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RELATIONS OF DAGUM DISTRIBUTION BASED ON DUAL GENERALIZED ORDER STATISTICS

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Abstract. The dual generalized order statistics is a unified model which contains the well known decreasingly ordered random variables like order statistics and lower record values. With this definition we give simple expressions for single and product moments of dual generalized order statistics from Dagum distribution. The results for order statistics and lower records are deduced from the relations derived and some computational works are also carried out. Further, a characterizing result of this distribution on using the conditional moment of the dual generalized order statistics is discussed. These recurrence relations enable computation of the means, variances and covariances of all order statistics for all sample sizes in a simple and efficient manner. By using these relations, we tabulate the means, variances, skewness and kurtosis of order statistics and record values of the Dagum distribution.

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1. Introduction

The Dagum distribution was introduced by Dagum (1977) it is also called the inverse Burr XII distribution. The Burr XII distribution is widely known in various fields of science, the Dagum distribution is not much popular, perhaps, because of its difficult mathematical tractability. Dagum proposed his model as income distribution, its properties have been appreciated in economics and financial fields and its features have been extensively discussed in the studies of income and wealth. For more details and its applications on this distribution one may refer to Kleiber and Kotz (2003) and Kleiber (2008).

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A random variable X is said to have Dagum distribution if its probability density function (*pdf*) is given by

$$f(x) = \alpha\beta\sigma x^{-(\sigma+1)}(1+\alpha x^{-\theta})^{-(\beta+1)},\ x>0,\ \alpha,\beta,\sigma>0 \tag{1}$$

and the corresponding cumulative distribution function (*cdf*) is

$$F(x) = (1+\alpha x^{-\sigma})^{-\beta},\ x>0,\ \alpha,\beta,\sigma>0. \tag{2}$$

Therefore, in view of (1) and (2), we have

$$\alpha\beta\sigma F(x) = x(\alpha + x^{\sigma})f(x). \tag{3}$$

Here α is the scale parameter, while β and σ are shape parameters. For $\beta = 1$, the above distribution corresponds to the log-logistic distribution. The Dagum distribution has positive asymmetry, and it is unimodal for $\beta\sigma > 1$ and zero-modal for $\beta\sigma \leq 1$. The relation (3) will be used to derive some simple relations for the single and product moments of DGOS from the Dagum distribution. These recurrence relations will enable one to obtain all the single and product moments in a simple recursive manner.

The concept of generalized order statistics GOS was introduced by Kamps (1995) as a general framework for models of ordered random variables. Moreover, many other models of ordered random variables, such as, order statistics, k-th upper record values, upper record values, progressively Type II censoring order statistics, Pfeifer records and sequential order statistics are seen to be particular cases of GOS. These models can be effectively applied, e.g., in reliability theory. However, random variables that are decreasingly ordered cannot be integrated into this framework. Consequently, this model is inappropriate to study, e.g. reversed ordered order statistic and lower record values models. Burkschat *et al.* (2003) introduced the concept of dual generalized order statistics ($DGOS$). The $DGOS$ models enable us to study decreasingly ordered random variables like reversed order statistics, lower k record values and lower Pfeirfer records, through a common approach below:

Suppose $X_d(1,n,m,k),\ldots,X_d(n,n,m,k)$, ($k \geq 1$, m is a real number), are n $DGOS$ from an absolutely continuous cumulative distribution function *cdf* $F(x)$ with probability density function *pdf* $f(x)$, if their joint *pdf* is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i)\right)[F(x_n)]^{k-1}f(x_n), \tag{4}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.
where $\gamma_j = k + (n-j)(m+1) > 0$ for all j, $1 \leq j \leq n$, k is a positive integer and $m \geq -1$.
If $m = 0$ and $k = 1$, then this model reduces to the $(n-r+1)$-th order statistic, from the sample $X_1, X_2, \ldots, X_n$ and (4) will be the joint *pdf* of n order statistics.

If $k = 1$ and $m = -1$, then (4) will be the joint *pdf* of the first n record values of the identically and independently distributed (iid) random variables with *cdf* $F(x)$ and corresponding *pdf* $f(x)$.
In view of (4), the marginal *pdf* of the rth $DGOS$, is given by

$$f_{X_d(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)). \tag{5}$$

The joint *pdf* of r-th and s-th $DGOS$, is

$$f_{X_d(r,n,m,k),X_d(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\cdot[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y),\ x > y, \tag{6}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}x^{m+1},\ m\neq -1 \\ -lnx,\ m=-1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1),\ x \in [0,1).$$

Order statistics and functions of these statistics play an important role in a wide range of theoretical and practical problems such as characterization of probability distributions and goodness-of-fit tests, entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials; see Arnold et al. (1992) and David and Nagaraja (2003) and the references therein for more details. The practicability of moments of order statistics can be seen in many areas such as quality control testing, reliability, etc. For instance, when the reliability of an item or product is high, the duration of the failed items will be high which in turn will make the product too expensive, both in terms of time and money. This fact prevents a practitioner from knowing enough about the product in a relatively short time. Therefore, a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictions are often based on moments of order statistics.

