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HEEGAARD SPLITTINGS OF BRANCHED CYCLIC COVERINGS OF CONNECTED SUMS OF LENS SPACES

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Abstract. We study relations between two descriptions of closed orientable 3-manifolds: as branched coverings and as Heegaard splittings. An explicit relation is presented for a class of 3-manifolds which are branched cyclic coverings of connected sums of lens spaces, where the branching set is an axis of a hyperelliptic involution of a Heegaard surface.

1. Introduction

Arbitrary closed orientable 3-manifold M can be described in various ways: by its triangulation, fundamental polyhedron, surgery on a link, Heegaard splitting, etc. Choosing of a way depends on a context as well as on a question asked about M.

Closed orientable 3-manifolds with cyclic symmetries are objects of intensive study in last decades. The initial and most known examples of closed orientable 3-manifolds belong to the class of branched cyclic covers of the 3-sphere S^3. Among them are the following:

- spherical and hyperbolic dodecahedral spaces, constructed by Weber and Seifert in 1933 [14], are the 3-fold cyclic cover of S^3 branched over the trefoil knot and the 5-fold cyclic cover of S^3 branched over the Whitehead link, respectively;
- Fibonacci manifolds, constructed by Helling, Kim, and Mennicke [8], are n-fold cyclic covers of S^3 branched over the figure-eight knot;
- Sieradski manifolds, constructed by Cavicchioli, Kim, and Hegenbarth [7], are n-fold cyclic covers of S^3 branched over the trefoil knot;
- the smallest volume closed orientable hyperbolic 3-manifold, constructed by Fomenko and Matveev [10] and by Weeks [15], is the 3-fold cyclic cover of S^3, branched over the two-bridge knot $7/3$.

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Here we are interested in the class of closed 3-manifolds which are branched cyclic coverings of connected sums of lens spaces. In particular, the class contains manifolds from [4] which are branched cyclic covers of connected sums of two lens spaces.

For lens spaces with arbitrary parameters we present Heegaard diagrams (see Theorem 4.2 and Fig. 4) for manifolds from the class in the case when the branching set is an axis of a hyperelliptic involution of a Heegaard surface, i.e., an involution with $2g+2$ fixed points for a surface of genus g.

2. Branched cyclic coverings and symmetric Heegaard splittings

Recall some basic definitions of the 3-manifold theory. Let M and N be triangulated 3-dimensional manifolds, and let $f: M \to N$ be a simplicial map. The map f is said to be a *branched covering space projection* if the restriction of f to the complement of the 1-dimensional skeleton of the triangulation is a covering space projection. The *branch set* $B \subset N$ is the set of points $z \in N$ which have the property that z has no neighborhood U such that the restriction of f to an arc-component of $f^{-1}(U)$ is a covering. The set $L = f^{-1}(B)$ is referred to as the *branch cover*, and we say that M is a covering space of N, *branched over* B. We refer to the pair (M, L) as a *branching covering* of (N, B), writing $f: (M, L) \to (N, B)$. Let $\hat{f}$ be the restriction of f to $M \setminus L$. Then $\hat{f}: (M \setminus L) \to (N \setminus B)$ is an ordinary covering space projection, which is referred to as the associate *unbranched covering space.* We say that M is an *n–fold cyclic covering of N branched over B* if the group $\hat{f}_* \pi_1(M \setminus L)$ is the kernel of a homomorphism from $\pi_1(N \setminus B)$ onto $\mathbb{Z}_n$, a cyclic group of order n.

We will work with representations of closed orientable 3-manifolds by Heegaard splittings. Consider H_g and H'_g, two solid handlebodies of genus g, and let $\tau: H_g \to H'_g$ be a map that identifies a point $z \in H_g$ with its corresponding point $z' \in H'_g$. Let Φ be an orientation-preserving self-homeomorphism of ∂H_g. We use Φ to define a map which "glue" ∂H_g to $\partial H'_g$ by the rule $\tau\Phi(z) = z$, where $z \in \partial H_g$. The space $H_g \cup_{\tau\Phi} H'_g$ is a closed orientable 3-manifold which is represented by a *Heegaard splitting* of genus g. It is well-known that every closed orientable 3-manifold can be presented in this way for some (nonunique) integer g and surface homeomorphism Φ. A 3-manifold which is so represented will be said to have *Heegaard genus* g if it admits a Heegaard splitting of genus g, but no Heegaard splittings of genus smaller than g. The only 3-manifold of genus 0 is S^3; 3-manifolds of Heegaard genus 1 are $S^2 \times S^1$ and lens spaces $L(p,q)$ defined below.

