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ON CARLITZ'S TYPE q-TANGENT NUMBERS AND POLYNOMIALS AND COMPUTATION OF THEIR ZEROS†

KYUNG-WON HWANG AND CHEON SEOUNG RYOO*

Abstract. In this paper we construct the Carlitz's type q-tangent numbers $T_{n,q}$ and polynomials $T_{n,q}(x)$. From these numbers and polynomials, we establish some interesting identities and relations.

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1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials(see [1-9]). In this paper, we define (p, q)-analogue of tangent polynomials and numbers and study some properties of the (p, q)-analogue of tangent polynomials and numbers.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -2, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

We remember that the classical tangent numbers T_n and tangent polynomials $T_n(x)$ are defined by the following generating functions(see [4])

$$\frac{2}{e^{2t}+1} = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!}, \quad (|2t| < \pi). \tag{1.1}$$

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and

$$\left(\frac{2}{e^{2t}+1}\right)e^{xt}=\sum_{n=0}^{\infty}T_n(x)\frac{t^n}{n!},\quad (|2t|<\pi). \tag{1.2}$$

respectively.

Some interesting properties of the classical tangent numbers and polynomials were first investigated by Ryoo[3, 4, 5]. Many kinds of of generalizations of these polynomials and numbers have been presented in the literature(see [3, 4, 5]). The q-number is defined by

$$[n]_q=\frac{1-q^n}{1-q}.$$

In particular, we can see $\lim_{q\to 1}[n]_q=n$.

By using q-number, we define the Carlitz's type q-tangent numbers and polynomials, which generalized the previously known numbers and polynomials, including the tangent numbers and polynomials. In the following section, we introduce the Carlitz's type q-tangent numbers and polynomials. After that we will investigate some their properties.

2. Carlitz's type q-tangent numbers and polynomials

In this section, we define the Carlitz's type q-tangent numbers and polynomials and provide some of their relevant properties.

Definition 2.1. For $|q|<1$, the Carlitz's type q-tangent numbers $T_{n,q}$ and polynomials $T_{n,q}(x)$ are defined by means of the generating functions

$$F_q(t)=\sum_{n=0}^{\infty}T_{n,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[2m]_q t}. \tag{2.1}$$

and

$$F_q(t,x)=\sum_{n=0}^{\infty}T_{n,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[2m+x]_q t}, \tag{2.2}$$

respectively.

Setting $q\to 1$ in (2.1) and (2.1), we can obtain the corresponding definitions for the tangent number T_n and tangent polynomials $T_n(x)$ respectively. By using above equation (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}T_{n,q}\frac{t^n}{n!}&=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[2m]_q t}\\&=\sum_{n=0}^{\infty}\left([2]_q\left(\frac{1}{1-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+q^{2l+1}}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, we have*

$$T_{n,q} = [2]_q \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{2l+1}}.$$

By (2.2), we obtain

$$T_{n,q}(x) = [2]_q \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} \frac{1}{1+q^{2l+1}}. \tag{2.4}$$

By using (2.2) and (2.4), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left([2]_q \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} \frac{1}{1+q^{2l+1}}\right) \frac{t^n}{n!} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[2m+x]_q t} \qquad (2.5) \\ &= [2]_q e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \left(\frac{1}{1-q}\right)^k \frac{(-1)^k q^{xk}}{1+q^{2k+1}} \frac{t^k}{k!}.\end{aligned}$$

Since $[x+2y]_q = [x]_q + q^x[2y]_q$, we see that

$$T_{n,q}(x) = [2]_q \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{xl} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \left(\frac{1}{1-q}\right)^l \frac{1}{1+q^{2k+1}}. \tag{2.6}$$

By using (2.6) and Theorem 2.2, we have the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_+$, we have*

$$\begin{aligned}T_{n,q}^{(h)}(x) &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{xl} T_{l,q} \\ &= (q^x T_q + [x]_q)^n \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [2m+x]_q^n,\end{aligned}$$

with the usual convention of replacing $(T_q)^n$ by $T_{n,q}$.

The following elementary properties of the Carlitz's type q-tangent numbers $T_{n,q}$ and polynomials $T_{n,q}(x)$ are readily derived form (2.1) and (2.2). We, therefore, choose to omit details involved.

