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ON THE MODAL OPERATORS OVER THE GENERALIZED INTERVAL VALUED INTUITIONISTIC FUZZY SETS

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Abstract. Interval valued intuitionistic fuzzy sets (IVIFSs) is widely used to model uncertainty, imprecise, incomplete and vague information. In this paper, newly defined modal operators over an extensional generalized interval valued intuitionistic fuzzy sets ($GIVIFS_Bs$) are proposed. Some of the basic properties of the new operators are discussed and few theorems were proved. The actual contribution in this paper is to discuss ten operators on $GIVIFS_Bs$.

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1. Introduction

Atanassov (1986) introduced the concept of intuitionistic fuzzy sets (IFSs), which is a generalization of fuzzy sets and defined new operations on IFSs. The theory of intuitionistic fuzzy sets has meaningful applications in many fields. In IFS, each element is assigned by membership and non-membership degrees, where the sum of the two degrees is between zero and one. However in reality, it may not always be true that the degree of membership and degree of non-membership of an element in IFS be real numbers. Therefore, a generalization of IFS was introduced by Atanassov and Gargov (1989) as interval valued intuitionistic fuzzy sets (IVIFSs) which its fundamental characteristic is that the values of it membership and non-membership degree are intervals rather than exact numbers. Atanassov (1994) introduced operators over interval valued intuitionistic fuzzy sets. In research literature, over the IFSs and IVIFSs different operators from modal, topological, level, negation and other types are defined. For example: topological operators (Atanassov (1986, 1989, 2001)); modal operators (Atanassov (2004)); different forms of negation operation (Atanassov and

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Dimitrov (2008)); relations between intuitionistic fuzzy negations and intuitionistic fuzzy modal operators (Hinde and Atanassov (2008)); some operators of IFS of root type (Srinivasan and Palaniappan (2012)) and etc. These operators not only important from the theoretical view but they are also important from point of view of applications.

Xu (2007a, 2007b) developed some arithmetic aggregation operators and some geometric aggregation operators of IVIFS for decision making. Wang and Liu (2013) considered the interval valued intuitionistic fuzzy hybrid weighted averaging operator based on Einstein operation and its application to decision making. They defined generalized interval valued intuitionistic fuzzy relation (GIVIF) with some results. Bhowmik and Pal (2009, 2010) defined generalized interval valued intuitionistic fuzzy sets (GIVIFSs). Bhowmik and Pal (2012) defined two composite relations, four types of reflexivity and irreflexivity of GIVIFSs with some of their properties. Chen et al. (2012), Yue (2011) and Li (2010a, 2010b, 2011) presented methods for multi-criteria fuzzy decision making based on IVIFS. Mondal and Samanta (2011) studied the topological properties and the category of topological spaces of IVIFSs. Zhang et al. (2011) introduced a generalized interval valued intuitionistic fuzzy sets. GIVIFS play a significant role in different science. One motivation for our study has been the significant performance achieved by the use of IVIFSs implications in some applications. For example, some recent applications of IVIFSs have been: medical diagnosis (Ahn et al. (2011)); multi-attribute decision making (Liu et al., (2012)); evaluation about the performance of E-government (Zhang et al. (2014)); reliability analysis (Tyagi, (2014)); medical diagnosis using logical operators (Pathinathan et al. (2015)) wind energy technology selection (Onar, (2015)); Kahraman et al. (2016)); decision and game theory in management (Li, (2013)) algorithm for solving an assignment model (Jose et al.(2013)); select supplier (Xiao and Wei (2011)); pattern recognition (Zhang et al. (2013)); exploitation investment evaluation (Qi et al. (2013)).

Baloui Jamkhaneh and Nadarajah (2015) considered a new generalized intuitionistic fuzzy sets ($GIFS_B$) and introduced some operators over $GIFS_B$. Shabani and Baloui Jamkhaneh (2014) introduced a new generalized intuitionistic fuzzy number based on generalization of the IFS related to Baloui Jamkhaneh and Nadaraja (2015). Recently, suitable means were defined by Baloui Jamkhaneh (2015), called generalized interval valued intuitionistic fuzzy sets ($GIVIFS_B$). In fact, all IVIFSs are GIVIFSs but all GIVIFSs are not IVIFSs. By analogy we shall introduce the some of operators (as $D_\alpha(A)$, $F_{\alpha,\beta}(A)$, $G_{\alpha,\beta}(A)$, $J_{\alpha,\beta}(A)$, $d_\alpha(A)$, $f_{\alpha,\beta}(A)$, $g_{\alpha,\beta}(A)$, $j_{\alpha,\beta}(A)$, $H_{\alpha,\beta}(A)$, $h_{\alpha,\beta}(A)$) over $GIVIFS_B$ and we will discuss their properties. Some of these properties are the following: i) All these operators are $GIVIFS_B$, ii) All these operators are increasing relative to α , iii) All these operators are decreasing relative to β, iv) $D_0(A) = F_{0,1}(A) = d_1(A) = \overline{f_{1,0}(A)} = H_{1,1}(A) = \overline{J_{1,1}(A)} = \Box A$, v) $D_1(A) = F_{1,0}(A) = J_{1,1}(A) = d_0(A) = \overline{f_{0,1}(A)} = \overline{h_{1,1}(A)} = \Diamond A$ and etc.

The originality of this study comes from the fact that, there was no previous work introduce operators for $GIVIFS_Bs$. This paper is organized as follows: In Section 2, we briefly introduce IFS and it generalizes. In Section 3 define operators over generalized interval valued intuitionistic fuzzy sets. The paper is concluded in Section 4.

2. Preliminaries

In this section, we give some basic definition. Let X be a non empty set.

Definition 2.1. (Atanassov,1986) An IFS A in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denotes the degree of membership and non-membership functions of A, respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Definition 2.2. (Atanassov & Gargov, 1989) Interval value intuitionistic fuzzy sets (IVIFS) A in X, is defined as an object of the form $A = \{\langle x, M_A(x), N_A(x)\rangle | x \in X\}$ where the functions $M_A(x) : X \to [I]$ and $N_A(x) : X \to [I]$, denote the degree of membership and degree of non-membership of A respectively, where $M_A(x) = [M_{AL}(x), M_{AU}(x)]$, $N_A(x) = [N_{AL}(x), N_{AU}(x)]$, $0 \leq M_{AU}(x) + N_{AU}(x) \leq 1$ for each $x \in X$.

Definition 2.3. (Baloui Jamkhaneh & Nadarajah (2015)) Generalized intuitionistic fuzzy sets ($GIFS_B$) A in X, is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$, denote the degree of membership and degree of non-membership functions of A respectively, and $0 \leq \mu_A(x)^\delta + \nu_A(x)^\delta \leq 1$ for each $x \in X$ and $\delta = n$ or $\frac{1}{n}$, $n = 1, 2, \cdots, N$.

Definition 2.4. Let $[I]$ be the set of all closed subintervals of the interval $[0,1]$ and $M_A(x) = [M_{AL}(x), M_{AU}(x)] \in [I]$ and $N_A(x) = [N_{AL}(x), N_{AU}(x)] \in [I]$ then $N_A(x) \leq M_A(x)$ if and only if $N_{AL}(x) \leq M_{AL}(x)$ and $N_{AU}(x) \leq M_{AU}(x)$.

Definition 2.5. Generalized interval valued intuitionistic fuzzy sets ($GIVIFS_B$) A in X, is defined as an object of the form $A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$ where the functions $M_A(x) : X \to [I]$ and $N_A(x) : X \to [I]$, denote the degree of membership and degree of non-membership of A respectively, and $M_A(x) = [M_{AL}(x), M_{AU}(x)]$, $N_A(x) = [N_{AL}(x), N_{AU}(x)]$, where $0 \leq M_{AU}(x)^\delta + N_{AU}(x)^\delta \leq 1$, for each $x \in X$ and $\delta = n$ or $\frac{1}{n}$, $n = 1, 2, \cdots, N$. The collection of all $GIVIFS_B(\delta)$ is denoted by $GIVIFS_B(\delta, X)$.

