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A TRACE-TYPE FUNCTIONAL METHOD FOR DETERMINATION OF A COEFFICIENT IN AN INVERSE HEAT CONDUCTION PROBLEM†

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Abstract. This paper investigates the inverse problem of determining an unknown heat radiative coefficient, which is only time-dependent. This is an ill-posed problem, that is, small errors in data may cause huge deviations in determining solution. In this paper, the existence and uniqueness of the problem is established by the second Volterra integral equation theory, and the method of trace-type functional formulation combined with finite difference scheme is studied. One typical numerical example using the proposed method is illustrated and discussed.

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1. Introduction

The identification of coefficients in parabolic equations is an ill-posed problem that has received considerable attention from several authors in a variety of fields. Some detailed treatments of problems in these areas can be referred to [3, 11, 14, 15, 20].

The inverse problem of identifying coefficient $q(x)$ in the following parabolic equation

$$u_t - u_{xx} + q(x)u = 0, \quad (x,t) \in Q \tag{1}$$

from final overspecified data $u(x,T)$ has been studied by several authors, and one can refer to [9, 10, 13, 14, 17–19, 21]. In [10], the authors used the Hölder

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space method to determine the unknown coefficient $q(x)$ from additional information at $t = T$. In [18], existence and uniqueness for the determination of $q(x)$ were derived by using the contracting mapping principle. In [9, 21], motivated by heuristic arguments, the optimization method was applied to stabilize the inverse coefficient problem. The [9] proved the existence of minimizer and the convergence of approximate solution in finite-dimensional space, while in [21], the authors constructed a new control functional and prove the existence and uniqueness of minimizing solution. In [19], numerical solution of $q(x)$ was given by using the method of quasi-reversibility.

In this paper, we will mainly study the following inverse problem:

$$u_t(x,t) = u_{xx}(x,t) + p(t)u(x,t), \quad (x,t) \in \Omega \times (0, t_{max}], \tag{2}$$

$$u(0,t) = h_0(t), \qquad u(1,t) = h_1(t), \tag{3}$$

$$u(x,0) = \varphi(x), \tag{4}$$

$$u(x^*,t) = \psi(t), \tag{5}$$

where, $\varphi(x)$, $h_0(t)$, $h_1(t)$ and $\psi(x)$ are known functions, and x^* is a fixed prescribed point in the admissible set. $p(t)$ is an unknown function to be determined.

The existence, uniqueness and some applications of this problem were presented in [1–7, 12]. For the conditions (3) changed by the Neumann's conditions, the authors in [16] considered the uniqueness theorem by the uniqueness of the solution of Volterra integral equation of second kind, and they used the method of fundamental solutions to give the numerical results.

The organization of this paper is as follows: in Section 2, we try to give the uniqueness of the proposed problem. Section 3 is devoted to the numerical technique of trace-type functional method by mathematical formula. A typical numerical experiment is given in Section 4 to illustrate the effectiveness of our proposed method. And Section 5 is the conclusion of this paper.

2. Existence and uniqueness

Theorem 1. *Suppose that $\varphi(x)$, $h_0(t)$, $h_1(t)$, and $\psi(t)$ are continuous functions such that, for some $R > 0$, $\epsilon > 0$,*

$$|h_i(t)| \le R, i = 0,1, \quad |\psi(t)| \ge \epsilon. \tag{6}$$

Then, the problem (2)–(5) has a unique solution (u,p).

Proof. According to the transformation in [8]:

$$u(x,t) = r(t)v(x,t), \tag{7}$$

where,

$$r(t) = e^{\int_0^t p(\tau)d\tau}, \tag{8}$$

so that

$$v(x,t) = \frac{u(x,t)}{r(t)}, \quad and \quad p(t) = \frac{r'(t)}{r(t)}. \tag{9}$$