Theorem 2.4. *(Distribution relation). For any positive integer m(=odd), we have*

$$T_{n,q}(x) = \frac{[2]_q}{[2]_{q^m}} [m]_q^n \sum_{a=0}^{m-1} (-1)^a q^a T_{n,q^m}\left(\frac{2a+x}{m}\right), n \in \mathbb{N}_0.$$

Theorem 2.5. *(Property of complement).*

$$T_{n,q^{-1}}(2-x)=(-1)^n q^n T_{n,q}(x)$$

By (2.1) and (2.2), we get

$$\begin{aligned}&-[2]_q\sum_{l=0}^{\infty}(-1)^{l+n}q^{l+n}e^{[2l+2n]_qt}+[2]_q\sum_{l=0}^{\infty}(-1)^l q^l e^{[2l]_qt}\\&=[2]_q\sum_{l=0}^{n-1}(-1)^l q^l e^{[2l]_qt}.\end{aligned}\tag{2.7}$$

Hence we have

$$\begin{aligned}&(-1)^{n+1}q^n\sum_{m=0}^{\infty}T_{m,q}(2n)\frac{t^m}{m!}+\sum_{m=0}^{\infty}T_{m,q}(2n)\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}\left([2]_q\sum_{l=0}^{n-l}(-1)^l q^l[2l]_q^m\right)\frac{t^m}{m!}.\end{aligned}\tag{2.8}$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of (2.8), we have the following theorem.

Theorem 2.6. *For $n\in\mathbb{Z}_+$, we have*

$$\sum_{l=0}^{n-l}(-1)^l q^l[2l]_q^m=\frac{(-1)^{n+1}q^nT_{mq}(2n)+T_{m,q}}{[2]_q}.$$

3. q-analogue of tangent zeta function

By using Carlitz's type q-tangent numbers and polynomials, q-tangent zeta function and Hurwitz q-tangent zeta functions are defined. These functions interpolate the Carlitz's type q-tangent numbers $T_{n,q}$, and polynomials $T_{n,q}(x)$, respectively. From (2.1), we note that

$$\begin{aligned}\left.\frac{d^k}{dt^k}F_q(t)\right|_{t=0}&=[2]_q\sum_{m=0}^{\infty}(-1)^n q^m[2m]_q^k\\&=T_{k,q},(k\in\mathbb{N}).\end{aligned}$$

By using the above equation, we are now ready to define q-tangent zeta functions.

Definition 3.1. Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$.

$$\zeta_{p,q}(s)=[2]_q\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{[2n]_q^s}.\tag{3.1}$$

Note that $\zeta_q(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $q\to 1$, then $\zeta_q(s)=\zeta_T(s)$ which is the tangent zeta functions(see [3]). Relation between $\zeta_q(s)$ and $T_{k,q}$ is given by the following theorem.

Theorem 3.2. *For $k \in \mathbb{N}$, we have*

$$\zeta_{p,q}(-k) = T_{k,p,q}.$$

Observe that $\zeta_q(s)$ function interpolates $T_{k,q}$ numbers at non-negative integers.

By using (2.2), we note that

$$\left.\frac{d^k}{dt^k} F_q(t,x)\right|_{t=0} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [2m+x]_q^k \tag{3.2}$$

and

$$\left.\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} T_{n,q}(x)\frac{t^n}{n!}\right)\right|_{t=0} = T_{k,q}(x), \text{ for } k \in \mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), we are now ready to define the Hurwitz q-tangent zeta functions.

Definition 3.3. Let $s \in \mathbb{C}$ with $\text{Re}(s) > 0$ and $x \notin \mathbb{Z}_0^-$.

$$\zeta_{p,q}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{[2n+x]_q^s}. \tag{3.4}$$

Note that $\zeta_q(s,x)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $q \to 1$, then $\zeta_q(s,x) = \zeta_T(s,x)$ which is the Hurwitz tangent zeta functions(see [3]). Relation between $\zeta_q(s,x)$ and $T_{k,q}(x)$ is given by the following theorem.

Theorem 3.4. *For $k \in \mathbb{N}$, we have*

$$\zeta_q(-k,x) = T_{k,q}(x).$$

Observe that $\zeta_q(-k,x)$ function interpolates $T_{k,q}(x)$ numbers at non-negative integers.

4. Carlitz's type q-tangent numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$

Throughout this section we use the notation: $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

Kim[1] defined the p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} g(x)(-q)^x. \tag{4.1}$$

From (4.1), we note that

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(g) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l), \tag{4.2}$$

where $g_n(x) = g(x+n)$ for $n \in \mathbb{N}$.

For $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, the Carlitz's type q-tangent are defined by

$$T_{n,q} = \int_{\mathbb{Z}_p} [2x]_q^n d\mu_{-q}(x). \tag{4.3}$$

By using p-adic q-integral on $\mathbb{Z}_p$, we obtain,

$$\begin{aligned}\int_{\mathbb{Z}_p} [2x]_q^n d\mu_{-q}(x) &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} [2x]_q^n (-q)^x \\ &= [2]_q \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{2l+1}} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [2m]_q^n.\end{aligned} \tag{4.4}$$

By using (2.1) and (4.3), we have

$$\sum_{n=0}^{\infty} T_{n,q} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [2x]_q^n d\mu_{-q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{[2x]_q t} d\mu_{-q}(x). \tag{4.5}$$