The theory of DGOS and their distributional properties has been extensively studied in statistics. See Pawlas and Szynal (2001), Ahsanullah (2004, 2005), Mbah and Ahsanullah (2007). Khan and Kumar (2010), AL-Hussaini et al. (2005), Kumar (2015a, 2015b), for reviews on various developments in the area of gos.

The only paper we were able to find on DGOS of the Dagum distribution is Domma *et al*. (2011). This paper gives some estimation based on maximum likelihood in Dagum distribution with censored samples. Kumar (2016) obtained the explicit expression and recurrence relation for kth record values from Dagum distribution. They did not consider moments of DGOS.

This paper is organized as follows: we describe some technical lemmas in section 2. In Section 3, explicit expressions and some recurrence for single and product moments of DGOS from Dagum distribution are presented. Then we show that results for order statistics and record values are deduced as special cases. In section, we prove a characterization result on Dagum distribution based on conditional moment of DGOS. The explicit expression in the case of order statistics and record values for $n = 1(1)5$ (see Tables 1, 2, 3, 4, 5, 6, 7 and 8) are shown computationally in section 5. Some final comments in section 6 conclude the papaer.

2. Technical lemmas

Here, we present and prove four technical lemmas.

Lemma 2.1. *For Dagum distribution as given in (2) and any non-negative and finite integers a and b*

$$J_j(a,0) = \beta\alpha^{j/\sigma}\sum_{p=0}^{\infty}\frac{(j/\sigma)_{(p)}}{p![\beta(a+1)+p+(j/\sigma)]},\ \sigma > j,\ j = 0,1,\ldots, \tag{7}$$

where

$$(\alpha)_{(i)} = \begin{cases}\alpha(\alpha+1)\ldots(\alpha+i-1),\ i>0\\ 1,\ i=0\end{cases}.$$

and

$$J_j(a,b) = \int_0^{\infty} x^j[F(x)]^a f(x) g_m^b(F(x))dx. \tag{8}$$

Proof. From (8), we have

$$\begin{aligned}
J_j(a,0) &= \int_0^{\infty} x^j[F(x)]^a f(x)dx \\
&= \beta\alpha^{j/\sigma}\int_0^1 (1-z)^{-j/\sigma} z^{\beta(a+1)+(j/\sigma)-1}dz \\
&= \beta\alpha^{j/\sigma}\sum_{p=0}^{\infty}\frac{(j/\beta)_{(p)}}{p!}\int_0^1 z^{\beta(a+1)+(j/\sigma)+p-1}dz,
\end{aligned}$$

where $z = [\bar{F}(x)]^{1/\beta}$. The proof is complete. □

Lemma 2.2. *For Dagum distribution as given in (2) and any non-negative and finite integers a and b*

$$\begin{aligned}
J_j(a,b) &= \frac{1}{(m+1)^b}\sum_{u=0}^{b}(-1)^u\binom{b}{u}J_j(a+u(m+1),0) \qquad (9)\\
&= \frac{\beta\alpha^{j/\sigma}}{(m+1)^b}\sum_{p=0}^{\infty}\sum_{u=0}^{b}(-1)^u\binom{b}{u}
\end{aligned}$$

$$\cdot \quad \frac{(j/\sigma)_{(p)}}{p![\beta\{a+(m+1)u+1\}+p+(j/\sigma)]}, \ m \neq -1 \qquad (10)$$

$$= \quad b!\beta^{b+1}\alpha^{j/\alpha}\sum_{p=0}^{\infty}\frac{(j/\sigma)_{(p)}}{p![\beta(a+1)+p(j/\sigma)]^{b+1}}, \ m=-1. \qquad (11)$$

where $J_j(a,b)$ *is as given in (8).*

Proof. On expanding $g_m^b(F(x)) = \left[\frac{1}{m+1}(1-(F(x))^{m+1})\right]^b$ binomially in (8), we get when $m \neq -1$

$$\begin{aligned} J_j(a,b) &= \frac{1}{(m+1)^b}\sum_{u=0}^{b}(-1)^u \binom{b}{u}\int_0^{\infty} x^j [F(x)]^{a+u(m+1)} f(x)dx \\ &= \frac{1}{(m+1)^b}\sum_{u=0}^{b}(-1)^u \binom{b}{u} J_j(a+u(m+1),0). \end{aligned}$$

Making use of Lemma 2.1, we establish the result given in (10)
and when $m=-1$ that
$J_j(a,b) = \frac{0}{0}$ as $\sum_{u=0}^{b}(-1)^u \binom{b}{u} = 0.$
Since (10) is of the form $\frac{0}{0}$ at $m=-1$, therefore, we have

$$J_j(a,b) = A\sum_{u=0}^{b}(-1)^u \binom{b}{u} \frac{[\beta\{a+u(m+1)+1\}+p+(j/\sigma)]^{-1}}{(m+1)^b}, \qquad (12)$$

where

$$A = \beta\alpha^{j/\sigma}\sum_{p=0}^{\infty}\frac{(j/\beta)_{(p)}}{p!}.$$

Differentiating numerator and denominator of (12) b times with respect to m, we get

$$J_j(a,b) = A\beta^b\sum_{u=0}^{b}(-1)^{u+b}\binom{b}{u}\frac{u^b}{[\beta\{a+u(m+1)+1\}+p+(j/\sigma)]^{b+1}}.$$

