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STRONG CONVERGENCE OF NEW VISCOSITY RULES OF NONEXPANSIVE MAPPINGS†

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ABSTRACT. The aim of this paper is to present two new viscosity rules for nonexpansive mappings in Hilbert spaces. Under some assumptions, the strong convergence theorems of the purposed new viscosity rules are proved. Some applications are also included.

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1. Introduction

In this paper, we shall take H as a real Hilbert space, $\langle \cdot, \cdot \rangle$ as inner product, $\| \cdot \|$ as the induced norm, and C as a nonempty closed subset of H.

Definition 1.1. Let $T : H \to H$ be a mapping. T is called *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Definition 1.2. A mapping $f : H \to H$ is called a *contraction* if for all $x, y \in H$ and $\theta \in [0, 1)$

$$\|f(x) - f(y)\| \leq \theta \|x - y\|.$$

Definition 1.3. $P_c : H \to C$ is called a *metric projection* if for every $x \in H$ there exists a unique nearest point in C, denoted by $P_c x$, such that

$$\|x - P_c x\| \leq \|x - y\|, \quad \forall y \in C.$$

The following theorem gives the condition for a projection mapping to be nonexpansive.

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Theorem 1.4. *Let C be a nonempty closed convex subset of the real Hilbert space H and $P_c : H \to H$ a metric projection. Then*

(1) $\|P_c x - P_c y\|^2 \leq \langle x - y, P_c x - P_c y\rangle$ *for all* $x, y \in H$.
(2) P_c *is a nonexpansive mapping, i.e.,* $\|x - P_c x\| \leq \|x - y\|$ *for all* $y \in C$.
(3) $\langle x - P_c x, y - P_c x\rangle \leq 0$ *for all* $x \in H$ *and* $y \in C$.

In order to verify the weak convergence of an algorithm to a fixed point of a nonexpansive mapping we need the demiclosedness principle:

Theorem 1.5. (*The demiclosedness principle*) [2] *Let C be a nonempty closed convex subset of the real Hilbert space H and $T : C \to C$ such that*

$$x_n \rightharpoonup x^* \in C \quad \text{and} \quad (I - T)x_n \to 0.$$

Then $x^* = Tx^*$. Here $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

Moreover, the following result gives the conditions for the convergence of a nonnegative real sequence.

Theorem 1.6. [9] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \forall n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence with*

(1) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $a_n \to 0$.

The following strong convergence theorem, which is also called the *viscosity approximation method*, for nonexpansive mappings in real Hilbert spaces is given by Moudafi [6] in 2000.

Theorem 1.7. *Let C be a noneempty closed convex subset of the real Hilbert space H. Let T be a nonexpansive mapping of C into itself such that $F(T) := \{x \in H : T(x) = x\}$ is nonempty. Let f be a contraction of C into itself. Consider the sequence*

$$x_{n+1} = \frac{\epsilon_n}{1 + \epsilon_n} f(x_n) + \frac{1}{1 + \epsilon_n} T(x_n), \quad n \geq 0,$$

where the sequence $\{\epsilon_n\}$ in $(0, 1)$ satisfies

(1) $\lim_{n\to\infty} \epsilon_n = 0$,
(2) $\sum_{n=0}^{\infty} \epsilon_n = \infty$, *and*
(3) $\lim_{n\to\infty} |\frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n}| = 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x\rangle \geq 0, \quad \forall y \in F(T). \tag{1}$$

In 2015, Xu et al. [9] applied viscosity method on the midpoint rule for nonexpansive mappings and give the generalized viscosity implicit rule:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \geq 0.$$

This, using contraction, regularizes the implicit midpoint rule for nonexpansive mappings. They also proved that the sequence generated by the generalized viscosity implicit rule converges strongly to a fixed point of T. Ke and Ma [5], motivated and inspired by the idea of Xu et al. [9], proposed two generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T\left(s_n x_n + (1 - s_n) x_{n+1}\right) \tag{2}$$

and

$$x_{n+1} = \alpha_n x_n + \beta f(x_n) + \gamma_n T(s_n x_n + (1 - s_n) x_{n+1}). \tag{3}$$

In this paper, we contribute the following two new viscosity rules:

$$\begin{cases} x_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) T(y_n), \\ y_n = s_n x_n + (1 - s_n) x_{n+1} \end{cases} \tag{4}$$

and

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = \alpha_n f(z_n) + (1 - \alpha_n) T(z_n), \\ z_n = s_n x_n + (1 - s_n) x_{n+1}. \end{cases} \tag{5}$$

2. Main Results

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) T(y_n), \\ y_n = s_n x_n + (1 - s_n) x_{n+1}, \end{cases} \tag{6}$$

where $\{\alpha_n\}$ and $\{s_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(i) $\alpha_n + s_n = 1$ *and* $\lim_{n\to\infty} s_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iv) $\sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T which is also the unique solution of the variational inequality $\langle (I - f)x, y - x\rangle \geq 0$ for all $y \in F(T)$.

