler polynomials of the second kind and find some basic properties of this polynomials. From this paper, we can see degenerated alternative power sum is defined and is related to degenerated Euler polynomials of the second kind. Using this power sum, we have a number of symmetric properties of degenerated Euler polynomials of the second kind.

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1. Introduction

The Euler numbers are the numbers $E_n \quad (n = 0, 1, 2, \cdots)$ in the Maclaurin Series representation

$$\sum_{n=0}^{\infty} \widetilde{E}_n \frac{t^n}{n!} = \frac{1}{\cosh t} = \frac{2e^t}{e^{2t}+1}, \quad (|t| < \frac{\pi}{2}).$$

We know that $E_{2n+1} = 0, (n = 0, 1, 2, \cdots)$. Sometimes, mathematicians refer to Euler numbers of the second kind in order to distinguish Euler numbers of the first kind. As well known, we can also represent Euler numbers of the second kind using exponential function .

Definition 1.1. The Euler numbers of the second kind define

$$\sum_{n=0}^{\infty} \widetilde{E}_n \frac{t^n}{n!} = \frac{2e^t}{e^{2t}+1}, \quad (|t| < \frac{\pi}{2}).$$

From these numbers, many mathematicians have studied the Euler, Bernoulli, and Genocchi polynomials. In 1961, L. Calitz introduced serveral properties of

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the Bernoulli and Euler polynomials of the second kind(see [3]). In [1-4,6-15], Mathematicians have also researched interesting relations between the Bernoulli, Euler, and Genocchi polynomials combining q-number or Apostol number or etc. Various numbers and polynomials are advanced, expanded by many mathematicians and taken a number of application and many branches of mathematics.

Definition 1.2. The Euler polynomials of the second kind define

$$\sum_{n=0}^{\infty} \widetilde{E}_n(x)\frac{t^n}{n!} = \frac{2e^t}{e^{2t}+1}e^{tx}, \quad |t| < \frac{\pi}{2}.$$

From Definition 1.2, mathematicians have observed various properties and identities(see [3,6,12]).

Theorem 1.3. For any positive integer n, we have

(i) For any positive integer m(=odd),

$$\widetilde{E}_n(x) = m^n \sum_{i=0}^{m-1}(-1)^i \widetilde{E}_n\left(\frac{2i+x+1-m}{m}\right) \text{ for } n \geq 0,$$

(ii) $$\widetilde{E}_l(x+y) = \sum_{n=0}^{l}\binom{l}{n}\widetilde{E}_n(x)y^{l-n},$$

(iii) $$\widetilde{E}_n(x) = (-1)^n \widetilde{E}_n(-x),$$

(iv)

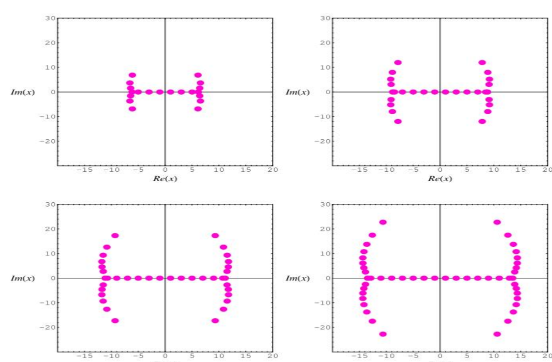

Figure 2: Zeros of $E_n(x)$ for $n = 20, 30, 40, 50$

In [2,10-12], we can see a numerical investigation on the zeros of various polynomials(see the above figure) and find some conjectures. In addition, we can also research the phenomenon of roots that form real numbers and complex numbers in their polynomials.

In this paper, the main aim is to find some identities of degenerated Euler polynomials of the second kind using degenerated alternative power sum. To make symmetric property of degenerated Euler polynomials of the second kind, we construct degenerated alternative power sum. We also have some basic properties of degenerated Euler polynomials of the second kind using various methods.

2. Some basic properties of degenerated Euler polynomials of the second kind

In this section we define degenerated Euler polynomials of the second kind. Fron this definition, we find serveral theorems that have important for finding these polynomials applications. Furthermore, we try to find a form, $(x|\lambda)_n$, which is related to degenerated Euler polynomials of the second kind.

