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POSITIVE SOLUTIONS FOR NONLINEAR m-POINT BVP WITH SIGN CHANGING NONLINEARITY ON TIME SCALES†

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ABSTRACT. In this paper, by using fixed point theorems in cones, the existence of positive solutions is considered for nonlinear m-point boundary value problem for the following second-order dynamic equations on time scales

$$u^{\Delta\nabla}(t) + a(t)f(t, u(t)) = 0, \quad t \in (0, T),$$

$$\beta u(0) - \gamma u^{\Delta}(0) = 0, \quad u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i), \ m \geq 3,$$

where $a(t) \in C_{ld}((0, T), \ [0, +\infty))$, $f \in C([0, T] \times [0, +\infty), \ (-\infty, +\infty))$, the nonlinear term f is allowed to change sign. We obtain several existence theorems of positive solutions for the above boundary value problems. In particular, our criteria generalize and improve some known results [15] and the obtained conditions are different from related literature [14]. As an application, an example to demonstrate our results is given.

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1. Introduction

A time scale **T** is a nonempty closed subset of R. We make the blanket assumption that 0, T are points in **T**. By an interval (0, T), we always mean the intersection of the real interval (0, T) with the given time scale, that is (0, T)∩**T**.

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In this paper, we will be concerned with the existence of positive solutions for the following dynamic equations on time scales:

$$u^{\Delta\nabla}(t)+a(t)f(t,u(t))=0,\quad t\in(0,T), \tag{1.1}$$

$$\beta u(0)-\gamma u^{\Delta}(0)=0,\quad u(T)=\sum_{i=1}^{m-2}a_iu(\xi_i),\ m\geq 3, \tag{1.2}$$

we will assume that the following conditions are satisfied throughout this paper:

(H_1) $0<\xi_1<\cdots<\xi_{m-2}<\rho(T)$, $\beta,\gamma\geq 0$, $\beta+\gamma>0$, $a_i\in[0,+\infty)$, $i=1,2,\cdots,m-3$, $a_{m-2}>0$, satisfy $0<\sum_{i=1}^{m-2}a_i\xi_i<T$, and $d=\beta(T-\sum_{i=1}^{m-2}a_i\xi_i)+\gamma(1-\sum_{i=1}^{m-2}a_i)>0$;

(H_2) $a(t)\in C_{ld}((0,T),\ [0,+\infty))$ and there exists $t_0\in[\xi_{m-2},T)$, such that $a(t_0)>0$;

(H_3) $f\in C([0,T]\times[0,+\infty),\ (-\infty,+\infty))$, $f(t,0)\geq 0$ *and* $f(t,0)\not\equiv 0$. (The Δ-derivative and the ∇-derivative in (1.1), (1.2) and the C_{ld} space in (H_2) are defined in Section 2.)

Recently, there has been much attention paid to the existence of positive solutions for second-order nonlinear boundary value problems on time scales, for examples, see [3, 8, 10, 14, 15] and references therein. But to the best of our knowledge, few people considered the second-order dynamic equations with sign changing nonlinear term on time scales.

In [3], Anderson discussed the following dynamic equation on time scales:

$$u^{\triangle\nabla}(t)+a(t)f(u(t))=0,t\in(0,T), \tag{1.3}$$

$$u(0)=0,\ \alpha u(\eta)=u(T). \tag{1.4}$$

He obtained some results for the existence of one positive solution of the problem (1.3) and (1.4) based on the limits $f_0=\lim\limits_{u\to 0^+}\dfrac{f(u)}{u}$ and $f_\infty=\lim\limits_{u\to 0}\dfrac{f(u)}{u}$.

In [8], Kaufmann studied the problem (1.3) and (1.4) and obtained the existence results of finitely many positive solutions and countably many positive solutions.