In other words, x^* is the unique fixed point of the contraction $P_{F(T)} f$, that is, $P_{F(T)} f(x^*) = x^*$.

Proof. We divide the proof into the following five steps.

STEP 1. Firstly, we show that $\{x_n\}$ is bounded.

Indeed, take $p \in F(T)$ arbitrarily, we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\alpha_n f(y_n) + (1-\alpha_n)T(y_n) - p\| \\
&= \|\alpha_n f(y_n) - \alpha_n p + (1-\alpha_n)T(y_n) - (1-\alpha_n)p\| \\
&\le \alpha_n\|f(y_n) - p\| + (1-\alpha_n)\|T(y_n) - p\| \\
&\le \alpha_n\|f(y_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|y_n - p\| \\
&\le \alpha_n\theta\|y_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|y_n - p\| \\
&= (\alpha_n\theta + 1 - \alpha_n)\|y_n - p\| + \alpha_n\|f(p) - p\| \\
&= (\alpha_n\theta + 1 - \alpha_n)\|s_n x_n + (1-s_n)x_{n+1} - p\| + \alpha_n\|f(p) - p\| \\
&\le (\alpha_n\theta + 1 - \alpha_n)[s_n\|x_n - p\| + (1-s_n)\|x_{n+1} - p\|] + \alpha_n\|f(p) - p\| \\
&= (\alpha_n\theta + 1 - \alpha_n)s_n\|x_n - p\| + (\alpha_n\theta + 1 - \alpha_n)(1-s_n)\|x_{n+1} - p\| \\
&\quad + \alpha_n\|f(p) - p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1 - (\alpha_n\theta + 1 - \alpha_n)(1-s_n))\|x_{n+1} - p\| \\
&\le (\alpha_n\theta + 1 - \alpha_n)s_n\|x_n - p\| + \alpha_n\|f(p) - p\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&[1 - (1 - \alpha_n(1-\theta))(1-s_n)]\|x_{n+1} - p\| \\
&\le s_n(1 - \alpha_n(1-\theta))\|x_n - p\| + \alpha_n\|f(p) - p\|. \qquad (7)
\end{aligned}$$

Since $\alpha_n, s_n, \theta \in (0,1)$, $1 - (\alpha_n\theta + 1 - \alpha_n)(1-s_n) \ge 0$. Moreover, by (7) we get

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\le \frac{s_n(1 - \alpha_n(1-\theta))}{1 - (1 - \alpha_n(1-\theta))(1-s_n)}\|x_n - p\| \\
&\quad + \frac{\alpha_n}{1 - (1 - \alpha_n(1-\theta))(1-s_n)}\|f(p) - p\| \\
&= \left[1 - \frac{1 - (1 - \alpha_n(1-\theta))(1-s_n) - s_n(1 - \alpha_n(1-\theta)}{1 - (1 - \alpha_n(1-\theta))(1-s_n)}\right]\|x_n - p\| \\
&\quad + \frac{\alpha_n}{1 - (1 - \alpha_n(1-\theta))(1-s_n)}\|f(p) - p\| \\
&= \left[1 - \frac{1 - (1 - \alpha_n(1-\theta))}{1 - (1 - \alpha_n(1-\theta))(1-s_n)}\right]\|x_n - p\| \\
&\quad + \frac{\alpha_n}{1 - (1 - \alpha_n(1-\theta))(1-s_n)}\|f(p) - p\| \\
&= \left[1 - \frac{\alpha_n(1-\theta)}{1 - (1 - \alpha_n(1-\theta))(1-s_n)}\right]\|x_n - p\|
\end{aligned}$$

$$+\frac{\alpha_n(1-\theta)}{1-(1-\alpha_n(1-\theta))(1-s_n)}\left(\frac{1}{1-\theta}\|f(p)-p\|\right).$$

Thus, we have $\|x_{n+1}-p\| \leq \max\left\{\|x_n-p\|, \frac{1}{1-\theta}\|f(p)-p\|\right\}$. By applying induction, we obtain $\|x_{n+1}-p\| \leq \max\left\{\|x_0-p\|, \frac{1}{1-\theta}\|f(p)-p\|\right\}$. Hence, we concluded that $\{x_n\}$is bounded. Consequently, we deduce immediately from it that $\{f(y_n)\}$, $\{T(y_n)\}$ are bounded.