In this paper we consider a special type of Heegaard splittings, which we will refer to as n-symmetric Heegaard splittings. Let H_g, H'_g, τ, and Φ be as defined above. It will be assumed further that H_g and H'_g are subsets of Euclidean 3-space E^3 and there is given a piecewise-linear homeomorphism $\mathcal{P}: E^3 \to E^3$ of period n, and that H_g is invariant under the action of $\mathcal{P}$. Note that the homeomorphisms $\mathcal{P}$ and τ define in a natural way a transformation

$\mathcal{P}' = \tau\mathcal{P}\tau^{-1}$ which acts on H'_g such that $\mathcal{P}'$ is also has period n, and that H'_g is left invariant under the action of $\mathcal{P}'$.

The Heegaard splitting $H_g \cup_{\tau\Phi} H'_g$ is said to be *n-symmetric* if

(i) There is n_0, with $1 \leq n_0 \leq n$, such that $(\Phi)(\mathcal{P}|_{\partial H_g})(\Phi^{-1}) = (\mathcal{P}|_{\partial H_g})^{n_0}$.
(ii) The orbit space of H_g under the action of $\mathcal{P}$ is a 3-ball.
(iii) The fixed point set of $\mathcal{P}$ coincides with the fixed point set of $\mathcal{P}^k$ for each k, $1 \leq k < n$.
(iv) The image of the fixed point set of $\mathcal{P}$ is an unknotted set of arcs in the ball $H_g/\mathcal{P}$.

The *n-symmetric Heegaard genus* of a 3-manifold M is the smallest integer g such that M admits a n-symmetric Heegaard splitting of genus g.

Remark 2.1. As observed in [13], by the positive solution of the Smith Conjecture [12] it is easy to see that necessary $n_0 \equiv \pm 1 \mod n$. We recall that a set of mutually disjoint arcs $\{\alpha_1, \dots, \alpha_\ell\}$ properly embedded in a handlebody H is *unknotted* if there is a set of mutually disjoint discs $\{D_1, \dots, D_\ell\}$ properly embedded in H such that $\alpha_i \cap D_i = \alpha_i \cap \partial D_i = \alpha_i$, $\alpha_i \cap D_j = \emptyset$ for $j \neq i$, and $\partial D_i \setminus \alpha_i \subset \partial H$ for $1 \leq i, j \leq \ell$.

A relation between cyclic coverings of S^3 branched over knots and symmetric Heegaard splittings was obtained in [1, Theorem 4]: every n-fold cyclic covering of S^3 branched over a knot of braid number b is a closed, orientable 3-manifold of n-symmetric Heegaard genus $g \leq (b-1)(n-1)$.

This result has a natural generalization from the case of knots to the case of links. Let $L = \cup_{j=1}^{\mu}$ be an oriented μ-component link in S^3. An n-fold cyclic covering of S^3 branched over L is completely determined by assigning to each component L_j an integer $c_j \in \mathbb{Z}_n \setminus \{0\}$, such that the set $\{c_1, \dots, c_\mu\}$ generates the group $\mathbb{Z}_n$, The monodromy associated to the covering sends each meridian of L_j, coherently oriented with the chosen orientations of L and S^3, to the permutation $(12 \cdots n)^{c_j} \in S_n$. Following [11] we shall call a branched cyclic covering *strictly-cyclic* if $c_{j'} = c_{j''}$, for every $j', j'' \in \{1, 2, \dots, \mu\}$. It was shown in [13] that every n-fold strictly-cyclic covering of S^3 branched over a link of bridge number b is a closed orientable 3-manifold of n-symmetric Heegaard genus $g \leq (b-1)(n-1)$.

3. Lens spaces and their cyclic branched covers

Let p and q be a pair of coprime integers, $p \geq 3$ and $p > q > 0$. Consider a p-gonal bipyramid, i.e., the union of two cones over a regular p-gon as presented in Fig. 1, where the vertices of the p-gon are denoted by $A_0, A_1, \dots, A_{p-1}$ and apex of cones are denoted by S_+ and S_-. For each $i = 0, \dots, n-1$ we glue the face $A_i S_+ A_{i+1}$ with the face $A_{i+q} S_- A_{i+q+1}$, where the indices are taken modulo p and the vertices are glued in the order in which they are written. The manifold obtained is the *lens space* $L(p, q)$. Lens spaces $L(p, q)$ and $L(p', q')$