Definition 2.6. The degree of non-determinacy (uncertainty) of an element $x \in X$ to the $GIVIFS_B$ A is defined by

$$\pi_A(x) = [\pi_{AL}(x), \pi_{AU}(x)] = [(1 - M_{AU}(x)^\delta - N_{AU}(x)^\delta)^{\frac{1}{\delta}},$$

$$(1 - M_{AL}(x)^\delta - N_{AL}(x)^\delta)^{\frac{1}{\delta}}].$$

Definition 2.7. For every $GIVIFS_B$ $A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$, we define the modal logic operators necessity and possibility.
The Necessity measure on A:

$$\Box A = \left\{\langle x, [M_{AL}(x), M_{AU}(x)], [(1 - M_{AU}(x)^\delta)^{\frac{1}{\delta}}, (1 - M_{AL}(x)^\delta)^{\frac{1}{\delta}}]\rangle : x \in X\right\},$$

The Possibility measure on A:

$$\Diamond A = \left\{\langle x, [(1 - N_{AU}(x)^\delta)^{\frac{1}{\delta}}, (1 - N_{AL}(x)^\delta)^{\frac{1}{\delta}}], [N_{AL}(x), N_{AU}(x)]\rangle : x \in X\right\}.$$

Definition 2.8. For every $GIVIFS_B$ $A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$, two analogues of the topological operators defined over the GIVIFSs:
The operator closure C:

$$C(A) = \left\{\langle x, \max_{y\in X} M_A(y), \min_{y\in X} N_A(y)\rangle : x \in X\right\},$$

The operator intersection I:

$$I(A) = \left\{\langle x, \min_{y\in X} M_A(y), \max_{y\in X} N_A(y)\rangle : x \in X\right\}.$$

3. The operators of $GIVIFS_B$

Here, we will introduce new operators over the $GIVIFS_B$ which extend some operators in the research literature related to IVIFSs. Let X is a non-empty finite set, $\lambda = \frac{N_{AU}(x)^\delta - N_{AL}(x)^\delta}{M_{AU}(x)^\delta - M_{AL}(x)^\delta}$ and $A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$ is a $GIVIFS_B$.

3.1. The first group operators.

Definition 3.1. Let $\alpha = \frac{1}{1+\lambda}$ and $A \in GIVIFS_B$, we define the operator of $D_\alpha(A)$ as follows

$$D_\alpha(A) = \{\langle x, M_{D_\alpha}(A), N_{D_\alpha}(A)\rangle : x \in X\},$$

$$M_{D_\alpha}(A) = \left[\left(M_{AL}(x)^\delta + \alpha\pi_{AU}(x)^\delta\right)^{\frac{1}{\delta}}, \left(M_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right],$$

$$N_{D_\alpha}(A) = \left[\left(N_{AL}(x)^\delta + (1-\alpha)\pi_{AU}(x)^\delta\right)^{\frac{1}{\delta}}, \left(N_{AU}(x)^\delta + (1-\alpha)\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right].$$

Clearly, $D_\alpha(A)$ is a $GIVIFS_B$.

Theorem 3.2. *For every $GIVIFS_B$ and A and $\alpha = \frac{1}{1+\lambda}$, it holds that*

i. *if $M_{AU}(x) = M_{AL}(x)$ then $\alpha = 0$ and $D_0(A) = \Box A$,*
ii. *if $N_{AU}(x) = N_{AL}(x)$ then $\alpha = 1$ and $D_1(A) = \Diamond A$.*

Proof. (i) it is clear that, if $M_{AU}(x) = M_{AL}(x)$ then $\alpha = 0$,

$$\begin{aligned} M_{D_0}(A) &= \left[\left(M_{AL}(x)^\delta + 0\times\pi_{AU}(x)^\delta\right)^{\frac{1}{\delta}}, \left(M_{AU}(x)^\delta + 0\times\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right], \\ &= [M_{AL}(x), M_{AU}(x)], \end{aligned}$$

$$N_{D_0}(A) = \left[\left(N_{AL}(x)^\delta + (1-0)\pi_{AU}(x)^\delta\right)^{\frac{1}{\delta}}, \left(N_{AU}(x)^\delta + (1-0)\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right],$$

$$= \left[\left(N_{AL}(x)^{\delta}+\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(N_{AU}(x)^{\delta}+\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right].$$

Since $\pi_{AU}(x)^{\delta}=1-M_{AL}(x)^{\delta}-N_{AL}(x)^{\delta}$, $\pi_{AL}(x)^{\delta}=1-M_{AU}(x)^{\delta}-N_{AU}(x)^{\delta}$, $N_{D_0}(A)=\left[\left(1-M_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(1-M_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}\right]$,
therefore

$$D_0(A)=$$
$$\left\{\langle x,[M_{AL}(x),M_{AU}(x)],\left[\left(1-M_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(1-M_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}\right]\rangle : x\in X\right\}$$
$$=\Box A.$$

Proof is complete.

(ii) it is clear that, if $N_{AU}(x)=N_{AL}(x)$ then $\alpha=1$,

$$D_1(A)=\{\langle x,M_{D_1}(A),N_{D_1}(A)\rangle : x\in X\},$$

$$\begin{aligned} N_{D_1}(A) &= \left[\left(N_{AL}(x)^{\delta}+(1-1)\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(N_{AU}(x)^{\delta}+(1-1)\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],\\ &= [N_{AL}(x),N_{AU}(x)],\end{aligned}$$

$$\begin{aligned} M_{D_1}(A) &= \left[\left(M_{AL}(x)^{\delta}+\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(M_{AU}(x)^{\delta}+\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],\\ &= \left[\left(1-N_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(1-N_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],\end{aligned}$$

$$D_1(A)=\left[\left[\left(1-N_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(1-N_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],[N_{AL}(x),N_{AU}(x)]\right]=\Diamond A.$$

Proof is complete. □

Definition 3.3. Let $0\le\alpha\le\frac{1}{1+\lambda}$, $0\le\beta\le\frac{\lambda}{1+\lambda}$, and $A\in GIVIFS_B$ we define the operator of $F_{\alpha,\beta}(A)$ as follows

$$F_{\alpha,\beta}(A)=\left\{\left\langle x,M_{F\alpha,\beta}(A),N_{F_{\alpha,\beta}}(A)\right\rangle : x\in X\right\},$$

$$M_{F_{\alpha,\beta}}(A)=\left[\left(M_{AL}(x)^{\delta}+\alpha\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(M_{AU}(x)^{\delta}+\alpha\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

$$N_{F_{\alpha,\beta}}(A)=\left[\left(N_{AL}(x)^{\delta}+\beta\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(N_{AU}(x)^{\delta}+\beta\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right].$$

Theorem 3.4. *For every $GIVIFS_B$ A and for some α,β, where $0\le\alpha\le\frac{1}{1+\lambda}$, $0\le\beta\le\frac{\lambda}{1+\lambda}$, it holds that*

i. $F_{\alpha,\beta}(A)\in GIVIFS_B$,
ii. $0\le\gamma\le\alpha\Rightarrow F_{\gamma,\beta}(A)\subset F_{\alpha,\beta}(A)$,
iii. $0\le\gamma\le\beta\Rightarrow F_{\alpha,\beta}(A)\supset F_{\alpha,\gamma}(A)$,
iv. $D_\alpha(A)=F_{\alpha,1-\alpha}(A)$, $\alpha=\frac{1}{1+\lambda}$,
v. $\Box A=F_{0,1}(A)$,
vi. $\Diamond A=F_{1,0}(A)$,
vii. $\overline{F_{\alpha,\beta}(\bar{A})}=F_{\beta,\alpha}(A)$, $0\le\alpha,\beta\le\min\{\frac{1}{1+\lambda},\frac{\lambda}{1+\lambda}\}$.

