J. Appl. Math. & Informatics Vol. **35***(2017), No. 5 - 6, pp. 505 - 512*
http://dx.doi.org/10.14317/jami.2017.505

ON THE SOLUTIONS OF THE PROBLEM OF VISCOUS FLOW OVER SHRINKING SHEET

LAZHAR BOUGOFFA* AND ABDUL-MAJID WAZWAZ

Abstract. In this paper, we explain how the exact closed-form solutions of the classical problems of viscous fluid flow over a heated stretching plate and shrinking sheet can be obtained by a reliable method.

AMS Mathematics Subject Classification : 65L10.
Key words and phrases : Stretching sheet flow, boundary value problem, exact solution.

1. Introduction

The problem of the layer flow of a viscous incompressible fluid comes in three distinct models. The first model is induced by the stretching sheet was examined by Crane [1], where under suitable assumptions, the first model can be reduced to the solution of the well-known nonlinear third-order differential equation:

$$\mu y''' + yy'' - (y')^2 = 0,\ 0 < x < \infty, \tag{1}$$

subject to the boundary conditions

$$y(0) = 0,\ y'(0) = \nu_0,\ y'(x) \to 0 \text{ as } x \to \infty, \tag{2}$$

where μ is a constant and $\nu_0 \geq 0$ is the coefficient of kinematic velocity.
The exact solution in exponential forms of this boundary value problem was originally proposed by Crane [1].
Recently, an exact solution of this boundary value problem in an explicit form was given by Aziz and Mahomed [2] using the compatibility and generalized group method.
However, in the study of the second model of flow due to a shrinking sheet

Received February 12, 2017. Revised June 7, 2017. Accepted June 13, 2017. *Corresponding author.

with suction, which reduces to the well-known nonlinear third-order differential equations [3]

$$y''' + myy'' - (y')^2 = 0,\ 0 < x < \infty, \tag{3}$$

subject to the boundary conditions

$$y(0) = s,\ y'(0) = -1,\ y'(x) \to 0 \text{ as } x \to \infty, \tag{4}$$

where $m = 1$ when the sheet shrinks in the x direction only, where $m = 2$ when the sheet shrinks axisymmetrically.
The authors [3] presented a special exact solution when $s = \sqrt{\frac{6}{2m-1}}$ as $y(x) = \frac{s^2}{x+s}$.
A new second order slip velocity model was proposed by Wu [4] as

$$y''' + yy'' - (y')^2 = 0,\ 0 < x < \infty, \tag{5}$$

subject to the boundary conditions

$$y(0) = s,\ y'(0) = -1 + \gamma y''(0) + \delta y'''(0),\ y'(x) \to 0 \text{ as } x \to \infty, \tag{6}$$

where $\gamma > 0$ is the first order velocity slip parameter and $\delta < 0$ is the second order velocity slip parameter. In [5], the authors have considered Prs.(5)-(6) and have presented the solution as $y(x) = a + be^{-\beta x}$ to this problem with the newly proposed Wu's slip velocity model, where the constants $a,\ b$ and β are determined by the wall mass transfer parameter s, $\gamma > 0$ and $\delta < 0$.
Various analytical approximations methods have also been proposed to solve these problems [6, 7] and see the references therein.
In general, there is no evidence of understanting in these explained solutions.
The purpose of this paper is to overcome the general difficulty and then present the exact solutions of Prs.(1)-(2), (3)-(4) and (5)-(6) using a reliable method.

2. The exact solutions

The most important feature of the proposed method is that it reduces these type of nonlinear ordinary differential equations into new equations that can be easily handled. For illustration, the above problems will be discussed to emphasize the use of the proposed method.

2.1. The Crane's Model [1]**.** We begin our method by considering the Crane's model.
First, differentiating Eq.(1) with respect to x, we get

$$\mu y^{(4)} - y'y'' + yy''' = 0 \tag{7}$$

and from Eq.(1), we have

$$y = \frac{y'^2 - \mu y'''}{y''}. \tag{8}$$

Substituting Eq.(8) into Eq.(7) to get

$$\mu y^{(4)} - y'y'' - \mu\frac{(y''')^2}{y''} + \frac{y'''}{y''}(y')^2 = 0, \tag{9}$$

or

$$\mu\left(y^{(4)} - \frac{(y''')^2}{y''}\right) - y'y'' + \frac{y'''}{y''}(y')^2 = 0. \tag{10}$$

Dividing both sides of Eq.(10) by $(y'')^2$, we obtain

$$\mu\left(\frac{y^{(4)}}{(y'')^2} - \frac{(y''')^2}{(y'')^3}\right) - \frac{y'}{y''} + \frac{y'''}{(y'')^3}(y')^2 = 0. \tag{11}$$