In [10], Jian-Ping Sun and Wan-Tong Li considered the following system on time scale $\mathbf{T}$:

$$u_i^{\Delta\Delta}(t)+f_i(u_1(t),u_2(t),\cdots,u_n(t))=0,\quad t\in(0,T), \tag{1.5}$$

$$u_i^{\Delta}(0)=0=u_i(\sigma^2(T))\ \ i=1,2,\cdots,n. \tag{1.6}$$

By using the theory of the fixed point index, the authors investigate the effect of $\sigma^2(T)$ on the existence and nonexistence of positive solution for the system (1.5) and (1.6) in sublinear cases.

In [12], Luo and Ma discussed the following dynamic equation on time scales:

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, t \in (0,T), \tag{1.7}$$

$$u(0) = \beta u(\eta),\ u(T) = \alpha u(\eta). \tag{1.8}$$

They obtained some results for the existence of one positive solution and of at least three positive solutions of the problem (1.7) and (1.8) by using a fixed point theorem and Leggett-Williams fixed point theorem, respectively.

Su et. al. [13] investigated the following singular m-point p-Laplacian boundary value problem on time scales with the sign changing nonlinearity:

$$(\varphi_p(u^{\Delta}(t)))^{\nabla} + a(t)f(t,u(t)) = 0, \quad t \in (0,T), \tag{1.9}$$

$$u(0) = 0, \quad u(T) - \sum_{i=1}^{m-2} \psi_i(u(\xi_i)) = 0, \tag{1.10}$$

where $\varphi_p(u) = |u|^{p-2}u$, $p > 1$, $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$, $a(t) \in C_{ld}((0,T), (0,+\infty))$, $f \in C_{ld}((0,T) \times (0,+\infty), (-\infty,+\infty))$. They presented some new existence criteria for positive solutions of the problem (1.9) and (1.10) by using the well-known Schauder fixed point theorem and upper and lower solutions method.

In [14], Sun and Li considered the existence of positive solutions of the following dynamic equations on time scales:

$$u^{\triangle\nabla}(t) + a(t)f(t,u(t)) = 0, t \in (0,T), \tag{1.11}$$

$$\beta u(0) - \gamma u^{\triangle}(0) = 0,\ \alpha u(\eta) = u(T). \tag{1.12}$$

They obtained the existence of single and multiple positive solutions of the problem (1.11) and (1.12) by using fixed point theorem and Leggett-Williams fixed point theorem, respectively.

In [16], Wang and Agarwal considered a general type of delay neural networks on time scales, by contraction principle and Gronwall-Bellman's inequality, they obtained some existence results of almost periodic solution for the problem.

The key conditions used in the above papers is that the nonlinearity is nonnegative, so the solution is concave down. If the nonlinear term is negative somewhere, then the solution needs no longer be concave down. As a result, it is difficult to find positive solutions for the dynamical equation when the nonlinearity change sign.

The present work is moviated by recent papers [13-16]. To date few paper has appeared in the literature which discusses the multipoint boundary value problem for second-order dynamic equations on time scales when nonlinear term may change sign. This paper attempts to fill this gap in the literature. In this paper, on the one hand, our work concentrates on the case when the nonlinear term may change sign, we will use the property of the solutions of the BVP (1.1) and (1.2) to overcome the difficulty. On the other hand, we will establish the key conditions in Theorem 3.1 and Theorem 3.2 to show the existence of positive solutions of the BVP (1.1) and (1.2).

The rest of the paper is arranged as follows. We state some basic time scale definitions and prove several preliminary results in Section 2, Section 3 is devoted to the existence of positive solution of (1.1) and (1.2), the main tool being the fixed point theorem in cone. At the end of the paper, we will give an example which illustrates that our work is true. We also point out that when $f \in C([0,T]\times[0,+\infty),\ [0,+\infty))$, i.e., the nonlinear term f is positive, (1.1) and (1.2) becomes a boundary value problem on time scales just considered in [15]. Our main results extend and include the main results of [14, 15].