STEP 2. Now, we prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

$$\begin{aligned}
&\|x_{n+1}-x_n\|\\
&= \|\alpha_n f(y_n)+(1-\alpha_n)T(y_n)-[\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T(y_{n-1})]\|\\
&= \|\alpha_n f(y_n)+(1-\alpha_n)T(y_n)-[\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T(y_{n-1})]\|\\
&= \|\alpha_n f(y_n)-\alpha_n f(y_{n-1})+(\alpha_n-\alpha_{n-1})f(y_n)+(1-\alpha_n)T(y_n)\\
&\quad -(1-\alpha_n)T(y_{n-1})-(\alpha_n-\alpha_{n-1})T(y_{n-1})\|\\
&= \|\alpha_n[f(y_n)-f(y_{n-1})]+(\alpha_n-\alpha_{n-1})[f(y_{n-1})-T(y_{n-1})]\\
&\quad +(1-\alpha_n)[T(y_n)-T(y_{n-1})\|\\
&\leq \alpha_n\|f(y_n)-f(y_{n-1})\|+(1-\alpha_n)\|T(y_n)-T(y_{n-1})\|\\
&\quad +|\alpha_n-\alpha_{n-1}|\|f(y_{n-1})-T(y_{n-1})\|\\
&\leq \alpha_n\theta\|y_n-y_{n-1}\|+(1-\alpha_n)\|y_n-y_{n-1}\|+|\alpha_n-\alpha_{n-1}|M\\
&\leq \alpha_n\theta\|(s_nx_n+(1-s_n)x_{n+1})-(s_{n-1}x_{n-1}+(1-s_{n-1})x_n)\|\\
&\quad +(1-\alpha_n)\|(s_nx_n+(1-s_n)x_{n+1})-(s_{n-1}x_{n-1}+(1-s_{n-1})x_n)\|\\
&\quad +|\alpha_n-\alpha_{n-1}|M\\
&\leq (\alpha_n\theta+1-\alpha_n)\|s_n(x_n-x_{n-1})+(1-s_n)(x_{n+1}-x_n)\\
&\quad +(s_n-s_{n-1})x_{n-1}-(s_n-s_{n-1})x_n)\|+|\alpha_n-\alpha_{n-1}|M\\
&= (1-\alpha_n(1-\theta))\|s_n(x_n-x_{n-1})+(1-s_n)(x_{n+1}-x_n)\\
&\quad +(s_n-s_{n-1})(x_{n-1}-x_n)\|+|\alpha_n-\alpha_{n-1}|M\\
&= (1-\alpha_n(1-\theta))\|s_{n-1}(x_n-x_{n-1})+(1-s_n)(x_{n+1}-x_n)\|\\
&\quad +|\alpha_n-\alpha_{n-1}|M\\
&\leq (1-\alpha_n(1-\theta))s_{n-1}\|x_n-x_{n-1}\|+(1-\alpha_n(1-\theta))(1-s_n)\|x_{n+1}-x_n\|\\
&\quad +|\alpha_n-\alpha_{n-1}|M,
\end{aligned}$$

where $M>0$ is a constant such that $M \geq \sup_{n\geq 0}\|f(y_{n-1})-T(y_{n-1})\|$. It give

$$\begin{aligned}
&(1-(1-\alpha_n(1-\theta))(1-s_n))\|x_{n+1}-x_n\|\\
&\leq (1-\alpha_n(1-\theta))s_{n-1}\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M,
\end{aligned}$$

that is,

$$\begin{aligned}
&\|x_{n+1}-x_n\|\\
&\leq \frac{(1-\alpha_n(1-\theta))s_{n-1}}{1-(1-\alpha_n(1-\theta))(1-s_n)}\|x_n-x_{n-1}\|
\end{aligned}$$

$$+\frac{M}{1-(1-\alpha_n(1-\theta))(1-s_n)}|\alpha_n-\alpha_{n-1}|$$
$$=\left[1-\frac{1-(1-\alpha_n(1-\theta))(1-s_n)-(1-\alpha_n(1-\theta))s_{n-1}}{1-(1-\alpha_n(1-\theta))(1-s_n)}\right]\|x_n-x_{n-1}\|$$
$$+\frac{M}{1-(1-\alpha_n(1-\theta))(1-s_n)}|\alpha_n-\alpha_{n-1}|$$
$$=\left[1-\frac{1-(1-\alpha_n(1-\theta))(1-s_n+s_{n-1})}{(1-(1-\alpha_n(1-\theta))(1-s_n))}\right]\|x_n-x_{n-1}\|$$
$$+\frac{M}{1-(1-\alpha_n(1-\theta))(1-s_n)}|\alpha_n-\alpha_{n-1}|$$
$$=\left[1-\frac{1-1+s_n-s_{n-1}+\alpha_n(1-\theta)-\alpha_n(1-\theta)(s_n-s_{n-1})}{1-(1-\alpha_n(1-\theta))(1-s_n)}\right]\|x_n-x_{n-1}\|$$
$$+\frac{M}{1-(1-\alpha_n(1-\theta))(1-s_n)}|\alpha_n-\alpha_{n-1}|$$
$$=\left[1-\frac{(1-\alpha_n(1-\theta))(s_n-s_{n-1})+\alpha_n(1-\theta)}{1-(1-\alpha_n(1-\theta))(1-s_n)}\right]\|x_n-x_{n-1}\|$$
$$+\frac{M}{(1-(1-\alpha_n(1-\theta)(1-s_n)}|\alpha_n-\alpha_{n-1}|.$$

