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A NUMERICAL INVESTIGATION ON THE STRUCTURE OF THE ROOT OF THE (p, q)-ANALOGUE OF BERNOULLI POLYNOMIALS†

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Abstract. In this paper we define the (p, q)-analogue of Bernoulli numbers and polynomials by generalizing the Bernoulli numbers and polynomials, Carlitz's type q-Bernoulli numbers and polynomials. We also give some interesting properties, explicit formulas, a connection with (p, q)-analogue of Bernoulli numbers and polynomials. Finally, we investigate the zeros of the (p, q)-analogue of Bernoulli polynomials by using computer.

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1. Introduction

Mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials(see [1-15]). Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -2, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

We remember that the classical Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ are defined by the following generating functions(see [1, 3, 9])

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{1.1}$$

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and

$$\left(\frac{t}{e^t-1}\right)e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{1.2}$$

respectively. The (p,q)-number is defined by

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}.$$

It is clear that (p,q)-number contains symmetric property, and this number is q-number when $p=1$. In particular, we can see $\lim_{q\to 1}[n]_{p,q}=n$ with $p=1$.

By using (p,q)-number, we define the (p,q)-analogue of Bernoulli polynomials and numbers, which generalized the previously known numbers and polynomials, including the Carlitz's type q-Bernoulli numbers and polynomials. We begin by recalling here the Carlitz's type q-Bernoulli numbers and polynomials(see [2]).

Definition 1.1. The Carlitz's type q-Bernoulli polynomials $B_{n,q}(x)$ are defined by means of the generating function

$$F_q(t,x)=\sum_{n=0}^{\infty}B_{n,q}(x)\frac{t^n}{n!}=-t\sum_{m=0}^{\infty}q^m e^{[m+x]_q t}, \tag{1.3}$$

and their values at $x=0$ are called the Carlitz's type q-Bernoulli numbers and denoted $B_{n,q}$.

Many kinds of of generalizations of these polynomials and numbers have been presented in the literature(see [1-15]). Based on this idea, we generalize the Carlitz's type q-Bernoullir number $B_{n,q}$ and q-Bernoulli polynomials $B_{n,q}(x)$. It follows that we define the following (p,q)-analogues of the the Carlitz's type q-Bernoulli number $B_{n,q}$ and q-Bernoulli polynomials $B_{n,q}(x)$.

In the following section, we introduce the (p,q)-analogue of Bernoulli polynomials and numbers. After that we define (p,q)-analogue of Riemann zeta function. Finally, we investigate the zeros of the (p,q)-analogue of Bernoulli polynomials by using computer.

2. (p,q)-analogue of Bernoulli numbers and polynomials

In this section, we define (p,q)-analogue of Bernoulli numbers and polynomials and provide some of their relevant properties.

Definition 2.1. For $0<q<p\le 1$, the Carlitz's type (p,q)-Bernoulli numbers $B_{n,p,q}$ and polynomials $B_{n,p,q}(x)$ are defined by means of the generating functions

$$F_{p,q}(t)=\sum_{n=0}^{\infty}B_{n,p,q}\frac{t^n}{n!}=-t\sum_{m=0}^{\infty}q^m e^{[m]_{p,q}t}, \tag{2.1}$$

and

$$F_{p,q}(t,x)=\sum_{n=0}^{\infty}B_{n,p,q}(x)\frac{t^n}{n!}=-t\sum_{m=0}^{\infty}q^m e^{[m+x]_{p,q}t}, \tag{2.2}$$

respectively.