Proof. (i)

$$\begin{aligned} M_{F_{\alpha,\beta}(A)U}\left(x\right)^{\delta}+N_{F_{\alpha,\beta}(A)U}\left(x\right)^{\delta} &= \left[\left(M_{AU}\left(x\right)^{\delta}+\alpha\pi_{AL}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}}\right]^{\delta} \\ &+ \left[\left(N_{AU}\left(x\right)^{\delta}+\beta\pi_{AL}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}}\right]^{\delta}, \\ &= M_{AU}\left(x\right)^{\delta}+\alpha\pi_{AL}\left(x\right)^{\delta}+N_{AU}\left(x\right)^{\delta} \\ &+ \beta\pi_{AL}\left(x\right)^{\delta}, \\ &= M_{AU}\left(x\right)^{\delta}+N_{AU}\left(x\right)^{\delta}+(\alpha+\beta)\pi_{AL}\left(x\right)^{\delta}, \\ &\leq M_{AU}\left(x\right)^{\delta}+N_{AU}\left(x\right)^{\delta}+\pi_{AL}\left(x\right)^{\delta}=1. \end{aligned}$$

Proof is complete for every α,β, where $0\leq\alpha\leq\frac{1}{1+\lambda}$, $0\leq\beta\leq\frac{\lambda}{1+\lambda}$.
Proofs (ii) and (iii) are clearly.
(iv) It is clear that if $0\leq\alpha\leq\frac{1}{1+\lambda}$ and $0\leq 1-\alpha\leq\frac{\lambda}{1+\lambda}$ then $\alpha=\frac{1}{1+\lambda}$.

$$M_{F_{\alpha,1-\alpha}}\left(A\right)=\left[\left(M_{AL}\left(x\right)^{\delta}+\alpha\pi_{AU}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}},\left(M_{AU}\left(x\right)^{\delta}+\alpha\pi_{AL}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

$$N_{F_{\alpha,1-\alpha}}\left(A\right)=$$
$$\left[\left(N_{AL}\left(x\right)^{\delta}+(1-\alpha)\pi_{AU}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}},\left(N_{AU}\left(x\right)^{\delta}+(1-\alpha)\pi_{AL}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

$$F_{\alpha,1-\alpha}\left(A\right)=\left\{\left\langle x,M_{F_{\alpha,1-\alpha}}\left(A\right),N_{F_{\alpha,1-\alpha}}\left(A\right)\right\rangle : x\in X\right\}=D_{\alpha}\left(A\right).$$

Remark 3.1. By using (iv) we have $D_0(A)=F_{0,1}(A)$ and $D_1(A)=F_{1,0}(A)$.

(v) By using Remark 3.1 and Theorem 3.2 it follows that $F_{0,1}(A)=\Box A$.
(vi) By using Remark 3.1 and Theorem 3.2 it follows that $F_{1,0}(A)=\Diamond A$.
(vii) Since

$$F_{\beta,\alpha}\left(A\right)=\left\{\left\langle x,M_{F_{\beta,\alpha}}\left(A\right),N_{F_{\beta,\alpha}}\left(A\right)\right\rangle : x\in X\right\},$$

$$M_{F_{\beta,\alpha}}\left(A\right)=\left[\left(M_{AL}\left(x\right)^{\delta}+\beta\pi_{AU}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}},\left(M_{AU}\left(x\right)^{\delta}+\beta\pi_{AL}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

$$N_{F_{\beta,\alpha}}\left(A\right)=\left[\left(N_{AL}\left(x\right)^{\delta}+\alpha\pi_{AU}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}},\left(N_{AU}\left(x\right)^{\delta}+\alpha\pi_{AL}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}}\right].$$

And

$$F_{\alpha,\beta}\left(\bar{A}\right)=\left\{\left\langle x,M_{F_{\alpha}}\left(\bar{A}\right),N_{F_{\beta}}\left(\bar{A}\right)\right\rangle : x\in X\right\},$$

$$M_{F_{\alpha,\beta}}\left(\bar{A}\right)=\left[\left(N_{AL}\left(x\right)^{\delta}+\alpha\pi_{AU}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}},\left(N_{AU}\left(x\right)^{\delta}+\alpha\pi_{AL}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

$$N_{F_{\alpha,\beta}}\left(\bar{A}\right)=\left[\left(M_{AL}\left(x\right)^{\delta}+\beta\pi_{AU}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}},\left(M_{AU}\left(x\right)^{\delta}+\beta\pi_{AL}\left(x\right)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

hence

$$\overline{F_{\alpha,\beta}(\bar{A})}=\left\{\left\langle x,N_{F_{\alpha,\beta}}\left(\bar{A}\right),M_{F_{\alpha,\beta}}\left(\bar{A}\right)\right\rangle : x\in X\right\}.$$

Finally, we have $\overline{F_{\alpha,\beta}(\bar{A})}=F_{\beta,\alpha}(A)$. $\Box$

Definition 3.5. Let $\alpha, \beta \in [0,1]$ and $A \in GIVIFS_B$, we define the operator of $G_{\alpha,\beta}(A)$ as follows

$$G_{\alpha,\beta}(A) = \left\{ \left\langle x, M_{G_{\alpha,\beta}}(A), N_{G_{\alpha,\beta}}(A) \right\rangle : x \in X \right\},$$

$$M_{G_{\alpha,\beta}}(A) = \alpha^{\frac{1}{\delta}} M_A(x) = [\alpha^{\frac{1}{\delta}} M_{AL}(x), \alpha^{\frac{1}{\delta}} \ M_{AU}(x)],$$

$$N_{G_{\alpha,\beta}}(A) = \beta^{\frac{1}{\delta}} N_A(x) = [\beta^{\frac{1}{\delta}} N_{AL}(x), \beta^{\frac{1}{\delta}} \ N_{AU}(x)].$$

Theorem 3.6. *For every $GIVIFS_B$ A, and for every three real numbers $\alpha, \beta, \gamma \in [0,1]$, it holds that*

i. $G_{\alpha,\beta}(A) \in GIVIFS_B$,
ii. $\alpha \leq \gamma \Rightarrow G_{\alpha,\beta}(A) \subset G_{\gamma,\beta}(A)$,
iii. $\beta \leq \gamma \Rightarrow G_{\alpha,\beta}(A) \supset G_{\alpha,\gamma}(A)$,
iv. $\tau \in [0,1] \Rightarrow G_{\alpha,\beta}(G_{\gamma,\tau}(A)) = G_{\alpha\gamma,\beta\tau}(A) = G_{\gamma,\tau}(G_{\alpha,\beta}(A))$,
v. $G_{\alpha,\beta}(C(A)) = C(G_{\alpha,\beta}(A))$,
vi. $G_{\alpha,\beta}(I(A)) = I(G_{\alpha,\beta}(A))$,
vii. $\overline{G_{\alpha,\beta}(\bar{A})} = G_{\beta,\alpha}(A)$.

Proof. (i)

$$\begin{aligned} M_{G_{\alpha,\beta}U}(A)(x)^{\delta} + N_{G_{\alpha,\beta}U}(A)(x)^{\delta} &= (\alpha^{\frac{1}{\delta}} M_{AU}(x))^{\delta} + \left(\beta^{\frac{1}{\delta}} N_{AU}(x)\right)^{\delta} \\ &= \alpha M_{AU}(x)^{\delta} + \beta N_{AU}(x)^{\delta} \\ &\leq M_{AU}(x)^{\delta} + N_{AU}(x)^{\delta} \leq 1. \end{aligned}$$