Since $\theta, \alpha_n, s_n \in (0,1)$,

$$0<\epsilon \le 1-(1-\alpha_n(1-\theta))(1-s_n) \le 1$$

and

$$\frac{(1-\alpha_n(1-\theta))(s_n-s_{n-1})+\alpha_n(1-\theta)}{(1-(1-\alpha_n(1-\theta))(1-s_n))} \ge \alpha_n(1-\theta).$$

Thus

$$\|x_{n+1}-x_n\| \le (1-\alpha_n(1-\theta))\|x_n-x_{n-1}\| + \frac{M}{\epsilon}|\alpha_n-\alpha_{n-1}|.$$

Since $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, by Theorem 1.6, we have $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$.

Step 3. Now, we prove that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Consider

$$\begin{aligned}
&\|x_n - T(x_n)\|\\
&= \|x_n - x_{n+1} + x_{n+1} - T(y_n) + T(y_n) - T(x_n)\|\\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - T(y_n)\| + \|T(y_n) - T(x_n)\|\\
&\le \|x_n - x_{n+1}\| + \|\alpha_n f(y_n) + (1-\alpha_n)T(y_n) - T(y_n)\| + \|y_n - x_n\|\\
&= \|x_n - x_{n+1}\| + \alpha_n\|f(y_n) - T(y_n)\| + (1-s_n)\|x_{n+1} - x_n\|\\
&\le (2-s_n)\|x_n - x_{n+1}\| + \alpha_n M\\
&\le 2\|x_n - x_{n+1}\| + \alpha_n M
\end{aligned}$$

Then by $\lim_{n\to\infty}\alpha_n = 0$, we get $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$.

Similarly we have $\|T(s_nx_n + (1-s_n)x_{n+1}) - x_n\| \to 0\|$ as $n \to \infty$.

STEP 4. In this step, we claim that $\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_n\rangle \le 0$, where $x^* = P_{F(T)}f(x^*)$.

Indeed, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a fixed point p of T. Without loss of generality, we may assume that $\{x_{n_i}\} \rightharpoonup p$. From $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and Theorem 1.5 we have $p = Tp$. This together with the property of the metric projection implies that

$$\begin{aligned}\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_n\rangle &= \limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_{n_i}\rangle \\ &= \langle x^* - f(x^*), x^* - p\rangle \\ &\le 0.\end{aligned}$$

STEP 5. Finally, we show that $x_n \to x^*$ as $n \to \infty$.

Now we again take $x^* \in F(T)$ is the unique fixed point of the contraction $P_{F(T)}f$. Consider

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &= \|\alpha_n f(y_n) + (1-\alpha_n)T(y_n) - x^*\|^2 \\ &= \|\alpha_n[f(y_n) - x^*] + (1-\alpha_n)[T(y_n) - x^*]\|^2 \\ &= \alpha_n^2\|f(y_n) - x^*\|^2 + (1-\alpha_n)^2\|T(y_n) - x^*\|^2 \\ &\quad +2\alpha_n(1-\alpha_n)\langle f(y_n) - x^*, T(y_n) - x^*\rangle \\ &\le \alpha_n^2\|f(y_n) - x^*\|^2 + (1-\alpha_n)^2\|y_n - x^*\|^2 \\ &\quad +2\alpha_n(1-\alpha_n)\langle f(y_n) - f(x^*), T(y_n) - x^*\rangle \\ &\quad +2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle \\ &\le (1-\alpha_n)^2\|y_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)\|f(y_n) - f(x^*)\|\|T(y_n) - x^*\| + K_n \\ &\le (1-\alpha_n)^2\|y_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)\theta\|y_n - x^*\|\|y_n - x^*\| + K_n \\ &= (1-\alpha_n)^2\|y_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)\theta\|y_n - x^*\|^2 + K_n \\ &= \left[(1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)\theta\right]\|y_n - x^*\|^2 + K_n \\ &= \left[(1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)\theta\right]\|s_nx_n + (1-s_n)x_{n+1} - x^*\|^2 + K_n \\ &= \left[(1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)\theta\right]\|s_n(x_n - x^*) + (1-s_n)(x_{n+1} - x^*)\|^2 + K_n \\ &\le \left[(1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)\theta\right][s_n^2\|x_n - x^*\|^2 + (1-s_n)^2\|x_{n+1} - x^*\| \\ &\quad +2s_n(1-s_n)\|x_n - x^*\|\|x_{n+1} - x^*\|] + K_n \\ &\le \left[(1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)\theta\right][s_n^2\|x_n - x^*\|^2 + (1-2s_n+s_n^2)\|x_{n+1} - x^*\| \\ &\quad +s_n(1-s_n)\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right)] + K_n \\ &= \left[(1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)\theta\right]\left[s_n\|x_n - x^*\|^2 + (1-s_n)\|x_{n+1} - x^*\|^2\right] + K_n \\ &= s_n\left[(1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)\theta\right]\|x_n - x^*\|^2 \\ &\quad +(1-s_n)\left[(1-\alpha_n)^2 + 2\alpha_n(1-\alpha)\theta\right]\|x_{n+1} - x^*\|^2 + K_n,\end{aligned}$$