2. Preliminaries and some Lemmas

For convenience, we list the following definitions which can be found in [1, 4, 6, 7].

Definition 2.1. A time scale $\mathbf{T}$ is a nonempty closed subset of real numbers R. For $t < \sup\mathbf{T}$ and $r > \inf\mathbf{T}$, define the forward jump operator σ and backward jump operator ρ, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T},$$

$$\rho(r) = \sup\{\tau \in \mathbf{T} \mid \tau < r\} \in \mathbf{T}.$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

Definition 2.2. For $f : \mathbf{T} \to R$ and $t \in \mathbf{T}^k$, the delta derivative of f at the point t is defined to be the number $f^\Delta(t)$, (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|,$$

for all $s \in U$.

For $f : \mathbf{T} \to R$ and $t \in \mathbf{T}_k$, the nabla derivative of f at t is the number $f^\nabla(t)$, (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \le \epsilon|\rho(t) - s|,$$

for all $s \in U$.

Definition 2.3. A function f is left-dense continuous (i.e. ld-continuous), if f is continuous at each left-dense point in $\mathbf{T}$ and its right-sided limit exists at each right-dense point in $\mathbf{T}$.

Definition 2.4. If $G^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = G(b) - G(a).$$

If $F^{\nabla}(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

To prove the main results in this paper, we will employ several lemmas. These lemmas are based on the linear BVP

$$u^{\Delta\nabla}(t) + h(t) = 0, \quad t \in (0, T), \tag{2.1}$$

$$\beta u(0) - \gamma u^{\Delta}(0) = 0, \quad u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i), \; m \geq 3. \tag{2.2}$$

Lemma 2.5. *(see [15]). If $d \neq 0$, then for $h \in C_{ld}[0,T]$ the BVP (2.1) and (2.2) has the unique solution*

$$u(t) = -\int_0^t (t-s)h(s)\nabla s + \frac{\beta t + \gamma}{d} \int_0^T (T-s)h(s)\nabla s$$

$$-\frac{\beta t + \gamma}{d} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)\nabla s. \tag{2.3}$$

Lemma 2.6. *(see [15]). Assume (H_1) holds, For $h \in C_{ld}[0,T]$ and $h \geq 0$, then the unique solution u of (2.1) and (2.2) satisfies*

$$u(t) \geq 0, \quad for \quad t \in [0, T].$$

Lemma 2.7. *(see [15]). Let $\sum_{i=1}^{m-2} a_i \xi_i > T$, $d \neq 0$. If $h \in C_{ld}[0,T]$ and $h \geq 0$, then (2.1) and (2.2) has no positive solution.*

Lemma 2.8. *(see [15]). Assume (H_1) holds, For $h \in C_{ld}[0,T]$ and $h \geq 0$, then the unique solution u of (2.1) and (2.2) satisfies*

$$\inf_{t \in [\xi_{m-2}, T]} u(t) \geq r\|u\|,$$

where

$$r = \min\left\{ \frac{a_{m-2}\,(T - \xi_{m-2})}{T - a_{m-2}\xi_{m-2}},\; \frac{a_{m-2}\xi_{m-2}}{T}, \frac{\xi_{m-2}}{T} \right\}, \; \|u\| = \sup_{t \in [0,T]} |u(t)|.$$

Let $E = C_{ld}([0,T], R)$ be the set of all ld-continuous functions from $[0,T]$ to R, and Let the norm on $C_{ld}([0,T], R)$ be the maximum norm. Then the $C_{ld}([0,T], R)$ is a Banach space. We define two cones by

$$P = \{u : \; u \in E, \; u(t) \geq 0, \; t \in [0, T]\},$$

and

$$K = \{u|u \in E, u(t) \text{ is nonnegative on } [0,T], \inf_{t\in[\xi_{m-2},T]} u(t) \geq r \parallel u \parallel\},$$

where r is the same as in Lemma 2.8.