By (2.1), (4.5), we have

$$\int_{\mathbb{Z}_p} e^{[2x]_q t} d\mu_{-1}(x) = 2 \sum_{m=0}^{\infty} (-1)^m q^m e^{[2m]_q t}.$$

The Carlitz's type q-tangent polynomials $T_{n,q}(x)$ are defined by

$$T_{n,q}(x) = \int_{\mathbb{Z}_p} [2y+x]_q d\mu_{-q}(y). \tag{4.6}$$

By using p-adic q-integral on $\mathbb{Z}_p$, we obtain

$$\begin{aligned}\int_{\mathbb{Z}_p} [2y+x]_q^n d\mu_{-q}(y) &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{y=0}^{p^N-1} [2y+x]_q^n (-q)^y \\ &= [2]_q \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} \frac{1}{1+q^{2l+1}} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [2m+x]_q^n.\end{aligned} \tag{4.7}$$

From (4.2), we note that

$$\begin{aligned}[2]_q &= q\int_{\mathbb{Z}_p} e^{[2x+2]_q t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} e^{[2x]_q t} d\mu_{-q}(x) \\ &= \sum_{n=0}^{\infty}\left(q\int_{\mathbb{Z}_p}[2x+2]_q^n d\mu_{-q}(x) + \int_{\mathbb{Z}_p}[2x]_q^n d\mu_{-q}(x)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(qT_{n,q}(2)+T_{n,q}\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 4.1. *For $n \in \mathbb{N}_0$, we have*

$$qT_{n,q}(2) + T_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

By Theorem 4.1, we have the following corollary.

Corollary 4.2. *For $n \in \mathbb{N}_0$, we have*

$$q\left(q^2T_q + [2]_q\right)^n + T_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention of replacing $(T_q)^n$ by $T_{n,q}$.

5. Zeros of the Carlitz's type q-tangent numbers and polynomials

In this section, we investigate the zeros of the Carlitz's type q-tangent polynomials $T_{n,q}(x)$. We investigate the beautiful zeros of the $T_{n,q}(x)$ by using a computer. We plot the zeros of the Carlitz's type q-tangent polynomials $T_{n,q}(x)$ for $n = 30, q = 1/2, 1/3, 1/4, 1/5$ and $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 30$ and $q = 1/2$. In Figure 1(top-right), we choose $n = 30$ and $q = 1/3$. In Figure 1(bottom-left), we choose $n = 30$ and $q = 1/4$. In Figure 1(bottom-right), we choose $n = 30$ and $q = 1/5$. Stacks of zeros of $T_{n,1/2}(x)$ for $1 \leq n \leq 30$ from a 3-D structure are presented(Figure 2). The plot of real zeros of $T_{n,q}(x)$ for $1 \leq n \leq 30$ and $q = 1/2, 1/5$ structure are presented(Figure 3). In Figure 3(left), we choose $1 \leq n \leq 30$ and $q = 1/2$. In Figure 3(right), we choose $1 \leq n \leq 30$ and $q = 1/3$. Our numerical results for approximate solutions of real zeros of $T_{n,q}(x)$ are displayed(Tables 1, 2).