Setting $p = 1$ in (2.1) and (2.2), we can obtain the corresponding definitions for the Carlitz's type q-Bernoulli number $B_{n,q}$ and q-Bernoulli polynomials $B_{n,q}(x)$ respectively. Obviously, if we put $p = 1$, then we have

$$B_{n,p,q}(x) = B_{n,q}(x), \quad B_{n,p,q} = B_{n,q}.$$

Putting $p = 1$, we have

$$\lim_{q\to 1} B_{n,p,q}(x) = B_n(x), \quad \lim_{q\to 1} B_{n,p,q} = B_n.$$

By using above equation (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,p,q}\frac{t^n}{n!} &= -t\sum_{m=0}^{\infty} q^m e^{[m]_{p,q}t} \\ &= \sum_{n=0}^{\infty}\left(\frac{-n}{(p-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\frac{1}{1-q^{l+1}p^{n-l-1}}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, we have*

$$B_{n,p,q} = -n\left(\frac{1}{p-q}\right)^{n-1}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\frac{1}{1-q^{l+1}p^{n-l-1}}.$$

If we put $p = 1$ in the above theorem we obtain

$$B_{n,q} = -n\left(\frac{1}{1-q}\right)^{n-1}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\frac{1}{1-q^{l+1}}.$$

By (2.2), we obtain

$$B_{n,p,q}(x) = \frac{-n}{(p-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l q^{xl}p^{(n-l-1)x}\frac{1}{1-q^{l+1}p^{n-l-1}}. \tag{2.4}$$

By using (2.2) and (2.4), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} B_{n,p,q}(x)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\frac{-n}{(p-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l q^{xl}p^{(n-l-1)x}\frac{1}{1-q^{l+1}p^{n-l-1}}\right)\frac{t^n}{n!} \\ &= -t\sum_{m=0}^{\infty} q^m e^{[m+x]_{p,q}t}.\end{aligned} \tag{2.5}$$

The following elementary properties of the (p, q)-analogue of Bernoulli numbers $B_{n,p,q}$ and polynomials $B_{n,p,q}(x)$ are readily derived form (2.1) and (2.2). We, therefore, choose to omit details involved.

Theorem 2.3. *(Distribution relation). For any positive integer m, we have*

$$B_{n,p,q}(x) = [m]_{p,q}^{n-1} \sum_{a=0}^{m-1} q^a B_{n,p^m,q^m}\left(\frac{a+x}{m}\right), n \in \mathbb{Z}_+.$$

Theorem 2.4. *(Property of complement). For $n \in \mathbb{Z}_+$, we have*

$$B_{n,p^{-1},q^{-1}}(1-x) = (-1)^n p^{n-1} q^n B_{n,p,q}(x).$$

Theorem 2.5. *For $n \in \mathbb{Z}_+$, we have*

$$qB_{n,p,q}(1) - B_{n,p,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases}$$

Theorem 2.6. *(Difference equation). For $n \in \mathbb{Z}_+$, we have*

$$qB_{n,p,q}(x+1) - B_{n,p,q}(x) = n[x]_{p,q}^{n-1}.$$

By (2.1) and (2.2), we get

$$-t\sum_{l=0}^{\infty} q^{l+n} e^{[x+n+l]_{p,q}t} + t\sum_{l=0}^{\infty} q^l e^{[x+l]_{p,q}t} = t\sum_{l=0}^{n-1} q^l e^{[x+l]_{p,q}t}. \tag{2.6}$$

Hence we have

$$\begin{aligned} &q^n \sum_{m=0}^{\infty} B_{m,p,q}(x+n)\frac{t^m}{m!} - \sum_{m=0}^{\infty} B_{m,p,q}(x)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(m\sum_{l=0}^{n-1} q^l [x+l]_{p,q}^{m-1}\right)\frac{t^m}{m!}. \end{aligned} \tag{2.7}$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of (2.7), we have the following theorem.