It is easy to see that the BVP (1.1) and (1.2) has a solution $u = u(t)$ if and only if u solves the equation

$$\begin{aligned} u(t) = & -\int_0^t (t-s)a(s)f(s,u(s))\nabla s + \frac{\beta t+\gamma}{d}\int_0^T (T-s)a(s)f(s,u(s))\nabla s \\ & -\frac{\beta t+\gamma}{d}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s,u(s))\nabla s. \end{aligned}$$

We define the operators $G: P \to E$ and $H: K \to E$ as follows

$$\begin{aligned} (Gu)(t) = & -\int_0^t (t-s)a(s)f(s,u(s))\nabla s + \frac{\beta t+\gamma}{d}\int_0^T (T-s)a(s)f(s,u(s))\nabla s \\ & -\frac{\beta t+\gamma}{d}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s,u(s))\nabla s, \end{aligned} \tag{2.4}$$

$$\begin{aligned} (Hu)(t) = & -\int_0^t (t-s)a(s)f^+(s,u(s))\nabla s \\ & +\frac{\beta t+\gamma}{d}\int_0^T (T-s)a(s)f^+(s,u(s))\nabla s \\ & -\frac{\beta t+\gamma}{d}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f^+(s,u(s))\nabla s, \end{aligned} \tag{2.5}$$

where $f^+(t,u(t)) = \max\{f(t,u(t)),\ 0\}, \quad t \in [0,T]$.

It is obvious that K is a cone in E. By Lemma 2.3, $H(K) \subset K$. So by applying Arzela-Ascoli theorem on time scales [2], we can obtain that $H(K)$ is relatively compact. In view of Lebesgue's dominated convergence theorem on time scales [6], it is easy to prove that H is continuous. Hence, $H : K \to K$ is completely continuous.

Lemma 2.9. *(see [9]) Let K be a cone in a Banach space X. Let D be an open bounded subset of X with $D_K = D \cap K \neq \phi$ and $\overline{D_K} \neq K$. Assume that $A : \overline{D_K} \longrightarrow K$ is a completely continuous map such that $x \neq Ax$ for $x \in \partial D_K$. Then the following results hold:*
(1) If $\|Ax\| \leq \|x\|$, $x \in \partial D_K$, then $i(A, D_K, K) = 1$;
(2) If there exists $x_0 \in K\backslash\{\theta\}$ such that $x \neq Ax + \lambda x_0$, for all $x \in \partial D_K$ and all $\lambda > 0$, then $i(A, D_K, K) = 0$;

(3) Let U_K be open in X such that $\overline{U_K} \subset D_K$. If $i(A, D_K, K) = 1$ and $i(A, U_K, K) = 0$, then A has a fixed point in $D_K \backslash \overline{U}_K$.

The same results holds, if $i(A, D_K, K) = 0$ and $i(A, U_K, K) = 1$.
We define

$$K_\rho = \{u(t) \in K : \|u\| < \rho\}, \quad \Omega_\rho = \{u(t) \in K : \min_{\xi_{m-2} \le t \le T} u(t) < r\rho\}.$$

Lemma 2.10. *(see [11]) Ω_ρ defined above has the following properties:*
(a) $K_{r\rho} \subset \Omega_\rho \subset K_\rho$;
(b) Ω_ρ is open relative to K;
(c) $x \in \partial\Omega_\rho$ if and only if $\min_{\xi_{m-2} \le t \le T} x(t) = r\rho$;
(d) If $x \in \partial\Omega_\rho$, then $r\rho \le x(t) \le \rho$ for $t \in [\xi_{m-2}, T]$.