Finally, it can be concluded that $G_{\alpha,\beta}(A) \in GIVIFS_B$.
(ii) Since $\alpha \leq \gamma$ then it is clear that $\alpha^{\frac{1}{\delta}} \leq \gamma^{\frac{1}{\delta}}$ hence $[\alpha^{\frac{1}{\delta}} M_{AL}(x), \alpha^{\frac{1}{\delta}} M_{AU}(x)] \leq [\gamma^{\frac{1}{\delta}} M_{AL}(x), \gamma^{\frac{1}{\delta}} M_{AU}(x)]$. Finally we have $G_{\alpha,\beta}(A) \subset G_{\gamma,\beta}(A)$.
The proof of (iii) is similar to (ii).
(iv) $A = \{\langle x, M_A(x), N_A(x)\rangle, x \in X\}$
$G_{\gamma,\tau}(A) = \left\{\left\langle x, M_{G_{\gamma,\tau}}(A), N_{G_{\gamma,\tau}}(A)\right\rangle : x \in X\right\}$,
$M_{G_{\gamma,\tau}}(A) = [\alpha^{\frac{1}{\delta}} M_{AL}(x), \alpha^{\frac{1}{\delta}} M_{AU}(x)]$,
$N_{G_{\gamma,\tau}}(A) = [\beta^{\frac{1}{\delta}} N_{AL}(x), \beta^{\frac{1}{\delta}} N_{AU}(x)]$,

$$\begin{aligned} G_{\alpha,\beta}(G_{\gamma,\tau}(A)) &= \left\{\left\langle x, \alpha^{\frac{1}{\delta}} \gamma^{\frac{1}{\delta}} M_A(x), \beta^{\frac{1}{\delta}} \tau^{\frac{1}{\delta}} N_A(x)\right\rangle : x \in X\right\}, \\ &= \left\{\left\langle x, (\alpha\gamma)^{\frac{1}{\delta}} M_A(x), (\beta\tau)^{\frac{1}{\delta}} N_A(x)\right\rangle : x \in X\right\}, \\ &= G_{\alpha\gamma,\beta\tau}(A), \end{aligned}$$

and

$$\begin{aligned} G_{\gamma,\tau}(G_{\alpha,\beta}(A)) &= \left\{\left\langle x, \gamma^{\frac{1}{\delta}} \alpha^{\frac{1}{\delta}} M_A(x), \tau^{\frac{1}{\delta}} \beta^{\frac{1}{\delta}} N_A(x)\right\rangle : x \in X\right\}, \\ &= \left\{\left\langle x, (\gamma\alpha)^{\frac{1}{\delta}} M_A(x), (\tau\beta)^{\frac{1}{\delta}} N_A(x)\right\rangle : x \in X\right\}, \\ &= \left\{\left\langle x, (\alpha\gamma)^{\frac{1}{\delta}} M_A(x), (\beta\tau)^{\frac{1}{\delta}} N_A(x)\right\rangle : x \in X\right\}, \\ &= G_{\alpha\gamma,\beta\tau}(A), \end{aligned}$$

hence
$$G_{\alpha,\beta}(G_{\gamma,\tau}(A)) = G_{\alpha\gamma,\beta\tau}(A) = G_{\gamma,\tau}(G_{\alpha,\beta}(A)).$$
(v)
$$\begin{aligned} G_{\alpha,\beta}(C(A)) &= \left\{\langle x, \alpha^{\frac{1}{\delta}} \max_{y\in X} M_A(y), \beta^{\frac{1}{\delta}} \min_{y\in X} N_A(y)\rangle : x \in X\right\}, \\ &= \left\{\langle x, \max_{y\in X} \alpha^{\frac{1}{\delta}} M_A(y), \min_{y\in X} \beta^{\frac{1}{\delta}} N_A(y)\rangle : x \in X\right\}, \\ &= C(G_{\alpha,\beta}(A)). \end{aligned}$$
(vi)
$$\begin{aligned} G_{\alpha,\beta}(I(A)) &= \left\{\langle x, \alpha^{\frac{1}{\delta}} \min_{y\in X} M_A(y), \beta^{\frac{1}{\delta}} \max_{y\in X} N_A(y)\rangle : x \in X\right\}, \\ &= \left\{\langle x, \min_{y\in X} \alpha^{\frac{1}{\delta}} M_A(y), \max_{y\in X} \beta^{\frac{1}{\delta}} N_A(y)\rangle : x \in X\right\}, \\ &= I(G_{\alpha,\beta}(A)). \end{aligned}$$
(vii) Let $A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$ is a $GIVIFS_B$ then
$\bar{A} = \{\ \langle x,\ N_A(x), M_A(y)\rangle\ : x \in X\}$,
$G_{\beta,\alpha}(A) = \left\{\left\langle x, \beta^{\frac{1}{\delta}} M_A(x), \alpha^{\frac{1}{\delta}} N_A(x)\right\rangle\ : x \in X\right\}$,
$G_{\alpha,\beta}(\bar{A}) = \left\{\left\langle x, \alpha^{\frac{1}{\delta}} N_A(x), \beta^{\frac{1}{\delta}} M_A(x)\ \right\rangle\ : x \in X\right\}$,
$\overline{G_{\alpha,\beta}(\bar{A})} = \left\{\left\langle x, \beta^{\frac{1}{\delta}} M_A(x), \alpha^{\frac{1}{\delta}} N_A(x)\right\rangle\ : x \in X\right\}$,
hence $\overline{G_{\alpha,\beta}(\bar{A})} = G_{\beta,\alpha}(A)$. □

Definition 3.7. Let $0 \le \alpha \le \frac{1}{1+\lambda}$, $\beta \in [0,1]$ and $A \in GIVIFS_B$, we define the operator of $J_{\alpha,\beta}(A)$ as follows
$J_{\alpha,\beta}(A) = \left\{\left\langle x, M_{J_{\alpha,\beta}}(A), N_{J_{\alpha,\beta}}(A)\right\rangle\ : x \in X\right\}$,
$M_{J_{\alpha,\beta}}(A) = \left[\left(M_{AL}(x)^\delta + \alpha\pi_{AU}(x)^\delta\right)^{\frac{1}{\delta}}, \left(M_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right]$,
$N_{J_{\alpha,\beta}}(A) = \left[\beta^{\frac{1}{\delta}} N_{AL}(x), \beta^{\frac{1}{\delta}} N_{AU}(x)\right]$.

Theorem 3.8. *For every $GIVIFS_B A$, and for some real numbers $0 \le \alpha \le \frac{1}{1+\lambda}$, $\beta \in [0,1]$, it holds that*

i. $J_{\alpha,\beta}(A) \in GIVIFS_B$,
ii. $\alpha \le \gamma \Rightarrow J_{\alpha,\beta}(A) \subset J_{\gamma,\beta}(A)$, $0 \le \gamma \le \frac{1}{1+\lambda}$,
iii. $\beta \le \gamma \Rightarrow J_{\alpha,\beta}(A) \supset J_{\alpha,\gamma}(A)$, $\gamma \in [0,1]$,
iv. *if* $N_{AU}(x) = N_{AL}(x)$ *then* $0 \le \alpha \le 1$ *and* $\Diamond A = J_{1,1}(A)$,
v. $A = J_{0,1}(A)$.

Proof. (i)
$$M_{J_{\alpha,\beta}(A)U}(x)^\delta + N_{J_{\alpha,\beta}(A)U}(x)^\delta \quad = \quad \left(\left(M_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right)^\delta$$

$$\begin{aligned} &+ \left(\beta^{\frac{1}{\delta}} N_{AU}(x)\right)^{\delta}, \\ &= \left(M_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta}\right) + \beta N_{AU}(x)^{\delta}, \\ &\leq M_{AU}(x)^{\delta} + \pi_{AL}(x)^{\delta} + N_{AU}(x)^{\delta} = 1. \end{aligned}$$

Finally, it can be concluded that $J_{\alpha,\beta}(A) \in IVGIFS_B$.
(ii) Since $\alpha \leq \gamma$ then it is clear that

$$\left[\left(M_{AL}(x)^{\delta} + \alpha\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(M_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],$$
$$\leq \left(M_{AL}(x)^{\delta} + \gamma\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(M_{AU}(x)^{\delta} + \gamma\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}.$$

Finally we have $J_{\alpha,\beta}(A) \subset J_{\gamma,\beta}(A)$. This completes the proof.
The proof of (iii) is similar (ii). Proofs (iv) and (v) are clearly. □

3.2. The second group operators.

Definition 3.9. Let $\alpha = \frac{\lambda}{1+\lambda}$ and $A \in GIVIFS_B$, we define the operator of $d_\alpha(A)$ as follows

$$d_\alpha(A) = \{\langle x, M_{d_\alpha}(A), N_{d_\alpha}(A)\rangle : x \in X\},$$
$$M_{d_\alpha}(A) = \left[\left(N_{AL}(x)^{\delta} + \alpha\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, (N_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}\right],$$
$$N_{d_\alpha}(A) = \left[\left(M_{AL}(x)^{\delta} + (1-\alpha)\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, (M_{AU}(x)^{\delta} + (1-\alpha)\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}\right].$$

Theorem 3.10. *For every $GIVIFS_B$ A and $\alpha = \frac{\lambda}{1+\lambda}$, it holds that*

i. $d_\alpha(A) \in GIVIFS_B$,
ii. $d_0(A) = \overline{\Diamond A}$, *(it is clear that, if $N_{AU}(x) = N_{AL}(x)$ then $\alpha = 0$)*,
iii. $d_1(A) = \overline{\Box A}$, *(it is clear that, if $M_{AU}(x) = M_{AL}(x)$ then $\alpha = 1$)*.