Theorem 2.7. *For $n \in \mathbb{Z}_+$, we have*

$$\frac{q^n B_{m,p,q}(x+n) - B_{m,p,q}(x)}{m} = \sum_{l=0}^{n-1} q^l [x+l]_{p,q}^{m-1}.$$

Setting $x = 0$ in Theorem 2.7, we obtain the sums of powers of consecutive (p,q)-numbers.

$$\sum_{l=0}^{n-1} q^l [l]_{p,q}^{m-1} = \frac{q^n B_{m,p,q}(n) - B_{m,p,q}}{m}. \tag{2.8}$$

Indeed, the formula (2.8) is a (p,q)-analogue of the well known formula

$$\sum_{l=0}^{n-1} l^{m-1} = \frac{B_m(n) - B_m}{m}.$$

In [11], Ryoo defined the Carlitz's type (p,q)-Euler polynomials $E_{n,p,q}(x)$ are defined by means of the generating functions

$$\sum_{n=0}^{\infty} E_{n,p,q}(x)\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[m+x]_{p,q}t}. \tag{2.9}$$

Let m be even. By (2.2) and (2.9), we obtain the following theorem.

Theorem 2.8. *Let m be even. For $n \in \mathbb{Z}_+$, we have*

$$E_{n,p,q}(x) = \frac{-[2]_q[m]_{p,q}^n}{n+1} \sum_{k=0}^{m-1} (-1)^k q^k B_{n+1,p^m,q^m}\left(\frac{x+k}{m}\right).$$

3. (p,q)-analogue of Riemann zeta function

By using (p,q)-analogue of Bernoulli numbers and polynomials, (p,q)-Riemann zeta function and Hurwitz (p,q)-Riemann zeta functions are defined. These functions interpolate the (p,q)-analogue of Bernoulli numbers $B_{n,p,q}$, and polynomials $B_{n,p,q}(x)$, respectively.

The Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ is defined by(cf. [1, 2, 3, 4, 5, 15])

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s},$$

$$(a \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ s \in \mathbb{C} \text{ when } |z| < 1 \quad \text{and} \quad \mathrm{Re}(s) > 1 \text{ when } |z| = 1).$$

contains, as its special cases, not only the Riemann and Hurwitz Zeta functions:

$$\zeta(s) = \Phi(1,s,1) = \zeta(s,1) = \frac{1}{2^s-1}\zeta\left(s,\frac{1}{2}\right),$$

$$\zeta(s,a) = \Phi(1,s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad (\ \mathrm{Re}(s) > 1\ ,\ a \notin \mathbb{Z}_0^-),$$

and the Lerch Zeta function:

$$l(s,\xi) = \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi}\Phi(e^{2\pi i\xi}, s, 1),\ (\xi \in \mathbb{R}\ ,\ \ \mathrm{Re}(s) > 1),$$

but also such other functions as the Polylogarithmic function:

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z,s,1),$$

$$(\ s \in \mathbb{C} \quad \text{when} \quad |z| < 1 \ \text{ and } \quad \mathrm{Re}(s) > 1 \quad \text{when} \quad |z| = 1),$$

and the Lipschitz-Lerch Zeta function:

$$\phi(\xi,a,s) = \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi(e^{2\pi i\xi}, s, a),$$

$$(a \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ \Re(s) > 0 \text{ when } \xi \in \mathbb{R}\backslash\mathbb{Z},\ \ \mathrm{Re}(s) > 1 \text{ when } \xi \in \mathbb{Z}).$$

We first define the (p,q)-Hurwitz-Lerch Zeta function as follows:

Definition 3.1. (p,q)-*Hurwitz-Lerch Zeta function* is defined by

$$\Phi_{p,q}(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{[n+a]_{p,q}^s},$$

$$(a\in\mathbb{C}\backslash\mathbb{Z}_0^-,\ s\in\mathbb{C}\text{ when }|z|<1\quad\text{and}\quad \mathrm{Re}(s)>1\text{ when }|z|=1).$$

Definition 3.2. (p,q)-*Lerch Zeta function* is defined by

$$l_{p,q}(s,\xi)=\sum_{n=1}^{\infty}\frac{e^{2n\pi i\xi}}{[n]_{p,q}^s},\quad (\xi\in\mathbb{R}\ ,\ \ \mathrm{Re}(s)>1)$$