Definition 2.1. Let n be any non-negative integer. For $x \in \mathbb{C}$, we define degenerated Euler polynomials of the second kind as

$$\sum_{n=0}^{\infty} \widetilde{E}_n(x,\lambda)\frac{t^n}{n!} = \frac{2}{(1+\lambda t)^{\frac{2}{\lambda}}+1}(1+\lambda t)^{\frac{1+x}{\lambda}}.$$

Setting $x=0$ in the degenerated Euler polynomials of the second kind, they can be simplified as follows:

$$\sum_{n=0}^{\infty} \widetilde{E}_n(0,\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \widetilde{E}_n(\lambda)\frac{t^n}{n!} = \frac{2}{(1+\lambda t)^{\frac{2}{\lambda}}+1}(1+\lambda t)^{\frac{1}{\lambda}},$$

where $\widetilde{E}_n(\lambda)$ is Euler numbers of the second kind. If $\lambda \to 0$, then we can find the classical Euler polynomials of the second kind as

$$\lim_{\lambda\to 0}\sum_{n=0}^{\infty} \widetilde{E}_n(x,\lambda)\frac{t^n}{n!} = \lim_{\lambda\to 0}\frac{2}{(1+\lambda t)^{\frac{2}{\lambda}}+1}(1+\lambda t)^{\frac{1+x}{\lambda}} = \frac{2}{e^t+e^{-t}}e^{tx} = \sum_{n=0}^{\infty} \widetilde{E}_n(x)\frac{t^n}{n!}.$$

Theorem 2.2. *Let x be any complex numbers. Then we have*

$$\text{(i)} \quad \widetilde{E}_n(x,\lambda) = \sum_{k=0}^{n}\binom{n}{k}(x|\lambda)_k \widetilde{E}_{n-k}(\lambda)$$

$$\text{(ii)} \quad \widetilde{E}_n(x+y,\lambda) = \sum_{k=0}^{n}\binom{n}{k}(y|\lambda)_k \widetilde{E}_{n-k}(x,\lambda).$$

Proof. (i) From the generating function of the degenerated Euler polynomials of the second kind, we can find

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{E}_n(x,\lambda)\frac{t^n}{n!}&=\frac{2}{(1+\lambda t)^{\frac{2}{\lambda}}+1}(1+\lambda t)^{\frac{1+x}{\lambda}}\\&=\sum_{n=0}^{\infty}\widetilde{E}_n(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(x|\lambda)_k\widetilde{E}_{n-k}(\lambda)\right)\frac{t^n}{n!},\end{aligned}$$

which gives the required result.
(ii) We omit a proof of (ii) due to its similarity to (i). ☐

Theorem 2.3. *For $x,\lambda\in\mathbb{C}$, the following holds*

$$\widetilde{E}_n(-x,-\lambda)=(-1)^n\widetilde{E}_n(x,\lambda).$$

Proof. Using the generating function of degenerated Euler polynomials of the second kind, we find

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{E}_n(-x,\lambda)\frac{t^n}{n!}&=\frac{2}{(1-\lambda t)^{-\frac{2}{\lambda}}+1}(1-\lambda t)^{\frac{1-x}{-\lambda}}\\&=\sum_{n=0}^{\infty}\widetilde{E}_n(x,\lambda)\frac{(-t)^n}{n!}.\end{aligned}$$

Now comparing the coefficients of t^n, we find the result. ☐

Theorem 2.4. *Let $n,x,\lambda\in\mathbb{Z}$. Then we have*

$$(x|\lambda)_n=\frac{1}{2}\left(\widetilde{E}_n(x+1,\lambda)+\widetilde{E}_n(x-1,\lambda)\right).$$

Proof. From the Definition 2.1, we can observe the following equation:

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\widetilde{E}_n(x+2,\lambda)+\widetilde{E}_n(x,\lambda)\right)\frac{t^n}{n!}\\&=\frac{2}{(1+\lambda t)^{\frac{2}{\lambda}}+1}(1+\lambda t)^{\frac{3+x}{\lambda}}+\frac{2}{(1+\lambda t)^{\frac{2}{\lambda}}+1}(1+\lambda t)^{\frac{1+x}{\lambda}}\\&=2(1+\lambda t)^{\frac{1+x}{\lambda}}=2\sum_{n=0}^{\infty}(x+1|\lambda)_n\frac{t^n}{n!}.\end{aligned}$$