Proof. Proof (i) is clearly.
(ii)

$$d_\alpha(A) = \{\langle x, M_{d_\alpha}(A), N_{d_\alpha}(A)\rangle : x \in X\},$$
$$M_{d_\alpha}(A) = \left[\left(N_{AL}(x)^{\delta} + \alpha\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, (N_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}\right],$$
$$N_{d_\alpha}(A) = \left[\left(M_{AL}(x)^{\delta} + (1-\alpha)\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, (M_{AU}(x)^{\delta} + (1-\alpha)\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}\right],$$

$$\begin{aligned} M_{d_0}(A) &= \left[\left(N_{AL}(x)^{\delta} + 0\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(N_{AU}(x)^{\delta} + 0\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right] \\ &= [N_{AL}(x), N_{AU}(x)], \end{aligned}$$

$$N_{d_0}(A) =$$

$$
\begin{aligned}
& \left[\left(M_{AL}(x)^{\delta}+(1-0)\,\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(M_{AU}(x)^{\delta}+(1-0)\,\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],\\
= & \left[\left(M_{AL}(x)^{\delta}+\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(M_{AU}(x)^{\delta}+\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],\\
= & \left[\left(1-N_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(1-N_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}\right].
\end{aligned}
$$

$d_0(A)=$

$$\left\{\left\langle x,\left[N_{AL}(x),N_{AU}(x)\right],\left[\left(1-N_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(1-N_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}\right]\right\rangle : x\in X\right\}$$

$=\overline{\lozenge A}$.

This completes the proof.

(iii) $d_1(A)=\{\langle x,M_{d_1}(A),N_{d_1}(A)\rangle : x\in X\}$,

$$
\begin{aligned}
N_{d_1}(A) &= \left[\left(M_{AL}(x)^{\delta}+(1-1)\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},(M_{AU}(x)^{\delta}+(1-1)\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}\right],\\
&= \left[M_{AL}(x),M_{AU}(x)\right],\\
M_{d_1}(A) &= \left[\left(N_{AL}(x)^{\delta}+\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},(N_{AU}(x)^{\delta}+\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}\right],\\
&= \left[\left(1-M_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}},(1-M_{AU}(x)^{\delta})^{\frac{1}{\delta}}\right],
\end{aligned}
$$

$$d_1(A)=\left[\left[\left(1-M_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}},(1-M_{AU}(x)^{\delta})^{\frac{1}{\delta}}\right],\left[M_{AL}(x),M_{AU}(x)\right]\right]=\overline{\square A}.$$

This completes the proof. $\square$

Definition 3.11. Let $0\leq\alpha\leq\frac{\lambda}{1+\lambda}$, $0\leq\beta\leq\frac{1}{1+\lambda}$ and $A\in GIVIFS_B$, we define the operator of $f_{\alpha,\beta}(A)$ as follows

$f_{\alpha,\beta}(A)=\left\{\left\langle x,M_{f_{\alpha,\beta}}(A),N_{f_{\alpha,\beta}}(A)\right\rangle : x\in X\right\}$,

$$M_{f_{\alpha,\beta}}(A)=\left[\left(N_{AL}(x)^{\delta}+\alpha\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},(N_{AU}(x)^{\delta}+\alpha\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}\right],$$

$$N_{f_{\alpha,\beta}}(A)=\left[\left(M_{AL}(x)^{\delta}+\beta\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},(M_{AU}(x)^{\delta}+\beta\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}\right].$$

Theorem 3.12. *For every $GIVIFS_B$ A and for some α,β where $0\leq\alpha\leq\frac{\lambda}{1+\lambda}$, $0\leq\beta\leq\frac{1}{1+\lambda}$, it holds that*

i. $f_{\alpha,\beta}(A)\in GIVIFS_B$,
ii. $0\leq\gamma\leq\alpha\Rightarrow f_{\gamma,\beta}(A)\subset f_{\alpha,\beta}(A)$,
iii. $0\leq\gamma\leq\beta\Rightarrow f_{\alpha,\beta}(A)\subset f_{\alpha,\gamma}(A)$,
iv. $D_\alpha(A)=f_{\alpha,1-\alpha}(A)$, $\alpha=\frac{\lambda}{1+\lambda}$,
v. $f_{0,1}(A)=\overline{\lozenge A}$, *(it is clear that, if $N_{AU}(x)=N_{AL}(x)$ then $0\leq\beta\leq 1$, $\alpha=0$),*
vi. $f_{1,0}(A)=\overline{\square A}$, *(it is clear that, if $M_{AU}(x)=M_{AL}(x)$ then $0\leq\alpha\leq 1$, $\beta=0$),*

vii. $\overline{f_{\alpha,\beta}(\bar{A})} = f_{\beta,\alpha}(A)$, $0 \leq \alpha, \beta \leq \min\{\frac{\lambda}{1+\lambda}, \frac{1}{1+\lambda}\}$.

Proof. (i)

$$\begin{aligned} M_{f_{\alpha,\beta}(A)U}(x)^{\delta} + N_{f_{\alpha,\beta}(A)U}(x)^{\delta} &= \left[\left(N_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right]^{\delta} \\ &+ \left[\left(M_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right]^{\delta}, \\ &= N_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta} + M_{AU}(x)^{\delta} \\ &+ \beta\pi_{AL}(x)^{\delta}, \\ &= M_{AU}(x)^{\delta} + N_{AU}(x)^{\delta} \\ &+ (\alpha+\beta)\pi_{AL}(x)^{\delta}, \\ &\leq M_{AU}(x)^{\delta} + N_{AU}(x)^{\delta} + \pi_{AL}(x)^{\delta} \\ &= 1. \end{aligned}$$

Proof (ii) and (iii) are clearly.
(iv.) It is clear, $0 \leq \alpha \leq \frac{\lambda}{1+\lambda}$ and $0 \leq 1-\alpha \leq \frac{1}{1+\lambda}$ then $\alpha = \frac{\lambda}{1+\lambda}$

$$M_{f_{\alpha,1-\alpha}}(A) = \left[\left(N_{AL}(x)^{\delta} + \alpha\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(N_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

$$N_{f_{\alpha,1-\alpha}}(A) = \\ \left[\left(M_{AL}(x)^{\delta} + (1-\alpha)\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(M_{AU}(x)^{\delta} + (1-\alpha)\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

$$f_{\alpha,1-\alpha}(A) = \left\{\left\langle x, M_{f_{\alpha}}(A), N_{f_{1-\alpha}}(A)\right\rangle : x \in X\right\} = d_{\alpha}(A).$$

Remark 3.2. By using (iv) we have $d_0(A) = f_{0,1}(A)$ and $d_1(A) = f_{1,0}(A)$.