On applying the L' Hospital rule, we have

$$lim_{m\to -1}J_j(a,b) = A\beta^b\sum_{u=0}^{b}(-1)^{u+b}\binom{b}{u}\frac{u^b}{[\beta(a+1)+p+(j/\sigma)]^{b+1}}. \qquad (13)$$

But for all integers $n \geq 0$ and for all real numbers x, we have Ruiz (1996)

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(x-i)^n = n!. \qquad (14)$$

Therefore,

$$\sum_{u=0}^{b}(-1)^{u+b}\begin{pmatrix} b \\ u \end{pmatrix} u^b = b!. \tag{15}$$

Now on substituting (15) in (13), we have the result given in (11). □

Lemma 2.3. *For Dagum distribution as given in (2) and any non-negative integers a, b, c with* $m \neq -1$

$$\begin{aligned} J_{i,j}(a,0,c) &= \beta^2\alpha^{(i+j)/\sigma}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(j/\sigma)_{(p)}(j/\sigma)_{(q)}}{p!\ q![\beta(c+1)+p+(j/\sigma)]} \\ &\cdot \frac{1}{[\beta(a+c+2)+p+q+\{(i+j)/\sigma\}+p+q]}, \end{aligned} \tag{16}$$

where

$$J_{i,j}(a,b,c) \tag{17}$$

$$= \int_0^\infty \int_x^\infty x^i y^j [F(x)]^a f(x)[h_m(F(y)) - h_m(F(x))]^b [F(y)]^c f(y) dy dx. \tag{18}$$

Proof. From (17), we have

$$J_{i,j}(a,0,c) = \int_0^\infty x^i [F(x)]^a f(x) G(x) dx, \tag{19}$$

where

$$G(x) = \int_x^\infty y^j [\bar{F}(y)]^c f(y) dy. \tag{20}$$

By setting $z = [\bar{F}(y)]^{1/\beta}$ in (19), we find that

$$G(x) = \beta\alpha^{j/\sigma}\sum_{p=0}^{\infty}\frac{(j/\sigma)_p [F(x)]^{c+1+\{p+(j/\sigma)\}/\beta}}{p![\beta(c+1)+p+(j/\sigma)]}.$$

On substituting the above expression of $G(x)$ in (18), we get

$$\begin{aligned} J_{i,j}(a,0,c) &= \beta\alpha^{j/\sigma}\sum_{p=0}^{\infty}\frac{(j/\sigma)_p}{p![\beta(c+1)+p+(j/\sigma)]} \\ &\cdot \int_0^\infty x^i [\bar{F}(x)]^{a+c+1+\{p+(j/\sigma)\}/\beta} f(x) dx. \end{aligned} \tag{21}$$

Again by setting $t = [F(x)]^{1/\beta}$ in (20) and simplifying the resulting expression, we derive the relation given in (16). □

Lemma 2.4. *For the distribution as given in (2) and any non-negative integers a, b, c and* $m \neq 1$

$$\begin{aligned} J_{i,j}(a,b,c) &= \frac{1}{(m+1)^b}\sum_{v=0}^{b}(-1)^v\begin{pmatrix} b \\ v \end{pmatrix} \\ &\cdot J_{i,j}(a+(b-v)(m+1), 0, c+v(m+1)) \end{aligned} \tag{22}$$

$$= \frac{\beta^2\sigma^{(i+j)/\sigma}}{(m+1)^b}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{v=0}^{b}(-1)^v\begin{pmatrix} b \\ v \end{pmatrix}$$

$$\cdot \frac{(j/\sigma)_p}{p!\ q![\beta\{c+(m+1)v+1\}+p+(j/\sigma)]}$$

$$\cdot \frac{(i/\sigma)_{(q)}}{[\beta\{a+c+(m+1)b+2\}+p+q+\{(i+j)/\sigma\}]} \tag{23}$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{b!\ \beta^{b+2}\sigma^{(i+j)/\sigma}(j/\sigma)_{(p)}}{p!q![\beta(c+1)+(j/\sigma)+p]^{b+1}}$$

$$\cdot \frac{(i/\sigma)_{(q)}}{[\beta(a+c+2)+p+q+\{(i+j)/\sigma\}]},\ m=-1 \tag{24}$$

where $J_{i,j}(a,b,c)$ is as given in (17).

Proof. When $m \neq -1$, we have

$$[h_m(F(y))-h_m(F(x))]^b = \frac{1}{(m+1)^b}[(F(x))^{m+1}-(F(y))^{m+1}]^b$$

$$= \frac{1}{(m+1)^b}\sum_{v=0}^{b}(-1)^v\begin{pmatrix} b \\ v \end{pmatrix}[F(y)]^{v(m+1)}[F(x)]^{(b-v)(m+1)}.$$

Now substituting for $[h_m(F(y))-h_m(F(x))]^b$ in (17), we get

$$J_{i,j}(a,b,c) = \frac{1}{(m+1)^b}\sum_{v=0}^{b}(-1)^v\begin{pmatrix} b \\ v \end{pmatrix}J_{i,j}(a+(b-v)(m+1),0,c+v(m+1)).$$

Making use of the Lemma 2.3, we derive the relation given in (22).