Now, for the convenience, we introduce the following notations. Let

$$f_{r\rho}^{\rho} = \min\left\{ \min_{\xi_{m-2} \le t \le T} \frac{f(t,u)}{\rho} : u \in [r\rho, \rho] \right\},$$

$$f_0^{\rho} = \max\left\{ \max_{0 \le t \le T} \frac{f(t,u)}{\rho} : u \in [0, \rho] \right\},$$

$$f^\alpha = \lim_{u \to \alpha} \sup \max_{0 \le t \le T} \frac{f(t,u)}{u}, \ f_\alpha = \lim_{u \to \alpha} \inf \min_{\xi_{m-2} \le t \le T} \frac{f(t,u)}{u}, \ (\alpha := \infty \ \ or \ \ 0^+),$$

$$m = d\left((\beta T + \gamma) \int_0^T (T-s) a(s) \nabla s \right)^{-1}, \tag{2.6}$$

$$M = d\left(\min\left\{ \beta\xi_{m-2} + \gamma, \beta \max\left\{ \sum_{i=1}^{m-2} a_i \xi_1, a_{m-2}\xi_{m-2} \right\} + \gamma \sum_{i=1}^{m-2} a_i \right\} \times \int_{\xi_{m-2}}^T (T-s) a(s) \nabla s \right)^{-1}. \tag{2.7}$$

Lemma 2.11. *If f satisfies the following conditions*

$$f_0^\rho \le m \ and \ \ u \ne Hu, for \ \ u \in \partial\ K_\rho, \tag{2.8}$$

then $i(H, K_\rho, K) = 1$.

Proof. By (2.6) and (2.8), we have for $\forall u \in \partial\ K_\rho$,

$$\begin{aligned} \| Hu \| &\le \frac{\beta T + \gamma}{d} \int_0^T (T-s) a(s) f(s, u(s)) \nabla s \\ &\le \frac{\rho m (\beta T + \gamma)}{d} \int_0^T (T-s) a(s) \nabla s \\ &= \rho = \|u\|. \end{aligned}$$

□

This implies that $\|Hu\| \leq \|u\|$ for $u \in \partial\, K_\rho$. By Lemma 2.5(1), we have

$$i(H, K_\rho, K) = 1.$$

Lemma 2.12. *If f satisfies the following conditions*

$$f_{r\rho}^{\rho} \geq Mr \quad and \quad u \neq Hu \; for \quad u \in \partial\Omega_\rho, \tag{2.9}$$

then $i(H, \Omega_\rho, K) = 0$.

The Proof is similar to Lemma 3.3 in [15], here we omit it.

3. Existence theorems of positive solutions

Main results are here

Theorem 3.1. *Assume $(H_1), (H_2)$ and (H_3) hold, and assume that one of the following conditions hold:*

(H_4) *There exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 < r\rho_2$ such that*

(1) $f(t, u) \geq 0,\ t \in [\xi_{m-2}, T], \quad u \in [r\rho_1, \rho_2]$;

(2) $f_0^{\rho_1} \leq m, \quad f_{r\rho_2}^{\rho_2} \geq Mr$;

(H_5) *There exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 < \rho_2$ such that*

(3) $f_0^{\rho_2} \leq m$;

(4) $f(t, u) \geq Mr\rho_1,\ t \in [\xi_{m-2}, T], \quad u \in [r^2\rho_1, \rho_2]$.

Then (1.1), (1.2) has a positive solution.

Proof. Assume that (H_4) holds. We show that H has a fixed point u_1 in $\Omega_{\rho_2} \backslash \overline{K}_{\rho_1}$. By Lemma 2.11, we have that

$$i(H, K_{\rho_1}, K) = 1.$$

By Lemma 2.12, we have that

$$i(H, K_{\rho_2}, K) = 0.$$

By Lemma 2.10 (a) and $\rho_1 < r\rho_2$, we have $\overline{K}_{\rho_1} \subset K_{r\rho_2} \subset \Omega_{\rho_2}$. It follows from Lemma 2.9(3) that A has a fixed point u_1 in $\Omega_{\rho_2} \backslash \overline{K}_{\rho_1}$, The proof is similar when H_5 holds, and we omit it here. The proof is complete. □