We next use the following transformations [8]:

$$z(\eta) = y''(x) \text{ and } \eta = y'(x). \tag{12}$$

By the chain rule, we have

$$z'(\eta) = \frac{dz}{d\eta} = \frac{y'''}{y''} \tag{13}$$

and

$$z''(\eta) = \frac{d^2z}{d\eta^2} = \frac{y^{(4)}}{(y'')^2} - \frac{(y''')^2}{y''^3}. \tag{14}$$

The substitution of Eqs.(12)-(14) into Eq.(10) yields

$$\mu z^2(\eta)z''(\eta) - \eta z(\eta) + \eta^2 z'(\eta) = 0,\ 0 < \eta < \nu_0, \tag{15}$$

that is, Eq.(1) can be transformed into this new equation.
This equation can be easily solved using its characteristic polynomial. In a parallel manner as in the Cauchy-Euler equation, a trial solution $z = \eta^r$ may be used to solve this equation. Thus it is possible to get exact solutions to this differential equation. For this, we need the following lemma.

Lemma 2.1. *For the general nonlinear differential equation*

$$\alpha z^2 z'' + \beta\eta z + \gamma\eta^2 z' = 0, \tag{16}$$

where $\alpha > 0$, β *and* γ *are constants, we have*

(1) *If* $\beta + \frac{3}{2}\gamma < 0$, *then*

$$z = a_1\eta^{\frac{3}{2}},\ \eta > 0, \tag{17}$$

where $a_1 = \pm\sqrt{-\frac{4}{3\alpha}(\beta + \frac{3}{2}\gamma)}$.

(2) *If* $\beta + \frac{3}{2}\gamma > 0$, *then*

$$z = a_2(-\eta)^{\frac{3}{2}},\ \eta < 0, \tag{18}$$

where $a_2 = \pm\sqrt{\frac{4}{3\alpha}(\beta + \frac{3}{2}\gamma)}$.

(3) *If* $\beta + \gamma = 0$, *then all solutions are of the form*
- *For* $\eta > 0$ *and* $\gamma > 0$, $z = a_3\eta$, *where* a_3 *is an arbitrary constant.*
- *For* $\eta > 0$ *and* $\gamma < 0$, $z = a_3\eta$ *and* $z = a_4\eta^{\frac{3}{2}}$, *where* $a_4 = \pm\sqrt{-\frac{2\gamma}{3\alpha}}$.

- *For* $\eta < 0$ *and* $\gamma > 0$, $z = a_3\eta$ *and* $z = a_5(-\eta)^{\frac{3}{2}}$, *where* $a_5 = \pm\sqrt{\frac{2\gamma}{3\alpha}}$.
- *For* $\eta < 0$ *and* $\gamma < 0$, $z = a_3\eta$.

Proof. Let's start off by assuming that $\beta + \frac{3}{2}\gamma < 0$ and all solutions are of the form

$$z(\eta) = a\eta^r, \text{ for } \eta > 0, \tag{19}$$

where $a,\ r \in R$ are two constants to be determined.
Differentiating, we have:

$$z'(\eta) = ar\eta^{r-1},\ z''(\eta) = ar(r-1)\eta^{r-2}. \tag{20}$$

Substituting into the original equation, we have

$$a^2\alpha r(r-1)\eta^{3r-2} + \beta\eta^{r+1} + \gamma r\eta^{r+1} = 0. \tag{21}$$

The solution $z(\eta)$ may be found by setting $3r - 2 = r + 1$, that is $r = \frac{3}{2}$.
Hence,

$$\frac{3}{4}a^2\alpha + \beta + \frac{3}{2}\gamma = 0, \tag{22}$$

and $a = \pm\sqrt{-\frac{4}{3\alpha}(\beta + \frac{3}{2}\gamma)} = a_1$.
Now for $\beta + \frac{3}{2}\gamma > 0$, plug

$$z(\eta) = a(-\eta)^r, \text{ for } \eta < 0 \tag{23}$$

into the differential equation to get

$$\frac{3}{4}a^2\alpha - \beta - \frac{3}{2}\gamma = 0. \tag{24}$$

Thus $a = \pm\sqrt{\frac{4}{3\alpha}(\beta + \frac{3}{2}\gamma)} = a_2$.
So solutions will be of these forms provided $r = \frac{3}{2}$.
For the rest parts when $\beta + \gamma = 0$ the proof is similar. □