Therefore, we find

$$(x+1|\lambda)_n=\frac{1}{2}\left(\widetilde{E}_n(x+2,\lambda)+\widetilde{E}_n(x,\lambda)\right),$$

which immediately gives the required result. ☐

Using Cauchy product on the above proof proccess, we can consider that

$$\begin{aligned}2(1+\lambda t)^{\frac{1+x}{\lambda}} &= 2(1+\lambda t)^{\frac{1}{\lambda}}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(1|\lambda)_{n-k}(x|\lambda)_k\right)\frac{t^n}{n!}.\end{aligned}$$

Corollary 2.5. *From Theorem 2.4 and the above equation, we have*

$$\sum_{k=0}^{n}\binom{n}{k}(1|\lambda)_{n-k}(x|\lambda)_k = \frac{1}{2}\left(\widetilde{E}_n(x+2,\lambda)+\widetilde{E}_n(x,\lambda)\right).$$

Corollary 2.6. *For $\lambda \to 0$, in Theorem 2.4, one holds*

$$x^n = \frac{1}{2}\left(\widetilde{E}_n(x+1)+\widetilde{E}_n(x-1)\right),$$

where $\widetilde{E}_n(x)$ is the classical Euler polynomials of the second kind.

Theorem 2.7. *For $x, \lambda \in \mathbb{C}$, we have*

$$(x|\lambda)_n = \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k}(2|\lambda)_{n-k}\widetilde{E}_k(x,\lambda)+\widetilde{E}_n(x,\lambda)\right).$$

Proof. Let $(1+\lambda t)^{\frac{2}{\lambda}}+1 \neq 0$. From the generating function of degenerated Euler polynomials of the second kind, we can find

$$\left((1+\lambda t)^{\frac{2}{\lambda}}+1\right)\sum_{n=0}^{\infty}\widetilde{E}_n(x,\lambda)\frac{t^n}{n!} = 2(1+\lambda t)^{\frac{1+x}{\lambda}}.$$

or, equivalently,

$$\begin{aligned}2(1+\lambda t)^{\frac{1+x}{\lambda}} &= 2\sum_{n=0}^{\infty}(1+x|\lambda)_n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{n=0}^{\infty}(2|\lambda)_n\frac{t^n}{n!}+1\right)\widetilde{E}_n(x,\lambda)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(2|\lambda)_{n-k}\widetilde{E}_k(x,\lambda)+\widetilde{E}_n(x,\lambda)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, the proof is complete. □

Corollary 2.8. *In Theorem 2.4 and Theorem 2.7, we can see*

$$\widetilde{E}_n(x+1,\lambda) = \sum_{k=0}^{n}\binom{n}{k}(2|\lambda)_{n-k}\widetilde{E}_k(x-1,\lambda).$$

We can use Theorem 2.2, to calculate some elements of $\widetilde{E}_n(x,\lambda)$. Here, we employ Mathematica to compute degenerated Euler polynomials of the second kind. The first few degenerated Euler polynomials of the second kind are

$$\begin{aligned}
&\widetilde{E}_1(x,0.1)=x,\\
&\widetilde{E}_2(x,0.1)=\frac{1}{10}(-10-x+10x^2),\\
&\widetilde{E}_3(x,0.1)=\frac{1}{50}(15-149x-15x^2+50x^2),\\
&\widetilde{E}_4(x,0.1)=\frac{1}{500}(2445+897x-2945x^2-300x^3+500x^4),\\
&\vdots
\end{aligned}$$

Using computer we investigate the zeros of $\widetilde{E}_n(x,\lambda)$. Here, our expectation is that a plot of $\widetilde{E}_n(x,\lambda)$ will approach to a plot of $\tilde{E}_n(x)$ when $\lambda\to 0$.