When $m=-1$, we have

$J_{i,j}(a,b,c) = \frac{0}{0}$ as $\sum_{v=0}^{b}(-1)^v\begin{pmatrix} b \\ v \end{pmatrix} = 0.$

On applying L' Hospital rule, (23) can be proved on the lines of (11). □

3. Momemts of dual generalized order statistics

In this section, we derive explicit expressions and recurrence relations for single and product moments of DGOS from the Dagum distribution.

3.1. Relations for single moments. The single moments of DGOS are very important to calculate the mean and variance of order statistics and record values. In the following, we derive the single moments of DGOS from the Dagum distribution.

Theorem 3.1. *For Dagum distribution as given in (2) and $1 \le r \le n$, $k = 1, 2, \ldots$ and $m \neq -1$*

$$E[X_d^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}J_j(\gamma_r-1,r-1)$$

$$= \frac{\beta\alpha^{j/\sigma}C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u\begin{pmatrix} r-1 \\ u \end{pmatrix}$$

$$\cdot \quad \frac{(j/\sigma)_(p)}{p![\beta\gamma_{r-u}+p+(j/\sigma)]}, \ \beta > j, \ j = 0, 1, 2, \ldots, \quad (25)$$

where $J_j(\gamma_r - 1, r-1)$ *is as defined in (8).*

Proof. From (5) and (8), we have

$$E[X_d^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}J_j(\gamma_r - 1, r-1). \quad (26)$$

Making use of Lemma 2.1, we establish the result given in (17).

□

Remark 3.1. Putting $m = 0$, $k = 1$ in (17), the explicit formula for the single moments of order statistics of the Dagum distribution can be obtained as

$$E[X_{n-r+1:n}^j] = C_{r:n}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u\begin{pmatrix} r-1 \\ u \end{pmatrix}\frac{\beta\alpha^{j/\sigma} \ (j/\sigma)_{(p)}}{p![\beta(n-r+u+1)+p+(j/\sigma)]}.$$

That is

$$E[X_{r:n}^j] = C_{r:n}\sum_{p=0}^{\infty}\sum_{u=0}^{n-r}(-1)^u\begin{pmatrix} n-r \\ u \end{pmatrix}\frac{\beta\alpha^{j/\sigma} \ (j/\sigma)_{(p)}}{p![\beta(r+u)+p+(j/\sigma)]},$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.$$

Remark 3.2. Putting $m = -1$ in (17), we deduce the explicit expression for the single moments of lower record values for Dagum distribution in view of (16) and (11) in the form

$$E[X_d^j(r,n,-1,k)] \quad = \quad E[(Z_r^{(k)})^j] = (\beta k)^r\alpha^{j/\sigma}\sum_{p=0}^{\infty}\frac{(j/\sigma)_{(p)}}{p![\beta k+p+(j/\sigma)]^r}$$

and hence for lower records

$$E[(Z_r^{(1)})^j] = E[X_{U(r)}^j] = \beta^r\alpha^{j/\sigma}\sum_{p=0}^{\infty}\frac{(j/\sigma)_{(p)}}{p![\beta+p+(j/\sigma)]^r},$$

as obtained Kumar (2016).

Recurrence relations for single moments of DGOS from (2) can be obtained in the following theorem.

Theorem 3.2. *For the distribution given in (2) and $2 \leq r \leq n$, $n \geq 2$ and $k = 1, 2, \ldots,$*

$$\begin{aligned} \frac{1}{\alpha} E[X_d^{j+\sigma}(r,n,m,k)] &= \frac{\beta\sigma\gamma_r}{j} E[X_d^j(r-1,n,m,k)] \\ &- \left(1 + \frac{\beta\sigma\gamma_r}{j}\right) E[X_d^j(r,n,m,k)]. \end{aligned} \tag{27}$$

Proof. From (4), we have

$$E[X_d^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx. \tag{28}$$

Integrating by parts treating $[F(x)]^{\gamma_r-1} f(x)$ for integration and rest of the integrand for differentiation, we get

$$\begin{aligned} E[X_d^j(r,n,m,k)] &= E[X_d^j(r-1,n,m,k)] \\ &- \frac{jC_{r-1}}{\gamma_r (r-1)!} \int_0^\infty x^{j-1} [F(x)]^{\gamma_r} g_m^{r-1}(F(x)) dx, \end{aligned}$$

the constant of integration vanishes since the integral considered in (19) is a definite integral. On using (3), we obtain

$$\begin{aligned} E[X_d^j(r,n,m,k)] &= E[X_d^j(r-1,n,m,k)] - \frac{jC_{r-1}}{\alpha\beta\sigma\gamma_r(r-1)!} \\ &\cdot \Bigg[\int_0^\infty x^{j+\sigma} [F(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) dx \\ &- \alpha \int_0^\infty x^j [F(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x)) dx \Bigg], \end{aligned}$$