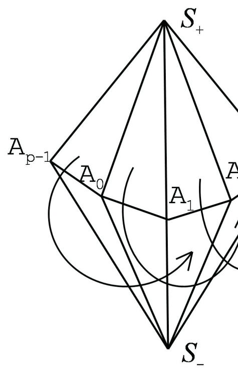

FIGURE 1. A p–gonal bipyramid.

are homeomorphic if and only if $p = p'$ and $q' = \pm q$, or $qq' = \pm 1 \bmod p$. Recall that $L(1, 0) = S^3$, $L(0, 1) = S^2 \times S^1$, $L(2, 1) = \mathbb{R}P^3$.

A convenient way of presenting a Heegaard diagram is given below. Let us cut the surface S_g along meridional disks $v_1, v_2, \ldots, v_g$. We obtain a sphere with $2g$ disks $D_1, \bar{D}_1, \ldots, D_g, \bar{D}_g$, where D_i and $\bar{D}_i$, which are joined by arcs obtained by cutting meridians of the system v_i. The meridians u will then be cut into arcs having endpoints $u \cap v_i$ on the boundaries of the disks D_i, $\bar{D}_i$, $i = 1, \ldots, g$. Let a direction to each v_i be chosen and let the point of set $u \cap v_i$ be numbered in the given direction. This numbering induces a numbering of the ends of arcs on the boundary of the disks D_i and $\bar{D}_i$. Heegaard diagram of the lens space $L(p, q)$ is presented in Fig. 2.

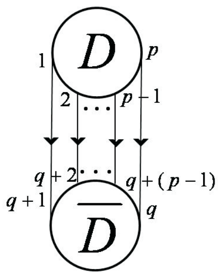

FIGURE 2. A Heegaard diagram for lens space $L(p, q)$.

Dunwoody introduced in [5] some infinite family of diagrams $D(a, b, c, n, r, s)$ with cyclic symmetry, depending on six integer parameters a, b, c, n, r, s, such that $n > 0$ and $a, b, c, r, s \geqslant 0$. Each manifold arising in this way is called a *Dunwoody manifold*. It was shown by Grasselli and Mulazzani [6] that Dunwoody manifolds are exactly the cyclic branched coverings of $(1, 1)$-knots, i.e., the knots that admit 1-bridge presentation of genus 1. In particular, the class of $(1,1)$-knots contains all two-bridge knots and all torus knots in S^3. The knot is called a (g,b)-*knot* if it admits a b-bridge presentation of genus g. Notice that $(1,1)$-knots are knots in a 3-sphere S^3 or in lens spaces. For generalization of Dunwoody manifolds see [2].

The existence and uniqueness of the cyclic branched coverings of $(g,1)$-knots were investigated in [4]. Moreover, there was given some algorithm of finding the fundamental group of a covering. We note that 3-manifolds from the class of cyclic branched coverings of $(1,b)$-links ($b \geqslant 2$) were considered in [9]. Some infinite families of 3-manifolds which are cyclic coverings of lens spaces $L(p,q)$, branched over two-component links, were constructed in [3].

4. Heegaard diagrams for connected sums of lens spaces and for cyclic branched covering spaces

We start with describing Heegaard diagrams for connected sums of lens spaces, i.e., for manifolds $N(p_1,q_1)\cdots(p_k,q_k) = L(p_1,q_1)\#\cdots\#L(p_k,q_k)$. We will start with a particular case of the connected sum of two lens spaces given by parameters $(p_i,q_i) = (3,1)$.

Example 4.1. Manifolds $N(3,1)(3,1)$ and $N(3,1)(3,1)(3,1)$ admit Heegaard diagrams presented in Fig. 3 on the left and on the right, respectively. Consider, firstly, $N(3,1)(3,1)$. Denote by A, B, $\overline{A}$ and $\overline{B}$ disjoint closed 2-discs on S^2. We orient boundaries $\partial\overline{A}$, $\partial\overline{B}$ in clockwise direction, and boundaries ∂A, ∂B – in counterclockwise direction. Then for each disc we fix five points on its boundary and numerate point in respect to the above orientation. Further we glue the disc A with $\overline{A}$, and the disc B with $\overline{B}$ in such a way that points with the same numbers will be identified. After that all arcs of the diagram will split into two classes of equivalent. These classes can be seen in Fig. 3.

A Heegaard diagram for $N(3,1)(3,1)(3,1)$ has similar description. A generalization to Heegaard diagrams of $N(p_1,q_1)\cdots(p_k,q_k)$ is straightforward.