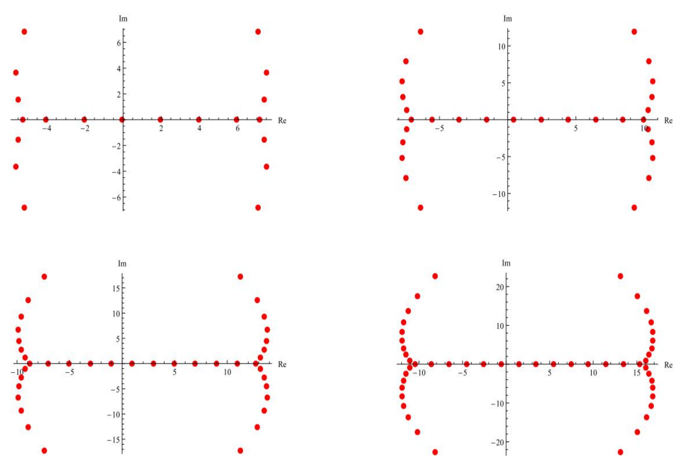

FIGURE 1. Zeros of $\widetilde{E}_n(x,0.1)$ for $n=20,30,40,50$

In Figure 1, for $n=20,30,40,50$, $\lambda=0.1$, we observe from top-left to the bottom-right that the zeros shape is similar to zeros shape of the classical Euler polynomials of the second kind(see [12]).

3. Some relations between degenerated Euler polynomials of the second kind and degenerated alternative power sum

In this section, we define the degenerated alternative power sum. From this power sum, we can find relation between this power sum and degenerated Euler polynomials of the second kind. We can also observe some symmetric properties of the degenerated Euler polynomials of the second kind using the degenerated alternative power sum.

Definition 3.1. Let $m \in \mathbb{N}$. We then define a degenerated alternative power sum as:

$$\sum_{n=0}^{\infty} \widetilde{P}_n(m-1;\lambda)\frac{t^n}{n!} = \frac{2(-1)^{m-1}(1+\lambda t)^{\frac{2m+1}{\lambda}}+1}{(1+\lambda t)^{\frac{2}{\lambda}}+1}.$$

Theorem 3.2. *For $m \in \mathbb{N}$, we hold*

$$\widetilde{P}_n(m-1;\lambda) = 2\sum_{i=0}^{m-1}(-1)^i(1+2i|\lambda)_n.$$

Proof. From Definition 3.1, we find

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{P}_n(m-1;\lambda)\frac{t^n}{n!} &= \frac{2(-1)^{m-1}(1+\lambda t)^{\frac{2m+1}{\lambda}}+1}{(1+\lambda t)^{\frac{2}{\lambda}}+1} \\ &= 2(1+\lambda t)^{\frac{1}{\lambda}}\sum_{i=0}^{m-1}(-1)^i(1+\lambda t)^{\frac{2}{\lambda}i} \\ &= \sum_{n=0}^{\infty} 2\sum_{i=0}^{m-1}(-1)^i(1+2i|\lambda)_n\frac{t^n}{n!}.\end{aligned}$$

The required relation now follows immediately. □

From Definition 3.1, we can note that

$$\begin{aligned}\lim_{\lambda\to 0}\sum_{n=0}^{\infty} \widetilde{P}_n(m-1;\lambda)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{2(-1)^{m-1}(1+\lambda t)^{\frac{2m+1}{\lambda}}+1}{(1+\lambda t)^{\frac{2}{\lambda}}+1} \\ &= 2e^t\frac{(-1)^{m-1}e^{2mt}+1}{e^{2t}+1} = 2\sum_{i=0}^{m-1}(-1)^i e^{(2i+1)t} \\ &= \sum_{n=0}^{\infty} 2\sum_{i=0}^{m-1}(-1)^i(2i+1)^n\frac{t^n}{n!},\end{aligned}$$

where $2\sum_{i=0}^{m-1}(-1)^i(2i+1)^n$ is the classical alternative power sum which is related to the classical Euler polynomials of the second kind.

Theorem 3.3. *For $\lambda \in \mathbb{C}$, the following relation holds:*

$$\widetilde{P}_n(m-1;\lambda) = \widetilde{E}_n(\lambda) + (-1)^{m-1}\widetilde{E}_n(2m,\lambda),$$

where $\widetilde{E}_n(\lambda)$ is the degenerated Euler numbers of the second kind.