Observe that

$$l_{p,q}(s,\xi)=e^{2\pi i\xi}\Phi_{p.q}(e^{2\pi i\xi},s,1).$$

Definition 3.3. (p,q)-*Polylogarithmic function* is defined by

$$Li_{p,q,s}(z)=\sum_{n=1}^{\infty}\frac{z^n}{[n]_{p,q}^s},$$

$$(\ s\in\mathbb{C}\quad\text{when }|z|<1\ \text{ and }\quad \mathrm{Re}(s)>1\quad\text{when }|z|=1).$$

Definition 3.4. (p,q)- *Lipschitz-Lerch Zeta function* is defined by

$$\phi_{p,q}(\xi,a,s)=\sum_{n=0}^{\infty}\frac{e^{2n\pi i\xi}}{[n+a]_{p,q}^s}=\Phi_{p,q}(e^{2\pi i\xi},s,a),$$

$$(a\in\mathbb{C}\backslash\mathbb{Z}_0^-,\ \Re(s)>0\text{ when }\xi\in\mathbb{R}\backslash\mathbb{Z},\ \ \mathrm{Re}(s)>1\text{ when }\xi\in\mathbb{Z}).$$

By using (2.2), we note that

$$\frac{d^k}{dt^k}F_{p,q}(t,x)\bigg|_{t=0}=-k\sum_{m=0}^{\infty}q^m[m+x]_{p,q}^{k-1}\tag{3.1}$$

and

$$\left(\frac{d}{dt}\right)^k\left(\sum_{n=0}^{\infty}B_{n,p,q}(x)\frac{t^n}{n!}\right)\bigg|_{t=0}=B_{k,p,q}(x),\ \text{ for }k\in\mathbb{N}.\tag{3.2}$$

By (3.1) and (3.2), we are now ready to define the (p,q)-Hurwitz zeta function.

Definition 3.5. Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$ and $x\notin\mathbb{Z}_0^-$.

$$\zeta_{p,q}(s,x)=\sum_{n=0}^{\infty}\frac{q^n}{[n+x]_{p,q}^s}.\tag{3.4}$$

Note that $\zeta_{p,q}(s,x)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $p=1$ and $q\to 1$, then $\zeta_{p,q}(s,x)=\zeta(s,x)$ which is the Hurwitz zeta function(see [1, 2, 3, 4, 14]). Relation between $\zeta_{p,q}(s,x)$ and $B_{k,p,q}(x)$ is given by the following theorem.

Theorem 3.6. *For $k \in \mathbb{N}$, we have*

$$\zeta_{p,q}(1-k,x) = -\frac{B_{k,p,q}(x)}{k}.$$

By Definition 3.1 and Theorem 3.6, we have

$$\begin{aligned} -\frac{B_{k,p,q}(a)}{k} &= \sum_{n=0}^{\infty} \frac{q^n}{[n+a]_{p,q}^{1-k}} \\ &= \Phi_{p,q}(q, 1-k, a). \end{aligned}$$

Hence, we have the following relationship:

Theorem 3.7. *Let $\Phi_{p,q}(q, 1-n, a)$ be the (p,q)-Hurwitz-Lerch Zeta function. For $n \in \mathbb{N}$, $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$, we have*

$$B_{n,p,q}(a) = -n\Phi_{p,q}(q, 1-n, a).$$

From (2.1), we note that

$$\begin{aligned} \frac{d^k}{dt^k} F_{p,q}(t)\bigg|_{t=0} &= -k \sum_{m=1}^{\infty} q^m [m]_{p,q}^{k-1} \\ &= B_{k,p,q}, (k \in \mathbb{N}). \end{aligned}$$

By using the above equation, we are now ready to define (p,q)-Riemann zeta function.