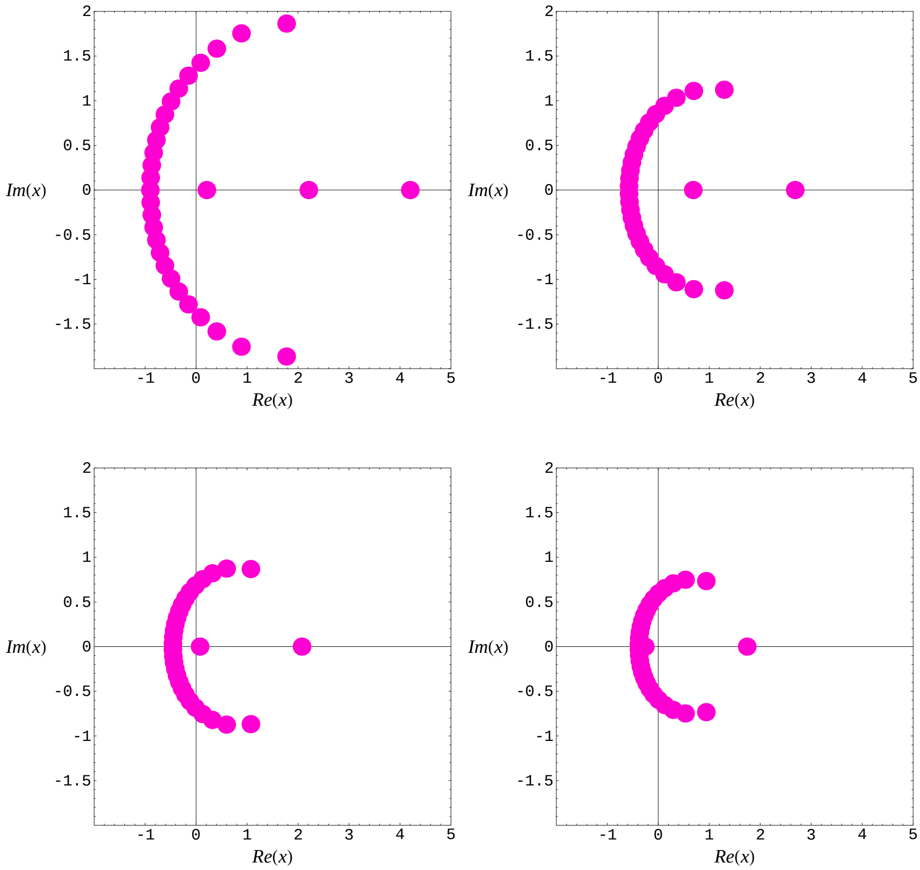

FIGURE 1. Zeros of $T_{n,q}(x)$

Table 1. Numbers of real and complex zeros of $T_{n,q}(x)$

degree n	$q = 1/2$		$q = 1/5$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	2	0
3	1	2	1	2
4	2	2	2	2
5	3	2	1	4
6	2	4	2	4
7	3	4	1	6
8	2	6	2	6
9	3	6	1	8
10	2	8	2	8

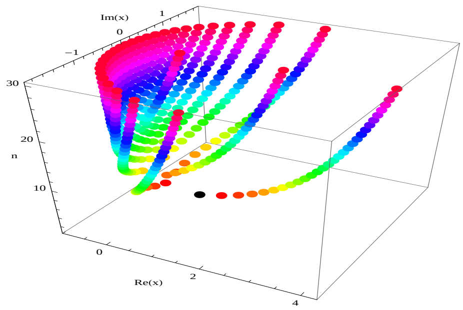

FIGURE 2. Stacks of zeros of $T_{n,1/2}(x)$ for $1 \leq n \leq 30$

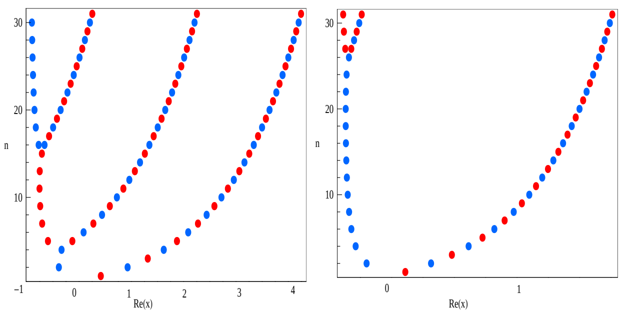

FIGURE 3. Real zeros of $T_{n,q}(x)$ for $1 \leq n \leq 30$

We observe a remarkably regular structure of the complex roots of the Carlitz's type q-tangent polynomials $T_{n,q}(x)$. We hope to verify a remarkably regular structure of the complex roots of the Carlitz's type q-tangent polynomials $T_{n,q}(x)$(Table 1). Next, we calculated an approximate solution satisfying $T_{n,q}(x), q = 1/2, x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $T_{n,q}(x) = 0, x \in \mathbb{R}$

degree n	x
1	0.415037
2	−0.386849, 0.927418
3	1.31328
4	−0.335264, 1.61879
5	−0.595106, −0.128945, 1.87113

Finally, we consider the more general problems. In general, how many zeros does $T_{n,q}(x)$ have? We are not able to decide if $T_{n,q}(x) = 0$ has n distinct solutions. We would like to know the number of complex zeros $C_{T_{n,q}(x)}$ of $T_{n,q}(x), Im(x) \neq 0$. Since n is the degree of the polynomial $T_{n,q}(x)$, the number of real zeros $R_{T_{n,q}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{T_{n,q}(x)} = n - C_{T_{n,q}(x)}$, where $C_{T_{n,q}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{T_{n,q}(x)}$ and $C_{T_{n,q}(x)}$. The reader may refer to [2, 3, 6, 7] for the details.

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Kyung-Won Hwang received Ph.D. degree from University of Illinois at Urbana-Champaign. His research interests focus on the combinatorics, graph theory,discrete geometry, scientific computing and p-adic functional analysis.

Department of Mathematics, Dang-A University, Pusan 49315, Korea
e-mail:khwang@dau.ac.kr

Cheon Seoung Ryoo received Ph.D. degree from Kyushu University. His research interests focus on the numerical verification method, scientific computing and p-adic functional analysis.

Department of Mathematics, Hannam University, Daejeon, 306-791, Korea
e-mail:ryoocs@hnu.kr