Proof. To obtain the relation between degenerated alternative power sum and Euler polynomials of the second kind, we can make

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{P}_n(m-1;\lambda)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}2\sum_{i=0}^{m-1}(-1)^i(2i+1|\lambda)_n\frac{t^n}{n!}\\&=2(1+\lambda t)^{\frac{1}{\lambda}}\sum_{i=0}^{m-1}\left(-(1+\lambda t)^{\frac{2}{\lambda}}\right)^i\\&=2(1+\lambda t)^{\frac{1}{\lambda}}\left(\frac{(-1)^{m-1}(1+\lambda t)^{\frac{2m}{\lambda}}+1}{(1+\lambda t)^{\frac{2}{\lambda}}+1}\right)\\&=\sum_{n=0}^{\infty}\left((-1)^{m-1}\widetilde{E}_n(2m,\lambda)+\widetilde{E}_n(\lambda)\right)\frac{t^n}{n!},\end{aligned}$$

which, on comparing the coefficients, the required relation at once. □

Theorem 3.4. *Let a, b be non-negative integers. We then have*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\widetilde{E}_{n-k}\left(bx,\frac{\lambda}{a}\right)\widetilde{E}_k\left(ay,\frac{\lambda}{b}\right)\\&=\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k\widetilde{E}_{n-k}\left(ax,\frac{\lambda}{b}\right)\widetilde{E}_k\left(by,\frac{\lambda}{a}\right).\end{aligned}$$

Proof. Cosider that

$$A(t)=\frac{4(1+\lambda t)^{\frac{a+b+ab(x+y)}{\lambda}}}{\left((1+\lambda t)^{\frac{2a}{\lambda}}+1\right)\left((1+\lambda t)^{\frac{2b}{\lambda}}+1\right)}.$$

The form A can turn to

$$\begin{aligned}A(t)&=\frac{2}{(1+\lambda)^{\frac{2a}{\lambda}}+1}(1+\lambda t)^{\frac{a(1+bx)}{\lambda}}\frac{2}{(1+\lambda t)^{\frac{2b}{\lambda}}+1}(1+\lambda t)^{\frac{b(1+ay)}{\lambda}}\\&=\sum_{n=0}^{\infty}\widetilde{E}_n\left(bx,\frac{\lambda}{a}\right)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}\widetilde{E}_n\left(ay,\frac{\lambda}{b}\right)\frac{(bt)^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\widetilde{E}_{n-k}\left(bx,\frac{\lambda}{a}\right)\widetilde{E}_k\left(ay,\frac{\lambda}{b}\right)\right)\frac{t^n}{n!},\end{aligned}\tag{3.1}$$

or, equivalently,

$$\begin{aligned}A(t)&=\frac{2}{(1+\lambda)^{\frac{2b}{\lambda}}+1}(1+\lambda t)^{\frac{b(1+ax)}{\lambda}}\frac{2}{(1+\lambda t)^{\frac{2a}{\lambda}}+1}(1+\lambda t)^{\frac{a(1+by)}{\lambda}}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k\widetilde{E}_{n-k}\left(ax,\frac{\lambda}{b}\right)\widetilde{E}_k\left(by,\frac{\lambda}{a}\right)\right)\frac{t^n}{n!},\end{aligned}\tag{3.2}$$

and the theorem is proved in (3.1) and (3.2). □

Corollary 3.5. *Setting $a = 1$ in Theorem 3.4, we can get*

$$\sum_{k=0}^{n}\binom{n}{k}b^{k}\widetilde{E}_{n-k}(bx,\lambda)\widetilde{E}_{k}\left(y,\frac{\lambda}{b}\right)=\sum_{k=0}^{n}\binom{n}{k}b^{n-k}\widetilde{E}_{n-k}\left(x,\frac{\lambda}{b}\right)\widetilde{E}_{k}(by,\lambda).$$

Theorem 3.6. *Let a, b be odd non-negative integers. We then have*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}\widetilde{E}_{n-k}\left(bx,\frac{\lambda}{a}\right)\widetilde{P}_{k}\left(a-1;\frac{\lambda}{b}\right)\\&=\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^{k}\widetilde{E}_{n-k}\left(ax,\frac{\lambda}{b}\right)\widetilde{P}_{k}\left(b-1;\frac{\lambda}{a}\right).\end{aligned}$$