Applying Lemma 2.1 to Eq.(15), we obtain

$$z(\eta) = a\eta, \tag{25}$$

where $\alpha = \mu$, $\beta = -\gamma = -1$ and a is an arbitrary constant.
Returning to the original dependent variable $y(x)$ to get

$$y''(x) = ay'(x). \tag{26}$$

Thus

$$y' = ay + b. \tag{27}$$

Consequently, the solution is given by

$$y = -\frac{b}{a} + ce^{ax}, \tag{28}$$

where b and c are two constants of integration.
Thus, it is required to find a, b and c by using the given boundary conditions.
For $y(0) = 0$ and $y'(0) = \nu_0$, we have $b = \nu_0$ and $c = \frac{\nu_0}{a}$. Thus

$$y = -\frac{\nu_0}{a} + \frac{\nu_0}{a} e^{ax}. \tag{29}$$

We must now determine a by using the remaining boundary condition $y'(x) \to 0$ as $x \to \infty$.
Substituting Eq.(29) into Eq.(1), we obtain $a = \pm\sqrt{\frac{\nu_0}{\mu}}$. Thus $a = -\sqrt{\frac{\nu_0}{\mu}}$ is suitable for this condition.
Accordingly, the exact solution of Pr.(1)-(2) is therefore given as

$$y = \sqrt{\nu_0 \mu}\left(1 - e^{-\sqrt{\frac{\nu_0}{\mu}}x}\right), \tag{30}$$

which is the same exact solution that was obtained by Crane [1].

2.2. The flow due to a shrinking sheet with suction [3]**.** Consider Pr.(3)-(4).
Proceeding as before, Eq.(3) reduces to the new equation

$$z^2(\eta)z''(\eta) + (m-2)\eta z(\eta) + \eta^2 z'(\eta) = 0, \ -1 < \eta < 0, \tag{31}$$

which we can solve by Lemma 2.1, where $\alpha = 1$, $\beta = m - 2$ and $\gamma = 1$ obtain,

$$z(\eta) = a(-\eta)^r, \ -1 < \eta < 0. \tag{32}$$

where $a = \pm\sqrt{\frac{4m-2}{3}}$ and $r = \frac{3}{2}$. Then the solution to the differential equation is,

$$z(\eta) = a(-\eta)^{\frac{3}{2}}, \ a = \pm\sqrt{\frac{4m-2}{3}}. \tag{33}$$

Returning to the original dependent variable $y(x)$ to get

$$y'' = a(-y')^{\frac{3}{2}}. \tag{34}$$

Thus

$$y' = \frac{-4}{(ax+b)^2}, \tag{35}$$

where b is a constant of integration.
Consequently, the solution is given by

$$y = \frac{4}{a}\frac{1}{ax+b} + c. \tag{36}$$

Note that the substitution of this solution into the original equation leads to $c = 0$.

Thus, it is required to find a and b by the given boundary conditions $y(0) = s$ and $y'(0) = -1$, we obtain $b = \pm 2$ and $s = \sqrt{\frac{6}{2m-1}}$. Thus

$$y = \frac{4}{\sqrt{\frac{4m-2}{3}}} \frac{1}{\sqrt{\frac{4m-2}{3}}x + 2} = \frac{s^2}{x+s}. \tag{37}$$

2.3. The problem with a second order slip flow model [4, 5]**.** In Pr.(5)-(6) two cases are necessary due to the presence of the unknown parameters in this model.

2.3.1. $-1 + \gamma y''(0) + \delta y'''(0) > 0$**.** Proceeding as before we get $y = -\frac{b}{a} + ce^{ax}$. The substitution of this solution into Eq. (1) with the application of boundary conditions (6) give

$$a = -\sqrt{b},\ -\frac{b}{a} + c = s,\ c = -\frac{-1 + \gamma y''(0) + \delta y'''(0)}{\sqrt{b}}, \tag{38}$$

that is

$$a = -\sqrt{b},\ c = s - \sqrt{b},\ c = -\frac{-1 + \gamma y''(0) + \delta y'''(0)}{\sqrt{b}}. \tag{39}$$

Accordingly, the exact solution of Pr.(5)-(6) is therefore given as

$$y = s + \frac{-1 + \gamma y''(0) + \delta y'''(0)}{\sqrt{b}} - \frac{-1 + \gamma y''(0) + \delta y'''(0)}{\sqrt{b}} e^{-\sqrt{b}x}. \tag{40}$$

Setting $0 < \beta = \sqrt{b}$, we find

$$y'(0) = -c\beta,\ y''(0) = c\beta^2,\ y'''(0) = -c\beta^3,\ \text{where } c = s - \beta. \tag{41}$$

Thus $\beta > 0$ must satisfy the following algebraic equation

$$\delta\beta^4 - (\gamma + \delta s)\beta^3 + (\gamma s - 1)\beta^2 + s\beta - 1 = 0. \tag{42}$$

Consequently, the solution is determined by the wall mass transfer parameter s, the first order slip parameter γ and the second order slip parameter δ and the only positive roots $\beta > 0$ of this algebraic equation.