Definition 3.8. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

$$\zeta_{p,q}(s) = \sum_{n=1}^{\infty} \frac{q^n}{[n]_{p,q}^s}.$$

Note that $\zeta_{p,q}(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $p = 1, q \to 1$, then $\zeta_{p,q}(s) = \zeta(s)$ which is the Riemann zeta function(see [3]). Relation between $\zeta_{p,q}(s)$ and $B_{k,p,q}$ is given by the following theorem.

Theorem 3.9. *For $k \in \mathbb{N}$, we have*

$$\zeta_{p,q}(1-k) = -\frac{B_{k,p,q}}{k}.$$

By Definition 3.3 and Theorem 3.9, we have

$$\begin{aligned} -\frac{B_{k,p,q}(a)}{k} &= \sum_{n=1}^{\infty} \frac{q^n}{[n]_{p,q}^{1-k}} \\ &= Li_{p,q,1-k}(q). \end{aligned}$$

Definition 3.10. The (p,q)-L-function is defined by

$$L_{p,q}(s,a) = \sum_{n=0}^{\infty} \frac{1}{[n+a]_{p,q}^s}, \quad (\mathrm{Re}(s) > 1\ ,\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-).$$

Obviously, $L_{p,q}(s,a) = \phi_{p,q}(1,a,s) = \Phi_{p,q}(e^{2\pi i}, s, a)$.

4. Zeros of the (p, q)-analogue of Bernoulli polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the (p, q)-analogue of Bernoulli polynomials $B_{n,p,q}(x)$. The (p, q)-analogue of Bernoulli polynomials $B_{n,p,q}(x)$ can be determined explicitly. A few of them are

$$B_{0,p,q}(x) = 0,$$

$$B_{1,p,q}(x) = \frac{1}{q-1},$$

$$B_{2,p,q}(x) = -\frac{2(p^x - p^x q^2 - q^x + pq^{1+x})}{(p-q)(-1+q)(1+q)(-1+pq)},$$

$$B_{2,p,q}(x) = \frac{3(p^{2x} - p^{1+2x}q^2 - p^{2x}q^3 + p^{1+2x}q^5 - 2p^x q^x + q^{2x} + 2p^{2+x}q^{1+x})}{(p-q)^2(-1+q)(-1+p^2q)(1+q+q^2)(-1+pq^2)} + \frac{3(2p^x q^{3+x} - 2p^{2+x}q^{4+x} - p^2q^{1+2x} - pq^{2+2x} + p^3q^{3+2x})}{(p-q)^2(-1+q)(-1+p^2q)(1+q+q^2)(-1+pq^2)}.$$

Our numerical results for approximate solutions of real zeros of $B_{n,p,q}(x)$ are displayed(Tables 1, 2).

Table 1. Numbers of real and complex zeros of $B_{n,p,q}(x)$

degree n	real zeros	complex zeros
2	1	0
3	0	2
4	1	2
5	0	4
6	1	4
7	0	6
8	1	6
9	0	0
10	1	8
11	0	10
12	1	10
13	0	12
14	1	12
15	0	0
16	1	14

In Table 1, we choose $p = 1/2$ and $q = 1/10$.

We investigate the beautiful zeros of the (p, q)-analogue of Bernoulli polynomials $B_{n,p,q}(x)$ by using a computer. We plot the zeros of the (p, q)-analogue of Bernoulli polynomials $B_{n,p,q}(x)$ for $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left),