where

$$K_n = \alpha_n^2 \|f(s_n x_n + (1-s_n)x_{n+1}) - x^*\|^2 \\ +2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(s_n x_n + (1-s_n)x_{n+1}) - x^*\rangle.$$

It will become as

$$\left[1-(1-s_n)\{(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\}\right]\|x_{n+1}-x^*\|^2 \\ \le s_n\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]\|x_n-x^*\|^2+K_n,$$

which implies that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\le \frac{s_n\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]}{1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]}\|x_n-x^*\|^2 \\
&\quad+\frac{K_n}{1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]} \\
&=\left[1-\frac{1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]-s_n\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]}{1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]}\right] \\
&\quad\times\|x_n-x^*\|^2+\frac{K_n}{1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]} \\
&=\left[1-\frac{1-\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]}{1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]}\right]\|x_n-x^*\|^2 \\
&\quad+\frac{K_n}{1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]} \\
&=\left[1-\frac{-\alpha_n^2+2\alpha_n^2\theta}{1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]}\right]\|x_n-x^*\|^2 \\
&\quad+\frac{K_n}{1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]}.
\end{aligned}$$

Note that $0 < 1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right] < 1$ and

$$\frac{-\alpha_n^2+2\alpha_n^2\theta}{1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]} \le -\alpha_n^2+2\alpha_n^2\theta < 2\alpha_n^2\theta.$$

Thus we have

$$\|x_{n+1}-x^*\|^2 \le [1-2\alpha_n^2\theta]\|x_n-x^*\|^2 \\ +\frac{K_n}{1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]}. \tag{8}$$

By $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty} s_n = 1$ we have

$$\lim_{n\to\infty}\frac{K_n}{1-(1-s_n)\left[(1-\alpha_n)^2+2\alpha_n(1-\alpha)\theta\right]}$$

$$= \lim_{n\to\infty} \left[\frac{\alpha_n^2 \|f(s_n x_n + (1-s_n)x_{n+1}) - x^*\|^2}{1-(1-s_n)\left[(1-\alpha_n)^2 + 2\alpha_n(1-\alpha)\theta\right]} \right.$$
$$\left. + \frac{2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(s_n x_n + (1-s_n)x_{n+1}) - x^*\rangle}{1-(1-s_n)\left[(1-\alpha_n)^2 + 2\alpha_n(1-\alpha)\theta\right]} \right]$$
$$\leq 0. \tag{9}$$

From (8), (9) and Theorem 1.6 we have $\lim_{n\to\infty} \|x_{n+1} - x_n\|^2 = 0$, which implies that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Theorem 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = \alpha_n f(z_n) + (1-\alpha_n)T(z_n), \\ z_n = s_n x_n + (1-s_n)x_{n+1}, \end{cases} \tag{10}$$

where $\{\alpha_n\}$ and $\{s_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(i) $\alpha_n + s_n = 1$ *and* $\lim_{n\to\infty} s_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iv) $\sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$,
(v) $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T which is also the unique solution of the variational inequality $\langle (I-f)x, y-x\rangle \geq 0$ for all $y \in F(T)$.

In other words, x^* is the unique fixed point of the contraction $P_{F(T)}f$, that is, $P_{F(T)}f(x^*) = x^*$.

Proof. The prove of this theorem is similar to the prove of Theorem 2.1. □

3. Applications

3.1. A more general system of variational inequalities. Let C be a nonempty closed convex subset of the real Hilbert Space H and $\{A_i\}_{i=1}^N : C \to H$ be a family if mappings. In [1], Cai and Bu considered the problem of finding $x_1^*, x_2^*, ..., x_N^* \in C \times C \times \cdots \times C$ such that

$$\begin{cases} \langle \lambda_N A_N x_N^* + x_1^* - x_N^*, x - x_1^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \lambda_{N-1} A_{N-1} x_{N-1}^* + x_N^* - x_{N-1}^*, x - x_N^* \rangle \geq 0, \quad \forall x \in C, \\ \ldots\ldots\ldots, \\ \langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{11}$$

Equation (11) can be written as

$$\begin{cases} \langle x_1^* - (I - \lambda_N A_N)x_N^*, x - x_1^* \rangle \geq 0, \quad \forall x \in C, \\ \langle x_N^* - (I - \lambda_{N-1} A_{N-1})x_{N-1}^*, x - x_N^* \rangle \geq 0, \quad \forall x \in C, \\ \dots\dots\dots, \\ \langle x_3^* - (I - \lambda_2 A_2)x_2^*, x - x_3^* \rangle \geq 0, \quad \forall x \in C, \\ \langle x_2^* - (I - \lambda_1 A_1)x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C, \end{cases}$$

which is more general system of variational inequalities in Hilbert spaces, where $\lambda_i > 0$ for all $i \in \{1, 2, 3, ..., N\}$. We also have following lemmas.