and hence the result. □

Remark 3.3. Putting $m = 0$, $k = 1$ in (18), we obtain a recurrence relation for single moments of order statistics of the Dagum distribution in the form

$$\begin{aligned} \frac{1}{\alpha} E\left(X_{n-r+1:n}^{j+\sigma}\right) &= \frac{\beta\sigma(n-r+1)}{j} E\left(X_{n-r-2:n}^j\right) \\ &- \left(1 + \frac{\beta\sigma(n-r+1)}{j}\right) E\left(X_{n-r+1:n}^j\right). \end{aligned}$$

That is

$$\frac{1}{\alpha} E\left(X_{r:n}^{j+\sigma}\right) = \frac{\beta\sigma(r-1)}{j} E\left(X_{r-1:n}^j\right) - \left(1 + \frac{\beta\sigma(r-)}{j}\right) E\left(X_{r:n}^j\right).$$

Remark 3.4. Setting $m = -1$ and $k \geq 1$, in Theorem 3.2, we get a recurrence relation for single moments of kth lower record values from Dagum distribution in the form

$$\frac{1}{\alpha} E\left(X_{L(n:k)}^{j+\sigma}\right) = \frac{\beta\sigma k}{j} E\left(X_{L(n-1:k)}^j\right) - \left(1 + \frac{\beta\sigma k}{j}\right) E\left(X_{L(n:k)}^j\right),$$

as obtained Kumar (2016).

3.2. Relations for product moments. The product moments of DGOS are very important to calculate the covariance of order statistics and record values. In the following, we derive the product moments of DGOS from the Dagum distribution.

Theorem 3.3. *For Dagum distribution as given in (2) and* $1 \leq r < s \leq n$, $k = 1, 2, \ldots$ *and* $m \neq -1$

$$E[X_d^i(r,n,m,k)X_d^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}}$$

$$\cdot \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} J_{i,j}(m+u(m+1), s-r-1, \gamma_s - 1) \quad (29)$$

$$= \frac{\beta^2 \sigma^{(i+j)/\sigma} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}$$

$$\cdot \binom{r-1}{u}\binom{s-r-1}{v} \frac{(j/\sigma)_{(p)}}{p!\ q![\beta\gamma_{s-v} + (j/\sigma) + p]}$$

$$\cdot \frac{(i/\sigma)_{(q)}}{[\beta\gamma_{r-u} + \{(i+j)/\sigma\} + p + q]}, \ \sigma > max(i,j), \ i,j = 0,1,2,\ldots. \quad (30)$$

Proof. From (6), we have

$$E[X_d^i(r,n,m,k)X_d^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty \int_0^x x^i y^j [F(x)]^m f(x)$$

$$\cdot g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1} f(y) dy dx. \quad (31)$$

On expanding $g_m^{r-1}(F(x))$ binomially in (30), we get

$$E[X_d^i(r,n,m,k)X_d^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{r-1}}$$

$$\cdot \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} J_{i,j}(m+u(m+1), s-r-1, \gamma_s - 1).$$

Making use of the Lemma 2.4, we derive the relation in (29). □

Remark 3.5. Putting $m = 0, \ k = 1$ in (29), the explicit formula for the product moments of order statistics of the Dagum distribution can be obtained as

$$E(X_{n-r+1:n}^i X_{n-s+1:n}^j) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$

$$\cdot \frac{\beta^2 \alpha^{(i+j)/\sigma}\ C_{r,s:n}\ (j/\sigma)_{(p)}\ (i/\sigma)_{(q)}}{p!q![\beta(n-s+1+v) + p + (j/\sigma)][\beta(n-r+1+u) + p + q + \{(i+j)/\sigma\}]}.$$

that is

$$E(X^i_{r:n}X^j_{s:n}) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{n-s}\sum_{v=0}^{s-r-1}(-1)^{u+v}\begin{pmatrix} n-s \\ u \end{pmatrix}\begin{pmatrix} s-r-1 \\ v \end{pmatrix}$$

$$\cdot\frac{\beta^2\alpha^{(i+j)/\sigma}\ C_{r,s:n}\ (j/\sigma)_{(p)}\ (i/\sigma)_{(q)}}{p!q![\beta(r+v)+p+(j/\sigma)][\beta(s+u)+p+q+\{(i+j)/\sigma\}]},$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

Remark 3.6. Putting $m=-1$ in (29), we deduce the explicit expression for the product moments of lower record values for the Dagum distribution in view of (28) and (23) in the form

$$E[(X^{(k)}_{L(r)})^i(X^{(k)}_{L(s)})^j)] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(\beta k)^s\alpha^{(i+j)/\sigma}(j/\sigma)_{(p)}(i/\sigma)_{(q)}}{p!q![\beta k+p+(j/\sigma)]^{s-r}[\beta k+p+q+\{(i+j)/\sigma\}]^r}$$

and hence for lower records

$$E[X^i_{L(r)}X^j_{L(s)})] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\beta^s\alpha^{(i+j)/\sigma}(j/\sigma)_{(p)}(i/\sigma)_{(q)}}{p!q![\beta+p+(j/\sigma)]^{s-r}[\beta+p+q+\{(i+j)/\sigma\}]^r}.$$

as obtained Kumar (2016).