Transformation (7)−(9) can eliminate $p(t)$ in (2) and rewrite (2)−(4) as follows:

$$v_t(x,t) = v_{xx}(x,t), \quad (x,t) \in \Omega \times (0, t_{max}], \tag{10}$$

$$v(0,t) = g(t), \quad v(1,t) = h(t)g(t), \tag{11}$$

$$v(x,0) = \varphi(x), \tag{12}$$

where, $g(t) = \frac{h_0(t)}{r(t)}$ is an unknown function and $h(t) = \frac{h_1(t)}{h_0(t)}$ is a known function. In this situation the over-specified condition becomes:

$$v(x^*,t) = \frac{\psi(t)}{h_0(t)} g(t). \tag{13}$$

Let us assume, for the moment, that $g(t)$ is known. If we define, for $x \in \mathbb{R}$ and $t > 0$,

$$K(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}, \tag{14}$$

$$\theta(x,t) = \sum_{m=-\infty}^{\infty} K(x+2m,t), \tag{15}$$

then, by Cannon ([8], Theorem 6.3.1), problem (10)−(12) has a unique solution of the form

$$v(x,t) = w(x,t) - 2\int_0^t \frac{\partial\theta}{\partial x}(x, t-\tau)g(\tau)d\tau + 2\int_0^t \frac{\partial\theta}{\partial x}(x-1, t-\tau)h(\tau)g(\tau)d\tau, \tag{16}$$

where,

$$w(x,t) = \int_0^1 [\theta(x-\xi,t) - \theta(x+\xi,t)]\varphi(\xi)d\xi$$

Now, in order to determine $g(t)$, we here impose the condition (13) to (16) and get the following

$$G(t) = w(x^*,t) + \int_0^t H(t,\tau,G(\tau))d\tau, \tag{17}$$

where, $G(t) = \frac{\psi(t)}{h_0(t)} g(t)$ and

$$H(t,\tau,s) = \frac{2[\frac{\partial\theta}{\partial x}(x^*-1, t-\tau)h_1(\tau) - \frac{\partial\theta}{\partial x}(x^*, t-\tau)h_0(\tau)]}{\psi(\tau)} s.$$

We can see that Eq.(17) is a linear Volterra integral equation of the second kind.

Obviously, $w(x^*,t)$ and $H(t,\tau,s)$ are continuous functions and thus, from the Theorem 8.2.1 in [8], the integral equation (17) has a unique solution if H satisfies the following Lipschitz condition

$$|H(t,\tau,s_1) - H(t,\tau,s_2)| \le L(t,\tau)|s_1 - s_2|, \tag{18}$$

where

$$\int_{t_0}^{t} L(t,\tau)d\tau \le \alpha(t-t_0) \quad (t > t_0) \tag{19}$$

for some monotone increasing function α, with $\lim_{\eta\downarrow 0}\alpha(\eta) = 0$, and if

$$\int_{t_0}^{t} |H(t,\tau,0)|d\tau \le \beta(t-t_0) \quad (t > t_0) \tag{20}$$

for some non-negative function β, with $\lim_{\eta\downarrow 0}\beta(\eta) = 0$. Since $H(t,\tau,0) = 0$, we observe that (20) holds for $\beta = 0$ and since $h_0(t)$, $h_1(t)$, and $\psi(t)$ satisfy the conditions (6), there exists a constant M, such that

$$|H(t,\tau,s_1) - H(t,\tau,s_2)| \le M[\frac{\partial\theta}{\partial x}(x^*-1,t-\tau) + \frac{\partial\theta}{\partial x}(x^*,t-\tau)]|s_1 - s_2|.$$

Clearly, for $m = 1, 2, \cdots$, and $x^* \in (0,1)$, we can obtain

$$-(x^*+2m)^2 \le -(2m)^2$$

and

$$-(x^*-2m)^2 \le -(2m-1)^2.$$

On the other hand, $e^{-x} < 1/x^2$, therefore

$$\begin{aligned}\frac{\partial\theta}{\partial x}(x^*-1,t) &\le \frac{1}{2\sqrt{\pi t}}\sum_{m=-\infty}^{\infty} \mid \frac{x^*-1+2m}{2t} \mid e^{-(x^*-1+2m)^2/4t} \\ &\le \frac{1}{2\sqrt{\pi t}}\sum_{m=-\infty}^{\infty} \mid \frac{x^*-1+2m}{2t} \mid \frac{16t^2}{(x^*-1+2m)^4} \\ &= \frac{4\sqrt{t}}{\sqrt{\pi}}\sum_{m=-\infty}^{\infty} \frac{1}{\mid x^*-1+2m \mid^3} < A\sqrt{t},\end{aligned}$$

for some constant A. Similarly, there exists a constant B, such that

$$\frac{\partial\theta}{\partial x}(x^*,t) < B\sqrt{t}.$$

Thus (19) holds if we take $\alpha(\eta) = M(A+B)\sqrt{\eta^3}$.