Lemma 3.1. [1] *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i \in \{1, 2, 3, ..., N\}$, let $A_i : C \to H$ be δ_i-inverse-strongly monotone for some positive real number δ_i, namely*

$$\langle A_i x - A_i y, x - y \rangle \geq \delta_i \|A_i x - A_i y\|^2, \quad \forall x, y \in C.$$

Let $G : C \to C$ be a mapping defined by

$$\begin{aligned} G(x) = {} & P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ & P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C. \end{aligned} \tag{12}$$

If $0 < \lambda_i \leq 2\delta_i$ for all $i \in \{1, 2, ..., N\}$, then G is nonexpansive.

Lemma 3.2. [4] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A_i : C \to H$ be a nonlinear mapping, where $i \in \{1, 2, 3, ..., N\}$. For given $x_i^* \in C$, $i \in \{1, 2, 3, ..., N\}$, $(x_1^*, x_2^*, x_3^*, ..., x_N^*)$ is a solution of the problem* (11) *if and only if*

$$\begin{aligned} x_1^* &= P_C(I - \lambda_N A_N)x_N^*, \\ x_i^* &= P_C(I - \lambda_{i-1} A_{i-1})x_{i-1}^*, \quad i = 2, 3, 4, ..., N, \end{aligned} \tag{13}$$

that is,

$$\begin{aligned} x_1^* = {} & P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ & P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x_1^*, \quad \forall x \in C. \end{aligned}$$

From Lemma 3.2, we know that $x_1^* = G(x_1^*)$, that is, x_1^* is a fixed point of the mapping G, where G is defined by (12). Moreover, if we find the fixed point x_1^*, it is easy to get the other points by (13). Applying Theorem 2.1 we get the following results.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i \in \{1, 2, 3, ..., N\}$, let $A_i : C \to H$ be δ_i-inverse-strongly monotone for some positive real number δ_i with $F(G) \neq \emptyset$, where $G : C \to C$ is defined by*

$$\begin{aligned} G(x) = {} & P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ & P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C. \end{aligned}$$

Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n)G(y_n), \\ y_n = s_n x_n + (1 - s_n)x_{n+1}, \end{cases}$$

where $\{\alpha_n\}$ and $\{s_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(i) $\alpha_n + s_n = 1$ *and* $\lim_{n\to\infty} s_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iv) $\sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping G which is also the unique solution of the variational inequality $\langle (I - f)x, y - x\rangle \geq 0, \forall y \in F(G)$.

In other words, x^* is the unique fixed point of the contraction $P_{F(G)}f$, that is, $P_{F(G)}f(x^*) = x^*$.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i \in \{1, 2, 3, ..., N\}$, let $A_i : C \to H$ be δ_i-inverse-strongly monotone for some positive real number δ_i with $F(G) \neq \emptyset$, where $G : C \to C$ is defined by*

$$G(x) = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1}) \cdots \\ P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C.$$

Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_{n+1} = G(y_n), \\ y_n = \alpha_n f(z_n) + (1 - \alpha_n)G(z_n), \\ z_n = s_n x_n + (1 - s_n)x_{n+1}, \end{cases}$$

where $\{\alpha_n\}$ and $\{s_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(i) $\alpha_n + s_n = 1$ *and* $\lim_{n\to\infty} s_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iv) $\sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$,
(v) $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping G which is also the unique solution of the variational inequality $\langle (I - f)x, y - x\rangle \geq 0, \forall y \in F(G)$.

In other words, x^* is the unique fixed point of the contraction $P_{F(G)}f$, that is, $P_{F(G)}f(x^*) = x^*$.

3.2. The constrained convex minimization problem. Now, we consider the following constrained convex minimization problem:

$$\min_{x \in C} \phi(x), \tag{14}$$

where $\phi : C \to R$ is a real-valued convex function and assumes that the problem (14) is consistent. Let Ω denote its solution set. For the minimization problem (14), if ϕ is (Fréchet) differentiable, then we have the following lemma.

Lemma 3.5. (*Optimality Condition*) [7] *A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem* (14) *is that x^* solves the variational inequality*

$$\langle \nabla\phi(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{15}$$

Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C\left(x^* - \lambda \nabla\phi(x^*)\right)$$

for every constant $\lambda > 0$.