Corollary 3.4. *For the distribution given in (2), we have*

$$\begin{aligned} E[X^i_d(r,n,m,k) &= \frac{\beta\alpha^{i/\sigma}C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u \\ &\cdot \begin{pmatrix} r-1 \\ u \end{pmatrix}\frac{(i/\sigma)_q}{q![\beta\gamma_{r-u}+q+(i/\sigma)]}. \end{aligned} \quad (32)$$

Proof. At $j=0$ in (29), we have

$$E[X^i_d(r,n,m,k) = \frac{\beta\alpha^{i/\sigma}C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}$$

$$\cdot\begin{pmatrix} r-1 \\ u \end{pmatrix}\begin{pmatrix} s-r-1 \\ v \end{pmatrix}\frac{(i/\sigma)_q}{q!\gamma_{s-v}[\beta\gamma_{r-u}+q+(i/\sigma)]}.$$

Simplifying the resulting expression, we get result given in (31). □

Making use of (2), we can derive recurrence relations for product moments of DGOS.

Theorem 3.5. *For the distribution given in (1.2) and $n \in N$, $m \in \Re$, $1 \le r < s \le n-1$*

$$\frac{1}{\alpha}E[X^i_d(r,n,m,k)X^j_d(s,n,m,k)] = \frac{\beta\sigma\gamma_s}{j}E[X^i_d(r,n,m,k)X^j_d(s-1,n,m,k)]$$

$$-\left(1+\frac{\beta\sigma\gamma_s}{j}\right)E[X_d^i(r,n,m,k)X_d^j(s,n,m,k)]. \tag{33}$$

Proof. From (6), we have

$$E[X_d^i(r,n,m,k)X_d^j(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}$$
$$\cdot\int_0^\infty x^i[F(x)]^m f(x)g_m^{r-1}(F(x))I(x)dx, \tag{34}$$

where

$$I(x) = \int_0^x y^j[F(y)]^{\gamma_s-1}[h_m(F(y))-h_m(F(x))]^{s-r-1}f(y)dy.$$

Solving the integral in $I(x)$ by parts and substituting the resulting expression in (33), we get

$$E[X_d^i(r,n,m,k)X_d^j(s,n,m,k)] = E[X_d^i(r,n,m,k)X_d^j(s-1,n,m,k)]$$
$$-\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_0^x x^iy^{j-1}[F(x)]^m f(x)g_m^{r-1}(F(x))$$
$$\cdot[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s}dydx$$

the constant of integration vanishes since the integral in $I(x)$ is a definite integral. On using the relation (3), we obtain

$$E[X_d^i(r,n,m,k)X_d^j(s,n,m,k)] = E[X_d^i(r,n,m,k)X_d^j(s-1,n,m,k)]$$
$$-\frac{jC_{s-1}}{\alpha\beta\sigma\gamma_s(r-1)!(s-r-1)!}\left\{\int_0^\infty\int_0^x x^iy^{j+\sigma}[F(x)]^m f(x)g_m^{r-1}(F(x))\right.$$
$$\cdot[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dydx$$
$$+\alpha\int_0^\infty\int_0^x x^iy^j[F(x)]^m f(x)g_m^{r-1}(F(x))[h_m(F(y))-h_m(F(x))]^{s-r-1}$$
$$\left.\cdot[F(y)]^{\gamma_s-1}f(y)dydx\right\},$$

and hence the result. □

Remark 3.7. Putting $m=0$, $k=1$ in (32), we obtain recurrence relations for product moments of order statistics of the Dagum distribution in the form

$$\frac{1}{\alpha}E\left(X_{n-r+1:n}^iX_{n-s+1:n}^{j+\sigma}\right) = \frac{\beta\sigma(n-r+1)}{j}E\left(X_{n-r+1:n}^iX_{n-s+2:n}^j\right)$$
$$-\left(1+\frac{\beta\sigma(n-s+1)}{j}\right)E\left(X_{n-r+1:n}^iX_{n-s+1:n}^j\right).$$

Remark 3.8. Setting $m = -1$ and $k \geq$, in Theorem 3.5, we obtain the recurrence relations for product moments of lower record values from Dagum distribution in the form

$$\begin{aligned}\frac{1}{\alpha}E\left(X_{L(r):k}^{(i)}X_{L(s):k}^{(j+\sigma)}\right) &= \frac{\beta\sigma k}{j}E\left(X_{L(r):k}^{(i)}X_{L(s-1):k}^{(j)}\right)\\ &- \left(1+\frac{\beta\sigma k}{j}\right)E\left(X_{L(r):k}^{(i)}X_{L(s):k}^{(j)}\right),\end{aligned}$$

as obtained by Kumar (2016).

4. Characterization

In this section, we shall characterize Dagum distribution based on conditional moment of the DGOS.