Therefore, the integral equation (17) has a unique solution $G(t)$ and thus problem (2)−(5) has a solution of the form (16) which is, in fact, unique by Cannon ([8], Theorem 6.4.1). □

3. Numerical technique

In this section, we will use the method of trace-type functional to solve the above inverse problem as follows.

If the functions pair (u,p) solves the inverse problem (2)−(5), then we have

$$\psi_t(t) = u_{xx}|_{x=x^*} + p(t)u(x^*,t). \tag{21}$$

From this, we can obtain

$$p(t) = \frac{\psi_t(t) - u_{xx}|_{x=x^*}}{\psi(t)}. \tag{22}$$

Substituting (22) to Equation (2) gives the following initial boundary value problem:

$$u_t(x,t) = u_{xx}(x,t) + \frac{\psi_t(t) - u_{xx}|_{x=x^*}}{\psi(t)} u(x,t), \quad (x,t) \in \Omega \times (0, t_{max}], \tag{23}$$

$$u(0,t) = h_0(t), \qquad u(1,t) = h_1(t) \tag{24}$$

$$u(x,0) = \varphi(x), \tag{25}$$

$$\frac{u_i^{j+1} - u_i^j}{\tau} = \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} + k(u_{i^*}^j), \qquad 1 \le i \le N-1, \;\; 0 \le j \le M-1, \tag{26}$$

$$u_0^j = h_0(t_j), \qquad u_N^j = h_1(t_j), \qquad 1 \le j \le M, \tag{27}$$

$$u_i^0 = \varphi(x_i), \qquad 0 \le i \le N, \tag{28}$$

$$k(u_{i^*}^j) = \frac{\frac{\psi_{j+1} - \psi_j}{\tau} - \frac{u_{i^*+1}^j - 2u_{i^*}^j + u_{i^*-1}^j}{h^2}}{\psi_j} u_i^j, \tag{29}$$

here,

$$u_{i^*}^j = \frac{x_{i_0+1} - x^*}{x_{i_0+1} - x_{i_0}} u_{i_0}^j + \frac{x_{i_0} - x^*}{x_{i_0} - x_{i_0+1}} u_{i_0+1}^j, \qquad if \quad i_0 \le i^* < i_0 + 1. \tag{30}$$

Now, we give the following finite difference Crank-Nicolson scheme about the above problem:

$$[A - \frac{1}{2}(\frac{\psi_{j+2}}{\psi_{j+1}} + \Phi_{j+1}U^j)]U^{j+1} = BU^j + \frac{1}{2}(\frac{\psi_{j+1}}{\psi_j} + \Phi_j U^j), \tag{31}$$

where

$$A = \begin{pmatrix} 1+r & -0.5r & 0 & 0 & \cdots \\ -0.5r & 1+r & -0.5r & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & -0.5r & 1+r & -0.5r \\ 0 & 0 & \cdots & -0.5r & 1+r \end{pmatrix},$$

$$B = \begin{pmatrix} 1-r & 0.5r & 0 & 0 & \cdots \\ 0.5r & 1-r & 0.5r & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0.5r & 1-r & 0.5r \\ 0 & 0 & \cdots & 0.5r & 1-r \end{pmatrix},$$

$$\Phi_j = \frac{1}{\psi_j}(\underbrace{0,\cdots,0}_{i_0-1}, -r\frac{x_{i_0}-(x^*-h)}{x_{i_0}-x_{i_0-1}}, -r(\frac{x_{i_0-1}-(x^*-h)}{x_{i_0-1}-x_{i_0}} - 2\frac{x_{i_0+1}-x^*}{x_{i_0+1}-x_{i_0}}),$$

$$-r(\frac{x_{i_0+2}-(x^*+h)}{x_{i_0+2}-x_{i_0+1}} - 2\frac{x_{i_0}-x^*}{x_{i_0}-x_{i_0+1}}), -r\frac{x_{i_0+1}-(x^*+h)}{x_{i_0+1}-x_{i_0+2}}, \underbrace{0,\cdots,0}_{M-i_0-2}),$$