If, in a addition ϕ is convex, then the optimality condition (15) *is also sufficient.*

It is well known that the mapping $P_C(I - \lambda A)$ is nonexpansive when the mapping A is δ-inverse-strongly monotone and $0 < \lambda < 2\delta$. We therefore have the following results.

Theorem 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. For the minimization problem* (14), *assume that ϕ is* (*Fréchet*) *differentiable and the gradient $\nabla\phi$ is a δ-inverse-strongly monotone mapping for some positive real number δ. Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) P_C(I - \lambda\nabla\phi)(y_n), \\ y_n = s_n x_n + (1 - s_n) x_{n+1}, \end{cases}$$

where $\{\alpha_n\}$ and $\{s_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(i) $\alpha_n + s_n = 1$ *and* $\lim_{n\to\infty} s_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iv) $\sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$.

Then $\{x_n\}$ converges strongly to a solution x^* of the minimization problem (14), *which is also the unique solution of the variational inequality $\langle (I - f)x, y - x \rangle \geq 0, \forall y \in \Omega$.*

In other words, x^* is the unique fixed point of the contraction $P_\Omega f$, that is, $P_\Omega f(x^*) = x^*$.

Theorem 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. For the minimization problem* (14), *assume that ϕ is* (*Fréchet*) *differentiable and the gradient $\nabla\phi$ is a δ-inverse-strongly monotone mapping for some positive* *real number* δ. *Let* $f : C \to C$ *be a contraction with coefficient* $\theta \in [0, 1)$. *Pick any* $x_0 \in C$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_{n+1} = P_C(I - \lambda\nabla\phi)(y_n), \\ y_n = \alpha_n f(z_n) + (1 - \alpha_n)P_C(I - \lambda\nabla\phi)(z_n), \\ z_n = s_n x_n + (1 - s_n)x_{n+1}, \end{cases}$$

where $\{\alpha_n\}$ *and* $\{s_n\}$ *are sequences in* $(0, 1)$ *satisfying the following conditions:*

(i) $\alpha_n + s_n = 1$ *and* $\lim_{n\to\infty} s_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iv) $\sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$,
(v) $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

Then $\{x_n\}$ *converges strongly to a solution* x^* *of the minimization problem* (14), *which is also the unique solution of the variational inequality* $\langle (I - f)x, y - x\rangle \geq 0, \forall y \in \Omega$.

In other words, x^* *is the unique fixed point of the contraction* $P_\Omega f$, *that is,* $P_\Omega f(x^*) = x^*$.

3.3. ***K*-mapping.** Kangtunyakarn and Suantai [3] in 2009 gave K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ as follows.

Definition 3.8. [3] Let C be a nonempty convex subset of the real Banach space. Let $\{T_i\}_{i=1}^N$ be a family of mappings of C into itself and let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, 2, 3, ..., N$. We define a mapping $K : C \to C$ as follows:

$$\begin{cases} U_1 = \lambda_1 T_1 + (1 - \lambda_1)I, \\ U_2 = \lambda_2 T_2 U_1 + (1 - \lambda_2)U_1, \\ \dots\dots\dots, \\ U_{N-1} = \lambda_{N-1}T_{N-1}U_{N-2} + (1 - \lambda_{N-1})U_{N-2}, \\ U_N = \lambda_N T_N U_{N-1} + (1 - \lambda_N)U_{N-1}. \end{cases}$$

Such a mapping is called a *K-mapping* generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$.

In 2014, Suwannaut and Kangtunyakarn [8] established the following result for K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$.

Lemma 3.9. [8] *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *For* $i = 1, 2, 3, ..., N$, *let* $\{T_i\}_{i=1}^N$ *be a finite family of* K_i*-strictly pseudo-contractive mapping of* C *into itself with* $K_i \leq \omega_i$ *and* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, *namely there exist constants* $K_i \in [0, 1)$ *such that*

$$\|T_i x - T_i y\|^2 \leq \|x - y\|^2 + K_i\|(I - T_i)x - (I - T_i)y\|^2, \quad \forall x, y \in C.$$

Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ *be real numbers with* $0 < \lambda_i < \omega_2, \forall i = 1, 2, 3, ..., N$ *and* $\omega_1 + \omega_2 < 1$. *Let* K *be the* K*-mapping generated by* $T_1, T_2, T_3, ..., T_N$ *and* $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$. *Then the following properties hold:*

(1) $F(K) = \bigcap_{i=1}^{N} F(T_i)$,

(2) K *is a nonexpansive mapping.*

On the bases of above lemma, we have the following results.