Proof. Suppose that

$$B(t)=\frac{4(1+\lambda t)^{\frac{a+b+abx}{\lambda}}\left((-1)^{a-1}(1+\lambda t)^{\frac{2ab}{\lambda}}+1\right)}{\left((1+\lambda t)^{\frac{2a}{\lambda}}+1\right)\left((1+\lambda t)^{\frac{2b}{\lambda}}+1\right)}.$$

Because a is odd integer, the form B can turn to

$$\begin{aligned}B(t)&=\frac{2(1+\lambda t)^{\frac{a(1+bx)}{\lambda}}}{(1+\lambda)^{\frac{2a}{\lambda}}+1}\left(\frac{(-1)^{a-1}(1+\lambda t)^{\frac{2ab}{\lambda}}+1}{(1+\lambda t)^{\frac{2b}{\lambda}}+1}\right)2(1+\lambda t)^{\frac{b}{\lambda}}\\&=\sum_{n=0}^{\infty}\widetilde{E}_{n}\left(bx,\frac{\lambda}{a}\right)\frac{(at)^{n}}{n!}2\sum_{i=0}^{a-1}(-1)^{i}\left(1+\lambda t\right)^{\frac{b(2i+1)}{\lambda}}.\end{aligned}$$

In here, we can note that

$$\begin{aligned}2\sum_{i=0}^{a-1}(-1)^{i}\left(1+\lambda t\right)^{\frac{b(2i+1)}{\lambda}}&=\sum_{n=0}^{\infty}2\sum_{i=0}^{a-1}(-1)^{i}\left(b(1+2i)|\lambda\right)_{n}\frac{t^{n}}{n!}\\&=\sum_{n=0}^{\infty}2\sum_{i=0}^{a-1}(-1)^{i}\left(1+2i|\frac{\lambda}{b}\right)_{n}\frac{(bt)^{n}}{n!}\\&=\sum_{n=0}^{\infty}\widetilde{P}_{n}\left(a-1;\frac{\lambda}{b}\right)\frac{(bt)^{n}}{n!}.\end{aligned}$$

Therefore, we have the following form B:

$$\begin{aligned}B(t)&=\sum_{n=0}^{\infty}\widetilde{E}_{n}\left(bx,\frac{\lambda}{a}\right)\frac{(at)^{n}}{n!}\sum_{n=0}^{\infty}\widetilde{P}_{n}\left(a-1;\frac{\lambda}{b}\right)\frac{(bt)^{n}}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}\widetilde{E}_{n-k}\left(bx,\frac{\lambda}{a}\right)\widetilde{P}_{k}\left(a-1;\frac{\lambda}{b}\right)\right)\frac{t^{n}}{n!}.\end{aligned}\tag{3.3}$$

Now following the same procedure when b is odd integer, we find the following other form B:

$$
\begin{aligned}
B(t) &= \frac{2(1+\lambda t)^{\frac{b(1+ax)}{\lambda}}}{(1+\lambda)^{\frac{2b}{\lambda}}+1}\left(\frac{(-1)^{b-1}(1+\lambda t)^{\frac{2ab}{\lambda}}+1}{(1+\lambda t)^{\frac{2a}{\lambda}}+1}\right)2(1+\lambda t)^{\frac{a}{\lambda}} \\
&= \sum_{n=0}^{\infty}\widetilde{E}_n\left(ax,\frac{\lambda}{b}\right)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}\widetilde{P}_n\left(b-1;\frac{\lambda}{a}\right)\frac{(at)^n}{n!} \qquad (3.4)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}b^{n-k}a^k\widetilde{E}_{n-k}\left(ax,\frac{\lambda}{b}\right)\widetilde{P}_k\left(b-1;\frac{\lambda}{a}\right)\right)\frac{t^n}{n!}
\end{aligned}
$$

Comparing (3.3) and (3.4), we can complete the proof at once. □

Corollary 3.7. *Putting $a=1$ in Theorem 3.6, we see*

$$
\sum_{k=0}^{n}\binom{n}{k}(b|\lambda)_k\widetilde{E}_{n-k}(bx,\lambda) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}b^{n-k}\widetilde{E}_{n-k}\left(x,\frac{\lambda}{b}\right)\widetilde{P}_k(b-1;\lambda).
$$

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