(v) By using Remark 3.2 and Theorem 3.10 it follows that $f_{0,1}(A) = \overline{\Diamond A}$.
(vi) By using Remark 3.2 and Theorem 3.10 it follows that $f_{1,0}(A) = \overline{\Box A}$.
(vii) Since

$$f_{\beta,\alpha}(A) = \left\{\left\langle x, M_{f_{\beta,\alpha}}(A), N_{f_{\beta,\alpha}}(A)\right\rangle : x \in X\right\},$$

$$M_{f_{\beta,\alpha}}(A) = \left[\left(N_{AL}(x)^{\delta} + \beta\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(N_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

$$N_{f_{\beta,\alpha}}(A) = \left[\left(M_{AL}(x)^{\delta} + \alpha\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(M_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right].$$

And

$$f_{\alpha,\beta}(\bar{A}) = \left\{\left\langle x, M_{f_{\alpha,\beta}}(\bar{A}), N_{f_{\alpha,\beta}}(\bar{A})\right\rangle : x \in X\right\},$$

$$M_{f_{\alpha,\beta}}(\bar{A}) = \left[\left(M_{AL}(x)^{\delta} + \alpha\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(M_{AU}(x)^{\delta} + \alpha\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

$$N_{f_{\alpha,\beta}}(\bar{A}) = \left[\left(N_{AL}(x)^{\delta} + \beta\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(N_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

hence
$$\overline{f_{\alpha,\beta}(\bar{A})} = \left\{\langle x, N_{f_\beta}(\bar{A}), M_{f_\alpha}(\bar{A})\rangle : x \in X\right\}.$$
Finally, we have $\overline{f_{\alpha,\beta}(\bar{A})} = f_{\beta,\alpha}(A)$. □

Definition 3.13. Let $\alpha, \beta \in [0,1]$ and $A \in GIVIFS_B$, we define the operator of $g_{\alpha,\beta}(A)$ as follows
$$g_{\alpha,\beta}\left(A\right) = \left\{\left\langle x, M_{g_{\alpha,\beta}}\left(A\right), N_{g_{\alpha,\beta}}\left(A\right)\right\rangle : x \in X\right\},$$
$$M_{g_{\alpha,\beta}}\left(A\right) = \alpha^{\frac{1}{\delta}} N_A\left(x\right) = [\alpha^{\frac{1}{\delta}} N_{AL}\left(x\right), \alpha^{\frac{1}{\delta}} N_{AU}\left(x\right)],$$
$$N_{g_{\alpha,\beta}}\left(A\right) = \beta^{\frac{1}{\delta}} M_A\left(x\right) = [\beta^{\frac{1}{\delta}} M_{AL}\left(x\right), \beta^{\frac{1}{\delta}} M_{AU}\left(x\right)].$$

Theorem 3.14. *For every $GIVIFS_B$ A, and for every three real numbers α, β, $\gamma \in [0,1]$, it holds that*

i. $g_{\alpha,\beta}(A) \in GIVIFS_B$,
ii. $\alpha \le \gamma \Rightarrow g_{\alpha,\beta}(A) \subset g_{\gamma,\beta}(A)$,
iii. $\beta \le \gamma \Rightarrow g_{\alpha,\beta}(A) \supset g_{\alpha,\gamma}(A)$,
iv. $g_{\alpha,\beta}(g_{\gamma,\tau}(A)) = g_{\alpha\gamma,\beta\tau}(A) = g_{\gamma,\tau(g_{\alpha,\beta}(A))}$, $\tau \in [0,1]$,
v. $g_{\alpha,\beta}(C(A)) = I(g_{\alpha,\beta}(A))$,
vi. $g_{\alpha,\beta}(I(A)) = C(g_{\alpha,\beta}(A))$,
vii. $\overline{g_{\alpha,\beta}(\bar{A})} = g_{\beta,\alpha}(A)$.

Proof. (i)
$$\begin{aligned} M_{g_{\alpha,\beta}U}\left(A\right)\left(x\right)^{\delta} + N_{g_{\alpha,\beta}U}\left(A\right)\left(x\right)^{\delta} &= \left(\alpha^{\frac{1}{\delta}} N_{AU}\left(x\right)\right)^{\delta} + \left(\beta^{\frac{1}{\delta}} M_{AU}\left(x\right)\right)^{\delta}, \\ &= \alpha N_{AU}\left(x\right)^{\delta} + \beta M_{AU}\left(x\right)^{\delta}, \\ &\le N_{AU}\left(x\right)^{\delta} + M_{AU}\left(x\right)^{\delta} \le 1. \end{aligned}$$
Finally, it can be concluded that $g_{\alpha,\beta}(A) \in GIVIFS_B$.
(ii) Since $\alpha \le \gamma$ then it is clear that $\alpha^{\frac{1}{\delta}} \le \gamma^{\frac{1}{\delta}}$
hence $[\alpha^{\frac{1}{\delta}} N_{AL}(x), \alpha^{\frac{1}{\delta}} N_{AU}(x)] \le [\gamma^{\frac{1}{\delta}} N_{AL}(x), \gamma^{\frac{1}{\delta}} N_{AU}(x)]$.

Finally we have $g_{\alpha,\beta}(A) \subset g_{\gamma,\beta}(A)$. The proof of (iii) is similar (ii).
(iv) $A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$,
$g_{\gamma,\tau}\left(A\right) = \left\{\left\langle x, M_{g_{\gamma,\tau}}\left(A\right), N_{g_{\gamma,\tau}}\left(A\right)\right\rangle : x \in X\right\}$.
$M_{g_{\gamma,\tau}}\left(A\right) = [\gamma^{\frac{1}{\delta}} N_{AL}\left(x\right), \gamma^{\frac{1}{\delta}} N_{AU}\left(x\right)]$,
$N_{g_{\gamma,\tau}}\left(A\right) = [\tau^{\frac{1}{\delta}} M_{AL}\left(x\right), \tau^{\frac{1}{\delta}} M_{AU}\left(x\right)]$,
$$\begin{aligned} g_{\alpha,\beta}\left(g_{\gamma,\tau}(A)\right) &= \left\{\left\langle x, \alpha^{\frac{1}{\delta}}\gamma^{\frac{1}{\delta}} N_A\left(x\right), \beta^{\frac{1}{\delta}}\tau^{\frac{1}{\delta}} M_A(x)\right\rangle : x \in X\right\}, \\ &= \left\{\left\langle x, (\alpha\gamma)^{\frac{1}{\delta}} N_A\left(x\right), (\beta\tau)^{\frac{1}{\delta}} M_A(x)\right\rangle : x \in X\right\}, \\ &= g_{\alpha\gamma,\beta,\tau}(A), \end{aligned}$$
and
$$\begin{aligned} g_{\gamma,\tau}\left(g_{\alpha,\beta}(A)\right) &= \left\{\left\langle x, \gamma^{\frac{1}{\delta}}\alpha^{\frac{1}{\delta}} N_A\left(x\right), \tau^{\frac{1}{\delta}}\beta^{\frac{1}{\delta}} M_A(x)\right\rangle : x \in X\right\}, \\ &= \left\{\left\langle x, (\gamma\alpha)^{\frac{1}{\delta}} N_A\left(x\right), (\tau\beta)^{\frac{1}{\delta}} M_A(x)\right\rangle : x \in X\right\}, \end{aligned}$$

$$\begin{aligned} &= \left\{\left\langle x, (\alpha\gamma)^{\frac{1}{\delta}} N_A(x), (\beta\tau)^{\frac{1}{\delta}} M_A(x)\right\rangle : x \in X\right\}, \\ &= g_{\alpha\gamma,\beta\tau}(A), \end{aligned}$$

hence

$$g_{\alpha,\beta}(g_{\gamma,\tau}(A)) = g_{\alpha\gamma,\beta\tau}(A) = g_{\gamma,\tau}(g_{\alpha,\beta}(A)).$$

(v)

$$C(A) = \{\langle x, \max_{y\in X} M_A(y), \min_{y\in X} N_A(y)\rangle : x \in X\},$$

$$\begin{aligned} g_{\alpha,\beta}(C(A)) &= \{\langle x, \alpha^{\frac{1}{\delta}} \min_{y\in X} N_A(y), \beta^{\frac{1}{\delta}} \max_{y\in X} M_A(y)\rangle : x \in X\}, \\ &= \{\langle x, \min_{y\in X} \alpha^{\frac{1}{\delta}} N_A(y), \max_{y\in X} \beta^{\frac{1}{\delta}} M_A(y)\rangle : x \in X\}, \\ &= I(g_{\alpha,\beta}(A)). \end{aligned}$$