2.3.2. $-1 + \gamma y''(0) + \delta y'''(0) < 0$**.** Proceeding as before, a special exact solution can be obtained when

$$s = \sqrt{-6(1 + \gamma y''(0) + \delta y'''(0))} \tag{43}$$

as follows

$$y(x) = \frac{4}{a(x+b)}, \tag{44}$$

where

$$a = \pm\sqrt{\frac{2}{3}},\ b = \pm\frac{2}{\sqrt{-(-1 + \gamma y''(0) + \delta y'''(0))}}. \tag{45}$$

Thus

$$y''(0) = -\sqrt{\frac{2}{3}}(-1 + \gamma y''(0) + \delta y'''(0))\sqrt{-(-1 + \gamma y''(0) + \delta y'''(0))} \tag{46}$$

and

$$y'''(0) = (-1 + \gamma y''(0) + \delta y'''(0))^2. \tag{47}$$

Setting $\beta = \sqrt{-(-1 + \gamma y''(0) + \delta y'''(0))}$. Thus the application of the boundary condition $y'(0) = -1 + \gamma y''(0) + \delta y'''(0)$ yields to

$$\delta\beta^4 - \sqrt{\frac{2}{3}}\gamma\beta^3 - \beta^2 - 1 = 0. \tag{48}$$

Consequently, the solution is determined by the parameters s, γ, δ and the only positive roots $\beta > 0$ of this algebraic equation.

3. Conclusion

In this paper, we have considered three models of viscous flow over shrinking sheet. We have demonstrated that the closed-form solutions of these problems can be obtained in a straightforward manner by a direct method. This method is completely different from the one presented in the literature. Therefore, it can be regarded as a new technique to obtain the exact solutions of the viscous flow induced by a shrinking sheet.

References

1. L.J. Crane, *Flow past stretching plate*, Z. Angew. Math. Phys. **21** (1970), 645-647.
2. Taha Aziz, F.M. Mahomed, *Remark on classical Crane's solution of viscous flow past a stretching plate*, Appl. Math. Letters **52** (2016), 205-211.
3. M. Miklavcic and C.Y. Wang, *Viscous flow due to a shrinking sheet*, Q. Appl. Mathematics **64** (2006), 283-290.
4. L. Wu, *A slip model for rarefied gas flows at arbitrary Knudsen number*, Appl. Phys. Lett. **93**(2008), 253103.
5. L. Fang, S. Yao, J. Zhang, A. Aziz, *Viscous flow over a shrinking sheet with a second order slip flow model*, Commun. Nonlinear Sci. Numer. Simulat. **15** (2010) 1831-1842.
6. M. Sajid, T. Javed, T. Hayat, *MHD rotating flow of a viscous fluid over a shrinking surface*, Nonlinear Dynamics **51** (2008), 259-265.
7. M.R. Mohyuddin, T. Hayat, F.M. Mahomed, S. Asghar, A.M. Siddiqui, *On Solutions of Some Non-Linear Differential Equations Arising in Newtonian and Non-Newtonian Fluids*, Nonlinear Dynamics **35** (2004), 229-248.
8. L. Wang, *A new algorithm for solving classical Blasius equation*, Appl. Math. and Computation **57** (2004), 1-9.

Lazhar Bougoffa is working as Professor of applied mathematics at department of mathematics, Al-Imam University, Kingdom Of Saudi Arabia. His current research interests include functional analysis and its applications to PDEs., nonlinear systems analysis, nonlinear differential equations and inequalities.

Department of Mathematics, Al Imam Mohammad Ibn Saud Islamic University IMSIU, P.O. Box 90950, Riyadh 11623, Saudi Arabia.
e-mail: lbbougoffa@imamu.edu.sa

Abdul-Majid Wazwaz is a Professor of Mathematics at Saint Xavier University in Chicago, Illinois, USA. He has both authored and co-authored more than 500 papers in applied mathematics and mathematical physics. He is the author of five books on the subjects of discrete mathematics, integral equations and partial differential equations. Furthermore, he has contributed extensively to theoretical advances in solitary waves theory, the Adomian decomposition method and the variational iteration method.

Department of Mathematics, Saint Xavier University, Chicago, IL 60655, U.S.A.
e-mail: wazwaz@sxu.edu