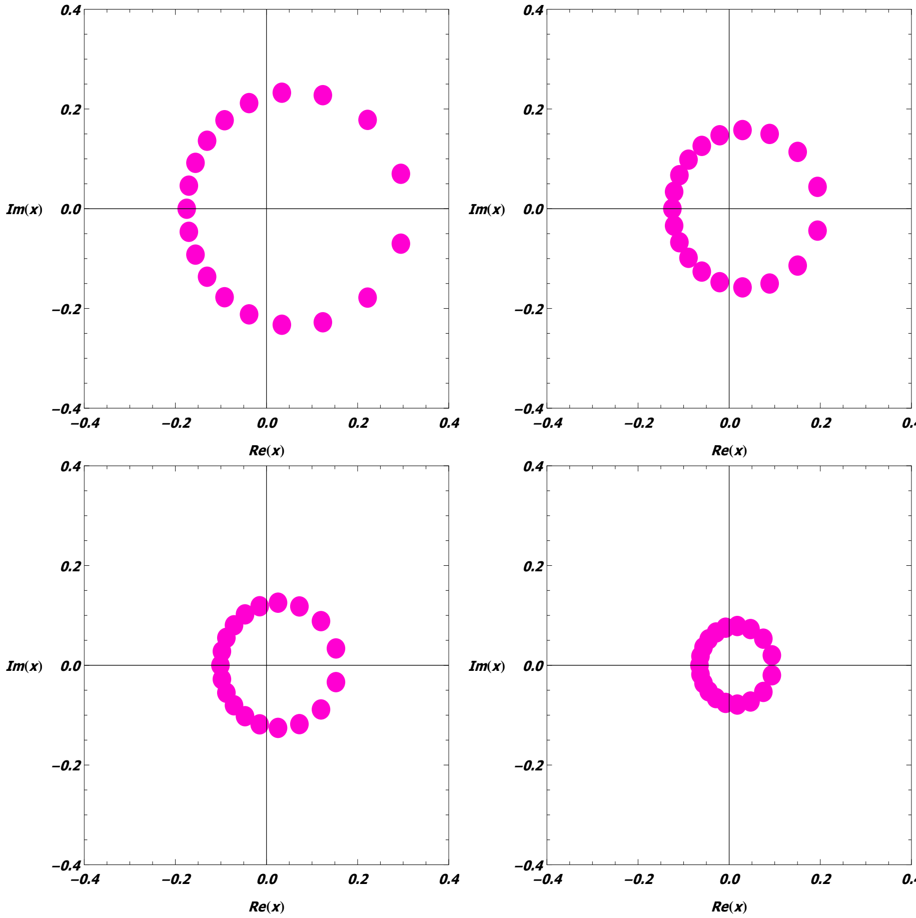

FIGURE 1. Zeros of $B_{n,p,q}(x)$

we choose $n = 20, p = 1/2$ and $q = 1/10$. In Figure 1(top-right), we choose $n = 20, p = 1/2$ and $q = 1/26$. In Figure 1(bottom-left), we choose $n = 20, p = 1/2$ and $q = 1/50$. In Figure 1(bottom-right), we choose $n = 20, p = 1/2$ and $q = 1/250$.

We observe a remarkable regular structure of the real roots of the (p, q)-analogue of Bernoulli polynomials $B_{n,p,q}(x)$. We also hope to verify a remarkable regular structure of the real roots of the (p, q)-analogue of Bernoulli polynomials $B_{n,p,q}(x)$(Table 1).

Next, we calculated an approximate solution satisfying (p, q)-analogue of Bernoulli polynomials $B_{n,p,q}(x) = 0$ for $x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $B_{n,p,q}(x) = 0, p = 1/2, q = 1/10$

degree n	x
2	-0.0256257
4	-0.10221
6	-0.132579
8	-0.147872
10	-0.157014
12	-0.163077
14	-0.167388
16	-0.170599
18	-0.157276

By numerical computations, we will make a series of the following conjectures:

Prove that $B_{n,p,q}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions. However, $B_{n,p,q}(x)$ has not $Re(x) = a$ reflection symmetry for $a \in \mathbb{R}$. Using computer, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value n(see Figures 1, 2, 3). We are able to decide if $B_{n,p,q}(x)) = 0$ has not $n-1$ distinct solutions(see Table 1). The author has no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the (p, q)-analogue of Bernoulli polynomials $B_{n,p,q}(x)$ which appear in mathematics and physics.

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