Let $X_d(r,n,m,k)$, $r = 1,2,\dots,n$ be DGOS, then from a continuous population with *cdf* $F(x)$ and *pdf* $f(x)$, then the conditional *pdf* of $X_d(s,n,m,k)$ given $X_d(r,n,m,k) = x$, $1 \leq r < s \leq n$, in view of (5) and (6), is

$$f_{X_d(s,n,m,k)|X_d(r,n,m,k)}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \cdot\frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}}{[F(x)]^{\gamma_{r+1}}}f(y). \quad (35)$$

Theorem 4.1. *Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then*

$$E[X_d(s,n,m,k)|X_d(r,n,m,k) = x] = \alpha^{1/\sigma}\sum_{p=0}^{\infty}\frac{(1/\sigma)_{(p)}(1+\alpha x^{-\sigma})^p}{p!} \cdot\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\beta}\right), \quad (36)$$

if and only if

$$F(x) = (1+\alpha x^{-\sigma})^{-\beta},\ x > 0,\ \alpha,\beta,\sigma > 0.$$

Proof. From (34), we have

$$E[X_d(s,n,m,k)|X_d(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \cdot\int_0^x y\left[1-\left(\frac{F(y)}{F(x)}\right)^{m+1}\right]^{s-r-1}\left(\frac{F(y)}{F(x)}\right)^{\gamma_s-1}\frac{f(y)}{F(x)}dy. \quad (37)$$

By setting $u = \frac{F(y)}{F(x)} = \left(\frac{1+\alpha x^{-\sigma}}{1+\alpha y^{-\sigma}}\right)^{\beta}$ from (2) in (36), we obtain

$$E[X_d(s,n,m,k)|X_d(r,n,m,k) = x] = \frac{\sigma C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\cdot \int_0^1 \left(\frac{1-(1+\alpha x^{-\sigma})\ u^{-1/\beta}}{\alpha} \right)^{-1/\sigma} \quad u^{\gamma_s - 1}(1-u^{m+1})^{s-r-1} du$$
$$= \frac{\alpha^{1/\sigma} C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}} I_1, \tag{38}$$

where

$$\begin{aligned} I_1 &= \int_0^1 \left(1-(1+\alpha x^{-\sigma})\ u^{-1/\beta}\right)^{-1/\sigma} \quad u^{\gamma_s - 1}(1-u^{m+1})^{s-r-1} du \\ &= \sum_{p=0}^{\infty} \frac{(1/\sigma)_{(p)}}{p!} (1+\alpha x^{-\sigma})^p \int_0^1 \quad u^{\gamma_s - (p/\beta) - 1}(1-u^{m+1})^{s-r-1} du. \end{aligned} \tag{39}$$

Again by setting $t = u^{m+1}$ in (38), we get

$$\begin{aligned} I_1 &= \sum_{p=0}^{\infty} \frac{(1/\sigma)_{(p)}}{p!} (1+\alpha x^{-\sigma})^p \int_0^1 \quad t^{\frac{\beta k + p}{\beta(m+1)} + n + s - 1}(1-t)^{s-r-1} dt \\ &= \sum_{p=0}^{\infty} \frac{(1/\sigma)_{(p)}}{p!(m+1)} (1+\alpha x^{-\sigma})^p \frac{\Gamma\left(\frac{\beta k + p}{\beta(m+1)} + n - s\right)\Gamma(s-r)}{\Gamma\left(\frac{\beta k + p}{\beta(m+1)} + n - r\right)} \\ &= \sum_{p=0}^{\infty} \frac{(1/\sigma)_{(p)}}{p!} (1+\alpha x^{-\sigma})^p \frac{(m+1)^{s-r-1}\Gamma(s-r)}{\prod_{j=1}^{s-r}(\gamma_{r+j} + (p/\beta))}. \end{aligned}$$

Substituting the value of I_1 in (33) and simplifying the resulting expression, we derive the relation in (35).

To prove sufficient part, we have from (33) and (34)

$$\frac{C_{s-1}}{(s-r-1)! C_{r-1} (m+1)^{s-r-1}} \int_0^x y[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1}$$
$$\cdot [F(y)]^{\gamma_s - 1} f(y) dy = [F(x)]^{\gamma_{r+1}} H_r(x), \tag{40}$$

where

$$H_r(x) = \alpha^{1/\sigma} \sum_{p=0}^{\infty} \frac{(1/\sigma)_{(p)}(1+\alpha x^{-\sigma})^p}{p!} \prod_{j=1}^{s-r} \left(\frac{\gamma_{r+j}}{\gamma_{r+j} + p/\beta} \right).$$

Differentiating (39) both sides with respect to x, we get

$$\frac{C_{s-1}[F(x)]^m f(x)}{(s-r-2)! C_{r-1} (m+1)^{s-r-2}} \int_0^x y[(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-2}$$
$$\cdot [F(y)]^{\gamma_s - 1} f(y) dy = H_r'(x)[F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x) [F(x)]^{\gamma_{r+1} - 1} f(x)$$

or

$$\gamma_{r+1} H_{r+1}(x) [F(x)]^{\gamma_{r+2} + m} f(x)$$
$$= H_r'(x)[F(x)]^{\gamma_{r+1}} + \gamma_{r+1} H_r(x) [F(x)]^{\gamma_{r+1} - 1} f(x).$$