Theorem 3.2. *Assume $(H_1), (H_2)$ and (H_3) hold, and suppose that one of the following conditions holds:*

(H_8) *There exist $\rho_1, \rho_2, \rho_3 \in (0, +\infty)$ with $\rho_1 < r\rho_2$ and $\rho_2 < \rho_3$ such that*

(1) $f_0^{\rho_1} \leq m, \quad f_{r\rho_2}^{\rho_2} \geq Mr, \quad u \neq Hu, \forall\, u \in \partial\Omega_{\rho_2},$ *and* $f_0^{\rho_3} \leq m$;

(2) $f(t, u) \geq 0,\ t \in [\xi_{m-2}, T], \quad u \in [r\rho_1, \rho_3]$.

(H_9) *There exist* $\rho_1, \rho_2, \rho_3 \in (0, +\infty)$ *with* $\rho_1 < \rho_2 < r\rho_3$ *such that*

(1) $f_0^{\rho_2} \leq m, \quad f_{r\rho_1}^{\rho_1} \geq Mr, \quad u \neq Hu, \forall\, u \in \partial K_{\rho_2},$ *and* $f_{r\rho_3}^{\rho_3} \geq Mr;$

(2) $f(t,u) \geq 0, \quad t \in [\xi_{m-2}, T], \ u \in [r\rho_2, \rho_3],$ *and* $f(t,u) \geq Mr\rho_1,$ $t \in [\xi_{m-2}, T], \quad u \in [r^2\rho_1, \rho_2].$

Then (1.1), (1.2) has two positive solutions. Moreover, if $(H_8) f_0^{\rho_1} \leq m$ *is replaced by* $f_0^{\rho_1} < m$*, then (1.1), (1.2) has a third positive solution* $u_3 \in K_{\rho_1}$*.*

Proof. Assume (H_8) holds, we show that H has a fixed point u_1 either in ∂K_{ρ_1} or u_1 in $\Omega_{\rho_2} \setminus \overline{K_{\rho_1}}$. If $u \neq Hu, \ u \in \partial K_{\rho_1} \bigcup \partial K_{\rho_3}$, by Lemmas 2.11 and 2.12, we have

$$i(H, K_{\rho_1}, K) = 1, \quad i(H, \Omega_{\rho_2}, K) = 0, \quad i(H, K_{\rho_3}, K) = 1.$$

By Lemma 2.10(a) and $\rho_1 < r\rho_2$, we have $\overline{K}_{\rho_1} \subset K_{r\rho_2} \subset \Omega_{\rho_2}$. By Lemma 2.9(3), we have H has a fixed point $u_1 \in \Omega_{\rho_2} \backslash \overline{K_{\rho_1}}$. Similarly, H has a fixed point $u_2 \in K_{\rho_3} \backslash \overline{\Omega_{\rho_2}}$. Clearly,

$$\|u_1\| > \rho_1, \quad \min_{t \in [\xi_{m-2}, T]} u_1(t) \geq r\|u_1\| > r\rho_1.$$

This implies that $r\rho_1 \leq u_1(t) \leq \rho_2, \ t \in [\xi_{m-2}, T]$. By $(H_8)(2)$, we have $f(t, u_1(t)) \geq 0, \ t \in [\xi_{m-2}, T]$, i.e. $f^+(t, u_1(t)) = f(t, u_1(t))$. Hence, we can get $Hu_1 = Gu_1$. That means u_1 is a fixed point of G. From $u_2 \in K_{\rho_3} \backslash \overline{\Omega_{\rho_2}}, \ \rho_2 < \rho_3$ and Lemma 2.10(a) we have $\overline{K}_{r\rho_2} \subset \Omega_{\rho_2} \subset K_{\rho_3}$. Obviously, $\|u_2\| > r\rho_2$. This implies that

$$\min_{t \in [\xi_{m-2}, T]} u_2(t) \geq r\|u_2\| > r^2\rho_2.$$

Therefore,

$$r^2\rho_2 \leq u_2(t) \leq \rho_3, \ t \in [\xi_{m-2}, T].$$

By $\rho_1 < r\rho_2$ and $(H_8)(2)$, we have $f(t, u_2(t)) \geq 0, \ t \in [\xi_{m-2}, T]$, i.e. $f^+(t, u_2(t)) = f(t, u_2(t))$. So u_2 is another fixed point of G. Thus, we have proved that (1.1) and (1.2) has at least two positive solutions u_1 and u_2. The proof is similar when (H_9) holds and we omit it here. The proof is completed. □