Theorem 3.10. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *For* $i = 1, 2, 3, ..., N$, *let* $\{T_i\}_{i=1}^{N}$ *be a finite family of* K_i*-strictly pseudo-contractive mapping of* C *into itself with* $K_i \leq \omega_i$ *and* $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ *be real numbers with* $0 < \lambda_i < \omega_2, \forall i = 1, 2, 3, ..., N$ *and* $\omega_1 + \omega_2 < 1$. *Let* K *be the* K*-mapping generated by* $T_1, T_2, T_3, ..., T_N$ *and* $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$. *Let* $f : C \to C$ *be a contraction with coefficient* $\theta \in [0, 1)$. *Pick any* $x_0 \in C$, *let* $\{x_n\}$ *be sequence generated by*

$$\begin{cases} x_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) K(y_n), \\ y_n = s_n x_n + (1 - s_n) x_{n+1}, \end{cases}$$

where $\{\alpha_n\}$ *and* $\{s_n\}$ *are sequences in* $(0, 1)$ *satisfying the following conditions:*

(i) $\alpha_n + s_n = 1$ *and* $\lim_{n\to\infty} s_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(iv) $\sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$.

Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of the mappings* $\{T_i\}_{i=1}^{N}$, *which is also the unique solution of the variational inequality* $\langle (I - f)x, y - x \rangle, \forall y \in F(K) = \bigcap_{i=1}^{N} F(T_i)$.

In other words, x^* *is the unique fixed point of the contraction* $P_{\bigcap_{i=1}^{N} F(T_i)} f$, *that is,* $P_{\bigcap_{i=1}^{N} F(T_i)} f(x^*) = x^*$.

Theorem 3.11. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *For* $i = 1, 2, 3, ..., N$, *let* $\{T_i\}_{i=1}^{N}$ *be a finite family of* K_i*-strictly pseudo-contractive mapping of* C *into itself with* $K_i \leq \omega_i$ *and* $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ *be real numbers with* $0 < \lambda_i < \omega_2, \forall i = 1, 2, 3, ..., N$ *and* $\omega_1 + \omega_2 < 1$. *Let* K *be the* K*-mapping generated by* $T_1, T_2, T_3, ..., T_N$ *and* $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$. *Let* $f : C \to C$ *be a contraction with coefficient* $\theta \in [0, 1)$. *Pick any* $x_0 \in C$, *let* $\{x_n\}$ *be sequence generated by*

$$\begin{cases} x_{n+1} = K(y_n), \\ y_n = \alpha_n f(z_n) + (1 - \alpha_n) K(z_n), \\ z_n = s_n x_n + (1 - s_n) x_{n+1}, \end{cases}$$

where $\{\alpha_n\}$ *and* $\{s_n\}$ *are sequences in* $(0, 1)$ *satisfying the following conditions:*

(i) $\alpha_n + s_n = 1$ *and* $\lim_{n\to\infty} s_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(iv) $\sum_{n=0}^{\infty} |s_{n+1} - s_n| < \infty$,
(v) $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of the mappings* $\{T_i\}_{i=1}^N$, *which is also the unique solution of the variational inequality* $\langle (I-f)x, y-x\rangle, \forall y \in F(K) = \bigcap_{i=1}^N F(T_i)$.

In other words, x^* *is the unique fixed point of the contraction* $P_{\bigcap_{i=1}^N F(T_i)} f$, *that is,* $P_{\bigcap_{i=1}^N F(T_i)} f(x^*) = x^*$.

References

1. G. Cai and S.Q. Bu, *Hybrid algorithm for generalized mixed equilibrium problems and variational inequality problems and fixed point problems*, Comput. Math. Appl. **62** (2011), 4772-4782.
2. K. Goebel and W. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, **28**. Cambridge University Press, Cambridge, 1990.
3. A. Kangtunyakarn and S. Suantai, *A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings*, Nonlinear Anal. **71** (2009), 4448-4460.
4. Y.F. Ke and C.F. Ma, *A new relaxed extragradient-like algorithm for approaching common solutions of generalized mixed equilibrium problems, a more general system of variational inequalities and a fixed point problem*, Fixed point Theory Appl. **126** (2013), 21 pages.
5. Y.F. Ke and C.F. Ma, *The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces*, Fixed point Theory Appl. **190** (2015), 21 pages.
6. A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl. **241** (2000), 46-55.
7. M. Su and H.K. Xu, *Remarks on gradient-projection algorithm*, J. Nonlinear Anal. Optim. **1** (2010), 35-43.
8. S. Suwannaut and A. Kangtunyakarn, *Strong convergence theorem for the modified generalized equilibrium problem and fixed point problem of strictly pseudo-contractive mappings*, Fixed Point Theory Appl. **86** (2014), 31 pages.
9. H.K. Xu, M.A. Alghamdi, and N. Shahzad, *The viscosity technique for the implicit mid point rule of nonexpansive mappings in Hilbert spaces*, Fixed point Theory Appl. **41** (2015), 12 pages.

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