(vi)

$$I(A) = \{\langle x, \min_{y\in X} M_A(y), \max_{y\in X} N_A(y)\rangle : x \in X\},$$

$$\begin{aligned} g_{\alpha,\beta}(I(A)) &= \{\langle x, \alpha^{\frac{1}{\delta}} \max_{y\in X} N_A(y), \beta^{\frac{1}{\delta}} \min_{y\in X} M_A(y)\rangle : x \in X\}, \\ &= \{\langle x, \max_{y\in X} \alpha^{\frac{1}{\delta}} N_A(y), \min_{y\in X} \beta^{\frac{1}{\delta}} M_A(y)\rangle : x \in X\}, \\ &= C(g_{\alpha,\beta}(A)), \quad \alpha, \beta \in [0,1]. \end{aligned}$$

(vii) Let $A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\}$ is a $GIVIFS_B$ then
$\bar{A} = \{\langle x, N_A(x), M_A(y)\rangle : x \in X\}$,
$g_{\beta,\alpha}(A) = \left\{\left\langle x, \beta^{\frac{1}{\delta}} N_A(x), \alpha^{\frac{1}{\delta}} M_A(x)\right\rangle : x \in X\right\}$,
$g_{\alpha,\beta}(\bar{A}) = \left\{\left\langle x, \alpha^{\frac{1}{\delta}} M_A(x), \beta^{\frac{1}{\delta}} N_A(x)\right\rangle : x \in X\right\}$,
$\overline{g_{\alpha,\beta}(\bar{A})} = \left\{\left\langle x, \beta^{\frac{1}{\delta}} N_A(x), \alpha^{\frac{1}{\delta}} M_A(x)\right\rangle : x \in X\right\}$,
hence $\overline{g_{\alpha,\beta}(\bar{A})} = g_{\beta,\alpha}(A)$. □

Definition 3.15. Let $0 \le \alpha \le \frac{\lambda}{1+\lambda}$, $\beta \in [0,1]$ and $A \in GIVIFS_B$, we define the operator of $j_{\alpha,\beta}(A)$ as follows
$j_{\alpha,\beta}(A) = \left\{\left\langle x, M_{j_{\alpha,\beta}}(A), N_{j_{\alpha,\beta}}(A)\right\rangle : x \in X\right\}$,
$M_{j_{\alpha,\beta}}(A) = \left[\left(N_{AL}(x)^\delta + \alpha\pi_{AU}(x)^\delta\right)^{\frac{1}{\delta}}, \left(N_{AU}(x)^\delta + \alpha\pi_{AL}(x)^\delta\right)^{\frac{1}{\delta}}\right]$,
$N_{j_{\alpha,\beta}}(A) = \left[\beta^{\frac{1}{\delta}} M_{AL}(x), \beta^{\frac{1}{\delta}} M_{AU}(x)\right]$.

Theorem 3.16. *For every $GIVIFS_B$ A, and for some real numbers $0 \le \alpha \le \frac{\lambda}{1+\lambda}$, $\beta \in [0,1]$, it holds that*

i. $j_{\alpha,\beta}(A) \in GIVIFS_B$,
ii. $\alpha \le \gamma \Rightarrow j_{\alpha,\beta}(A) \subset j_{\gamma,\beta}(A)$, $0 \le \gamma \le \frac{\lambda}{1+\lambda}$,
iii. $\beta \le \gamma \Rightarrow j_{\alpha,\beta}(A) \supset j_{\alpha,\gamma}(A)$, $0 \le \gamma \le 1$,
iv. $j_{1,1}(A) = \overline{\Box A}$, *(it is clear that, if $M_{AU}(x) = M_{AL}(x)$ then, $0 \le \alpha \le 1$)*,
v. $j_{0,1}(A) = \bar{A}$, *(it is clear that, if $N_{AU}(x) = N_{AL}(x)$ then, $\alpha = 0$)*.

Proof. (i)

$$\begin{aligned} M_{j_{\alpha,\beta}(A)U}(x)^{\delta}+N_{j_{\alpha,\beta}(A)U}(x)^{\delta} &= \left(\left(N_{AU}(x)^{\delta}+\alpha\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right)^{\delta} \\ &+ \left(\beta^{\frac{1}{\delta}}M_{AU}(x)\right)^{\delta}, \\ &= \left(N_{AU}(x)^{\delta}+\alpha\pi_{AL}(x)^{\delta}\right)+\beta M_{AU}(x)^{\delta}, \\ &\leq N_{AU}(x)^{\delta}+\pi_{AL}(x)^{\delta}+M_{AU}(x)^{\delta}=1. \end{aligned}$$

Finally, it can be concluded that $j_{\alpha,\beta}(A)\in GIVIFS_B$.
(ii) Since $\alpha\leq\gamma$ then it is clear that

$$\begin{aligned} &\left[\left(N_{AL}(x)^{\delta}+\alpha\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(N_{AU}(x)^{\delta}+\alpha\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right] \\ &\qquad\leq\left[\left(N_{AL}(x)^{\delta}+\gamma\pi_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(N_{AU}(x)^{\delta}+\gamma\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right]. \end{aligned}$$

Finally we have $j_{\alpha,\beta}(A)\subset j_{\gamma,\beta}(A)$.
The proof of (iii) is similar (ii). Proofs (iv) and (v) are clearly. $\square$

Definition 3.17. Let $\alpha\in[0,1]$, $0\leq\beta\leq\frac{\lambda}{1+\lambda}$ and $A\in GIVIFS_B$, we define the operator of $H_{\alpha,\beta}(A)$ as follows
$H_{\alpha,\beta}(A)=\left\{\left\langle x,M_{H_{\alpha,\beta}}(A),N_{H_{\alpha,\beta}}(A)\right\rangle : x\in X\right\}$,
$M_{H_{\alpha,\beta}}(A)=\left[\left(\alpha M_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}},\left(\alpha M_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}\right]$,
$N_{H_{\alpha,\beta}}(A)=\left[(N_{AL}(x)^{\delta}+\beta\pi_{AU}(x)^{\delta})^{\frac{1}{\delta}},(N_{AU}(x)^{\delta}+\beta\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}\right]$.

Theorem 3.18. *For every $GIVIFS_B$ A, and for every real numbers $\alpha\in[0,1]$, $0\leq\beta\leq\frac{\lambda}{1+\lambda}$, it holds that*

i. $H_{\alpha,\beta}(A)\in GIVIFS_B$,
ii. $\alpha\leq\gamma\Rightarrow H_{\alpha,\beta}(A)\subset H_{\gamma,\beta}(A)$, $\quad\gamma\in[0,1]$,
iii. $\beta\leq\gamma\Rightarrow H_{\alpha,\beta}(A)\supset H_{\alpha,\gamma}(A)$, $\quad 0\leq\gamma\leq\frac{\lambda}{1+\lambda}$,
iv. $H_{1,0}(A)=A$,
v. $H_{1,1}(A)=\Box A$, *(it is clear that, if $M_{AU}(x)=M_{AL}(x)$ then $0\leq\beta\leq 1$).*

Proof.

$$\begin{aligned} M_{H_{\alpha,\beta}(A)U}(x)^{\delta}+N_{H_{\alpha,\beta}(A)U}(x)^{\delta} &= \left(\alpha^{\frac{1}{\delta}}M_{AU}(x)\right)^{\delta} \\ &+ \left(\left(N_{AU}(x)^{\delta}+\alpha\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right)^{\delta}, \\ &= \beta M_{AU}(x)^{\delta}+\left(N_{AU}(x)^{\delta}+\alpha\pi_{AL}(x)^{\delta}\right), \\ &\leq M_{AU}(x)^{\delta}+\pi_{AL}(x)^{\delta}+N_{AU}(x)^{\delta}=1. \end{aligned}$$