Remark 3.1. If $f \in C([0,T] \times [0, +\infty), \ [0, +\infty))$, i.e., the nonlinear term f is positive. Theorem 3.1 and 3.2 improve Theorem 3.1 in [15]. When $m = 3$, the BVP (1.1) and (1.2) becomes the problem considered in [14].

4. Example

In this section, we present a simple example to explain our results.

Let $\mathbf{T} = \{(\frac{1}{2})^n : n \in N\} \bigcup \{1\}$, $T = 1$. Consider the following BVP on time scales

$$u^{\Delta\nabla}(t) + f(t, u(t)) = 0, \qquad t \in (0, T), \tag{4.1}$$

$$\frac{1}{2}u(0) - \frac{1}{3}u^{\Delta}(0) = 0, \quad u(T) = \frac{1}{4}u(\frac{1}{3}), \tag{4.2}$$

where

$$f(t,u):=f(u)=\begin{cases}\left(u-\frac{1}{12}\right)^3, & u\in[0,\frac{1}{12}];\\ \frac{1}{2}\sin(\frac{6\pi}{11}u-\frac{\pi}{22}), & u\in[\frac{1}{12},1];\\ \frac{1}{2}u, & u\in[1,2];\\ 1+\frac{1}{22^4}(u-2)^2, & u\in[2,24];\\ 1+\frac{1}{484}[1+(u-24)(30-u)], & u\in[24,+\infty).\end{cases}$$

It is easy to check that $f:[0,1]\times[0,+\infty)\longrightarrow(-\infty,+\infty)$ is continuous. In this case, $a(t)\equiv 1$, $\beta=\frac{1}{2}$, $\gamma=\frac{1}{3}$, $a_1=\frac{1}{4}$, $\xi_1=\frac{1}{3}$, and $m=3$, it follows from a direct calculation that

$$0<a_1\xi_1=\frac{1}{4}\cdot\frac{1}{3}=\frac{1}{12}<1,$$

$$d=\beta\left(T-\sum_{i=1}^{m-2}a_i\xi_i\right)+\gamma\left(1-\sum_{i=1}^{m-2}a_i\right)=\frac{1}{2}(1-\frac{1}{12})+\frac{1}{3}(1-\frac{1}{4})=\frac{17}{24}>0,$$

we can easily find that $(H_1),(H_2)$ and (H_3) hold, and

$$m=d\left((\beta T+\gamma)\int_0^T(T-s)a(s)\nabla s\right)^{-1}=\frac{17}{24}\left((\frac{1}{2}+\frac{1}{3})\int_0^1(1-s)ds\right)^{-1}=\frac{17}{10},$$

$$\begin{aligned}M&=d\left(\min\left\{\beta\xi_{m-2}+\gamma,\beta\max\left\{\sum_{i=1}^{m-2}a_i\xi_1,a_{m-2}\xi_{m-2}\right\}+\gamma\sum_{i=1}^{m-2}a_i\right\}\right.\\&\qquad\left.\times\int_{\xi_{m-2}}^T(T-s)a(s)\nabla s\right)^{-1}\\&=\frac{17}{24}\left(\min\left\{\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{3},\ \frac{1}{2}\cdot\frac{1}{12}+\frac{1}{3}\cdot\frac{1}{4}\right\}\int_{\frac{1}{3}}^1(1-s)ds\right)^{-1}\\&=\frac{51}{2},\end{aligned}$$