$U^j = \begin{pmatrix} u_0^j \\ u_1^j \\ \vdots \\ u_N^j \end{pmatrix}$ and $r = \frac{\tau}{h^2}$.

After solving the above difference equation, we can give the approximate solution of $p(t)$ as follows:

$$p^\delta(t) = \frac{\psi_t(t) - u_{xx}|_{x=x^*}}{\psi(t)} \approx \frac{\psi_t^\delta(t) - u_{xx}^\delta|_{x=x^*}}{\psi^\delta(t)}, \tag{32}$$

where, the superscript δ means that the approximate solution is obtained by the noisy data, in which the noisy level is δ.

4. Numerical verification

In this section we test numerical examples to demonstrate the feasibility of our approach. The examples are performed using MATLAB.

Example: Take the exact solution for the problem (2)–(5) as

$$u(x,t) = \exp(-t^2)x, \tag{33}$$

and we can compute the given data of boundary conditions $h_0(t)$, $h_1(t)$, $\psi(t)$ and heat radiative coefficient $p(t)$ directly from (33) as follows:

$$h_0(t) = 0, \tag{34}$$

$$h_1(t) = \exp(-t^2), \tag{35}$$

$$\psi(t) = \exp(-0.25)x, \tag{36}$$

$$p(t) = -2t. \tag{37}$$

The artificial error is introduced into the additional specification data by defining the function:

$$\psi_\delta = \psi(1+\delta)$$

where δ denotes the noisy level.

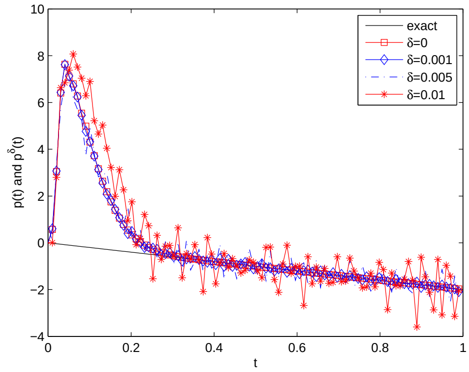

FIGURE 1. The approximate solutions solved by the TTF and FDM schemes for Example 1 when we choose 100×100 grid points.

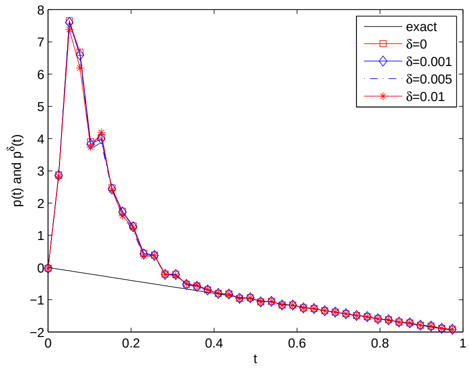

FIGURE 2. The approximate solutions solved by the TTF and FDM schemes for Example 1 when we choose 40×40 grid points.

As shown in Figs. 1 and 2, our method solves the model problem quite effectively in the interval $[0.3, 1]$, but not so well in the rest interval $[0, 0.3]$. The main reason is that the finite difference method works not well near the left end of the time interval. We can see that the numerical accuracy is not too sensitive to the numbers of grid points.

5. Conclusions

In this paper, we give a simple method to solve an inverse coefficient problem in a one-dimensional heat equation, in which we combine the trace-type functional method and finite difference scheme to obtain the numerical solution to the proposed problem. Numerical experiment shows that our method is stable and efficient. Of course, the method proposed in our paper can be generalized to fractional differential equations. That is our future work.

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