Proofs (ii), (iii), (iv) and (v) are clearly. $\square$

Definition 3.19. Let $\alpha \in [0,1]$, $0 \leq \beta \leq \frac{1}{1+\lambda}$ and $A \in GIVIFS_B$, we define the operator of $h_{\alpha,\beta}(A)$ as follows

$$h_{\alpha,\beta}(A) = \left\{\left\langle x, M_{h_{\alpha,\beta}}(A), N_{h_{\alpha,\beta}}(A)\right\rangle : x \in X\right\},$$

$$M_{h_{\alpha,\beta}}(A) = \left[\left(\alpha N_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}, \left(\alpha N_{AU}(x)^{\delta}\right)^{\frac{1}{\delta}}\right],$$

$$N_{h_{\alpha,\beta}}(A) = [(M_{AL}(x)^{\delta} + \beta\pi_{AU}(x)^{\delta})^{\frac{1}{\delta}}, (M_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta})^{\frac{1}{\delta}}].$$

Theorem 3.20. *For every $GIVIFS_B$ A, and for every real numbers $\alpha \in [0,1]$, $0 \leq \beta \leq \frac{1}{1+\lambda}$, it holds that*

i. $h_{\alpha,\beta}(A) \in GIVIFS_B$,
ii. $\alpha \leq \gamma \Rightarrow h_{\alpha,\beta}(A) \subset h_{\gamma,\beta}(A)$, $\quad \gamma \in [0,1]$,
iii. $\beta \leq \gamma \Rightarrow h_{\alpha,\beta}(A) \supset h_{\alpha,\gamma}(A)$, $\quad 0 \leq \gamma \leq \frac{1}{1+\lambda}$,
iv. $h_{\alpha,\beta}(\bar{A}) = H_{\alpha,\beta}(A)$, $\quad 0 \leq \beta \leq \min\{\frac{\lambda}{1+\lambda}, \frac{1}{1+\lambda}\}$,
v. $h_{1,0}(A) = \bar{A}$,
vi. $h_{1,1}(A) = \overline{\Diamond A}$. *(it is clear that, if $N_{AU}(x) = N_{AL}(x)$ then $0 \leq \beta \leq 1$).*

Proof. (i)

$$\begin{aligned} M_{h_{\alpha,\beta}(A)U}(x)^{\delta} + N_{h_{\alpha,\beta}(A)U}(x)^{\delta} &= \left(\alpha^{\frac{1}{\delta}} N_{AU}(x)\right)^{\delta} \\ &+ \left(\left(M_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta}\right)^{\frac{1}{\delta}}\right)^{\delta}, \\ &= \alpha N_{AU}(x)^{\delta} + \left(M_{AU}(x)^{\delta} + \beta\pi_{AL}(x)^{\delta}\right), \\ &\leq N_{AU}(x)^{\delta} + \pi_{AL}(x)^{\delta} + M_{AU}(x)^{\delta} = 1. \end{aligned}$$

The proofs (iii), (ii), (iv), (v) and (vi) are clearly. □

Remark 3.3. According to definition, the operators of $D_\alpha(A)$ and $F_{\alpha,\beta}$ (A) increases the membership and non-membership degree$A$, the operators of $d_\alpha(A)$and $f_{\alpha,\beta}(A)$ increases the membership and non-membership degree $\bar{A}$, the operator of $G_{\alpha,\beta}(A)$ reduces the membership and non-membership degree A, the operator of $g_{\alpha,\beta}(A)$ reduces the membership and non-membership degree $\bar{A}$, the operator of $h_{\alpha,\beta}(A)$ reduces the membership degree $\bar{A}$ and increases non-membership degree $\bar{A}$, the operator of $H_{\alpha,\beta}(A)$ reduces the membership degree A and increases non-membership degree A, the operator of $j_{\alpha,\beta}(A)$ increases the membership degree $\bar{A}$ and reduces non-membership degree $\bar{A}$, the operator of $J_{\alpha,\beta}(A)$ increases the membership degree A and reduces non-membership degree A.

Example 3.21. Let $A = \{\langle x_1, [0.5, 0.6], [0.2, 0.3]\rangle\}$, $\delta = 2$, then
$\pi_A(x_1) = [0.7416, 0.8426]$, $\lambda_A = 0.4545$ and
$F_{\alpha,\beta}(A) = \left\{\langle x_1, [(0.25+0.71\alpha)^{\frac{1}{2}}, (0.36+0.55\alpha)^{\frac{1}{2}}], [(0.04+0.71\beta)^{\frac{1}{2}}, (0.09+0.55\beta)^{\frac{1}{2}}]\rangle\right\}$
, $0 \leq \alpha \leq 0.6875$, $0 \leq \beta \leq 0.3125$.
$G_{\alpha,\beta}(A) = \left\{\langle x_1, [0.5\sqrt{\alpha}, 0.6\sqrt{\alpha}], [0.2\sqrt{\beta}, 0.3\sqrt{\beta}]\rangle\right\}$, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$.
$J_{\alpha,\beta}(A) = \left\{\left\langle x_1, [(0.25 + 0.71\alpha)^{\frac{1}{2}}, (0.36 + 0.55\alpha)^{\frac{1}{2}}], [0.2\sqrt{\beta}, 0.3\sqrt{\beta}]\right\rangle\right\}$,

$0 \le \alpha \le 0.6875,\ 0 \le \beta\ \le 1.$
$f_{\alpha,\beta}(A)=\left\{\left\langle x_1,\left[(0.04+0.71\alpha)^{\frac{1}{2}},(0.09+0.55\alpha)^{\frac{1}{2}}\right],\left[(0.25+0.71\beta)^{\frac{1}{2}},(0.36+0.55\beta)^{\frac{1}{2}}\right]\right\rangle\right\},$
$0 \le \alpha \le 0.3125,\ 0 \le \beta \le 0.6875.$
$g_{\alpha,\beta}(A) = \left\{\left\langle x_1, \left[0.2\sqrt{\alpha}, 0.3\sqrt{\alpha}\right], \left[0.5\sqrt{\beta}, 0.6\sqrt{\beta}\right]\right\rangle\right\}, 0 \le \alpha \le 1,\ 0 \le \beta \le 1.$
$j_{\alpha,\beta}(A) = \left\{\left\langle x_1, \left[(0.04 + 0.71\alpha\ )^{\frac{1}{2}}, (0.09 + 0.55\alpha)^{\frac{1}{2}}\right], \left[0.5\sqrt{\beta}, 0.6\sqrt{\beta}\right]\right\rangle\right\},$
$0 \le \alpha \le 0.3125,\ 0 \le \beta \le 1.$
$H_{\alpha,\beta}(A) = \left\{\left\langle x_1\ , \left[0.5\sqrt{\alpha},\ 0.6\sqrt{\alpha}\right], \left[(0.04 + 0.71\beta)^{\frac{1}{2}}, (0.09 + 0.55\beta)^{\frac{1}{2}}\right]\right\rangle\right\},$
$0 \le \alpha \le 1,\ 0 \le \beta \le 0.3125.$
$h_{\alpha,\beta}(A) = \left\{\left\langle x_1, \left[0.2\sqrt{\alpha}, 0.3\sqrt{\alpha}\right], \left[(0.25 + 0.71\beta)^{\frac{1}{2}}, (0.36 + 0.55\beta)^{\frac{1}{2}}\right]\right\rangle\right\},$
$0 \le \alpha \le 1,\ 0 \le \beta \le 0.6875.$

4. Conclusion

We have introduced newly defined modal types of operators over Baloui's generalized interval valued intuitionistic fuzzy sets and their relationships are proved. We show that these operators are $GIVIFS_B$. These operators are well defined since, if $\delta = 1$, $M_{AU}(x) = M_{AL}(x)$, and $N_{AU}(x) = N_{AL}(x)$, the results agree with IFS by replacing the new requirements related to α and β. In future, the application of these operators will be proposed and other operators over $GIVIFS_B$ are to be introduced. Some open problems are as follows: definition of level operator, negation operator and other types over $GIVIFS_B$ and study their properties.

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