$$\begin{aligned}r&=\min\left\{\frac{a_{m-2}\left(T-\xi_{m-2}\right)}{T-a_{m-2}\xi_{m-2}},\ \frac{a_{m-2}\xi_{m-2}}{T},\frac{\xi_{m-2}}{T}\right\}\\&=\min\left\{\frac{\frac{1}{4}\left(1-\frac{1}{3}\right)}{1-\frac{1}{4}\cdot\frac{1}{3}},\ \frac{\frac{1}{4}\cdot\frac{1}{3}}{1},\ \frac{\frac{1}{3}}{1}\right\}\\&=\frac{1}{12}.\end{aligned}$$

Choose $\rho_1 = 1,\ \rho_2 = 24,\ \ \rho_3 = 30$, it is easy to check that $1 = \rho_1 < r\rho_2 = \dfrac{1}{12} \times 24 = 2 < \rho_2 < \rho_3,\ f(t,u) = f(u) \geq 0$, for $t \in [\frac{1}{3}, 1]$ and $u \in [\frac{1}{12} \cdot 1, 30]$, moreover,

$$\begin{aligned} f_0^{\rho_1} &= \max\left\{\max_{0\leq t\leq 1} \frac{f(t,u)}{1} :\ u \in [0,1]\right\} \\ &= \frac{1}{2} \leq m = \frac{17}{10}, \\ f_{r\rho_2}^{\rho_2} &= \min\left\{\min_{\frac{1}{3}\leq t\leq 1} \frac{f(t,u)}{24} :\ u \in [2,24]\right\} \\ &= \frac{\frac{1}{2}\cdot 2^{10}}{24} \approx 21.3333 \geq Mr = \frac{51}{2}\cdot\frac{1}{12} = 2.1250, \\ f_0^{\rho_3} &= \max\left\{\max_{0\leq t\leq 1} \frac{f(t,u)}{30} :\ u \in [0,30]\right\} \\ &= \frac{247}{7260} \leq m = \frac{17}{10}. \end{aligned}$$

By (2.5), we have

$$\begin{aligned} (Hu)(t) &= -\int_0^t (t-s)a(s)f^+(s,u(s))\nabla s \\ &\quad + \frac{\beta t + \gamma}{d}\int_0^T (T-s)a(s)f^+(s,u(s))\nabla s \\ &\quad - \frac{\beta t+\gamma}{d}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f^+(s,u(s))\nabla s \\ &= -\int_0^t (t-s)f^+(s,u(s))\nabla s + \frac{\frac{1}{2}t+\frac{1}{3}}{\frac{17}{24}}\int_0^1 (1-s)f^+(s,u(s))\nabla s \\ &\quad - \frac{\frac{1}{2}t+\frac{1}{3}}{\frac{17}{24}}\cdot\frac{1}{4}\cdot\int_0^{\frac{1}{3}} (\frac{1}{3}-s)f^+(s,u(s))\nabla s. \end{aligned}$$

Since

$$f(t,u) = f(u) \leq \frac{485}{484},\ t \in [0,1],\ u \in [0,24],\ \text{for } u \in \partial K_{24}.$$

We have

$$\|Hu\| \leq \frac{\frac{1}{2}+\frac{1}{3}}{\frac{17}{24}}\int_0^1 (1-s)f^+(s,u(s))\nabla s \leq \frac{2425}{4114} < 24 = \|u\|.$$

This implies $Hu \neq u$, for $u \in \partial\Omega_{24}$. Thus, (H_8) of Theorem 3.2 is satisfied. Then the BVP (4.1) and (4.2) has two positive solutions $u_1,\ u_2$ satisfying

$$\|u_1\| \leq 24, \qquad \|u_2\| > 24.$$

Remark 4.1. We note that Theorem 3.1 in [14, 15] can not apply to our example. Hence, we generalize the results[14, 15].

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