Therefore,

$$\frac{f(x)}{F(x)} = \frac{H_r'(x)}{\gamma_{r+1}[H_{r+1}(x) - H_r(x)]} = \frac{\alpha\beta\sigma}{(\alpha + x^\sigma)\ x}$$

which proves that

$$F(x) = (1 + \alpha x^{-\sigma})^{-\beta},\ x > 0,\ \alpha, \beta, \sigma > 0.$$

□

5. Numerical Results

The recurrence relations obtained in the preceding sections allow us to evaluate the means, variances and covariances of all order statistics for all sample sizes in a simple recursive manner. Means, variances, covariances and skewness and kurtosis of all order statistics and record values can be used for various inferential purposes; for example, they are useful in determining best linear unbiased estimators of location/scale parameters and best linear unbiased predictors of failure times. More details on BLUEs and BLUPs based on order statistics can be seen in Balakrishnan and Cohen (1991) and Arnold et al. (1992).

Tables for variance, skewness, kurtosis based on order statistics and mean, variance, skewness, kurtosis based record values are not presented here but are available from the authors on request. All computations here were performed using Mathematica. Mathematica like other algebraic manipulation packages allows for arbitrary precision, so the accuracy of the given values is not an issue.

6. Conclusions

We have derived explicit expressions for single moments and product moments DGOS from the Dagum distribution. Also, we have given tabulations of the mean, variance, skewness and kurtosis based on order statistics and record values. Tabulations for the covariances of order statistics and record values are not presented here but are available from the authors on request. All computations here we performed using Mathematica. Mathematica like other algebraic manipulation packages allows for arbitrary precision, so the accuracy of the given values is not an issue.

A future work may be to derive estimation procedures for the Dagum distribution based on record values, order statistics and generalized order statistics. Another future work may be to characterize the Dagum distribution based on record values, order statistics and generalized order statistics.

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Table 1. Mean based on order statistics

		$\sigma = 3$			
		$\alpha = 1$		$\alpha = 2$	
n	r	$\beta = 1$	$\beta = 2$	$\beta = 1$	$\beta = 2$
1	1	1.19098	1.57591	1.50054	1.98552
2	1	0.80605	1.13422	1.01556	1.42903
	2	1.57591	2.01760	1.98552	2.54202
3	1	0.67178	0.98195	0.84639	1.23718
	2	1.07459	1.43876	1.35390	1.81273
	3	1.82657	2.30702	2.30134	2.90667
4	1	0.59714	0.89890	0.75234	1.13254
	2	0.89570	1.23110	1.12851	1.55109
	3	1.25348	1.64643	1.57929	2.07437
	4	2.01760	2.52722	2.54202	3.18410
5	1	0.54737	0.84413	0.68965	1.06354
	2	0.79618	1.11797	1.00312	1.40856
	3	1.04498	1.40080	1.31659	1.76490
	4	1.39248	1.81018	1.75442	2.28068
	5	2.17388	2.70648	2.73892	3.40995
		$\sigma = 4$			
1	1	1.10292	1.37284	1.31160	1.63259
2	1	0.83300	1.08453	0.99061	1.28974
	2	1.37284	1.66114	1.63259	1.97544
3	1	0.72891	0.97662	0.86683	1.16141
	2	1.04118	1.30036	1.23818	1.54639
	3	1.53866	1.84153	1.82979	2.18996
4	1	0.66817	0.91533	0.79459	1.08852
	2	0.91114	1.16051	1.08353	1.38008
	3	1.17123	1.44021	1.39284	1.71270
	4	1.66114	1.97530	1.97544	2.34905
5	1	0.62641	0.87390	0.74493	1.03924
	2	0.83521	1.08106	0.99324	1.28560
	3	1.02503	1.27968	1.21897	1.52180
	4	1.26870	1.54722	1.50875	1.83997
	5	1.75925	2.08232	2.09211	2.47631
		$\sigma = 5$			
1	1	1.06449	1.27384	1.22278	1.46325
2	1	0.85514	1.06032	0.9823	1.21798
	2	1.27384	1.48736	1.46325	1.70852
3	1	0.76965	0.97653	0.88410	1.12174
	2	1.02613	1.22789	1.17872	1.41047
	3	1.39769	1.61709	1.60552	1.85755
4	1	0.71834	0.92782	0.82516	1.06578
	2	0.92358	1.12267	1.06091	1.28961
	3	1.12868	1.33310	1.29652	1.53133
	4	1.48736	1.71176	1.70852	1.96629
5	1	0.68242	0.89441	0.78390	1.02741
	2	0.86201	1.06144	0.99019	1.21927
	3	1.01594	1.21453	1.16700	1.39513
	4	1.20385	1.41215	1.38286	1.62213
	5	1.55823	1.78666	1.78994	2.05233

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