Now we will describe n-symmetric Heegaard diagram for the n-fold branched cyclic cover $N^n(p_1,q_1)\cdots(p_k,q_k)$ of the 3-manifold $N(p_1,q_1)\cdots(p_k,q_k)$ with branching set, corresponding the axis of the hyperelliptic involution of the Heegaard surface.

Theorem 4.2. *A manifold $N^n(p_1,q_1)\cdots(p_k,q_k)$ admits an n-symmetric Heegaard diagram presented in Figure 4.*

The construction of a Heegaard diagram is as follows. Denote by $A,B,\ldots,Z$ and $\overline{A},\overline{B},\ldots,\overline{Z}$ discs of the Heegaard diagram of $N(p_1,q_1)\cdots(p_k,q_k)$, which arises as a generalization of Fig. 3. Denote by $A_i,B_i,\ldots,Z_i$, $\overline{A}_i,\overline{B}_i,\ldots,\overline{Z}_i$ $i=1,\ldots,n$ discs, obtained by the n-cyclic symmetry. The discs $A_i,B_i,\ldots,Z_i$ and $\overline{A}_i,\overline{B}_i,\ldots,\overline{Z}_i$, $i=1,\ldots,n$, will serve as discs of the Heegaard diagram of $N^n(p_1,q_1)\cdots(p_k,q_k)$. We label points on the boundary of $A_i,B_i,\ldots,Z_i$ (resp. $\overline{A}_i,\overline{B}_i,\ldots,\overline{Z}_i$) by $p_1,p_2,p_3,\ldots,p_k$ with a counterclockwise orientation (resp.

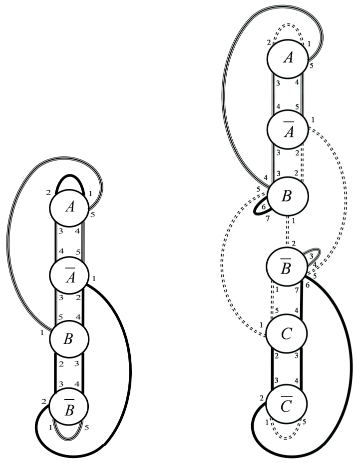

FIGURE 3. Heegaard diagrams for $N_{(3,1)(3,1)}$ and $N_{(3,1)(3,1)(3,1)}$.

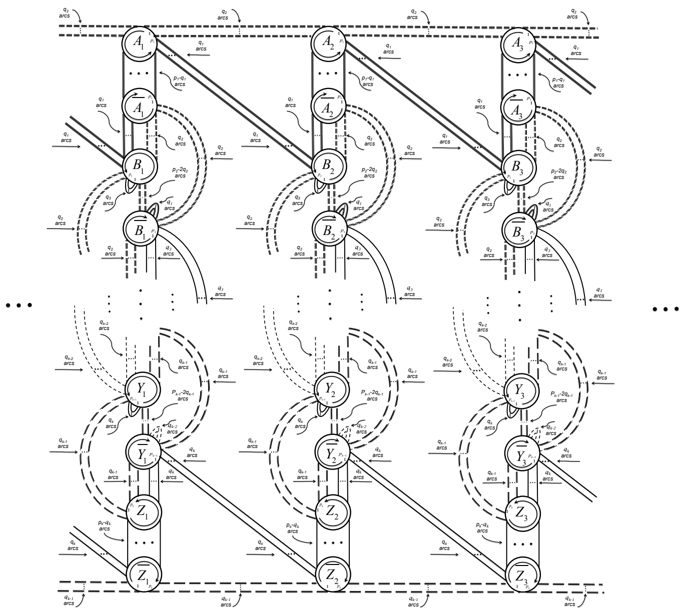

FIGURE 4. A Heegaard diagram for $N^n(p_1, q_1)(p_2, q_2) \cdots (p_k, q_k)$.

clockwise orientation). These points are numerated in according to the chosen direction of the diversion. We identify the discs in pairs A_i and $\overline{A}_i, \ldots, Z_i$ and $\overline{Z}_i$ in such a way that points with the same numbers will coincide. The arcs of the diagram are drawn in Fig. 4.

The proof consists of two steps. Firstly, we show that the presented diagram is a Heegaard diagram of some 3-manifold. Secondly, we show the this manifold is a branched cyclic cover of a connected sum of lens spaces $N(p_1, q_1) \cdots (p_k, q_k)$. Since these two steps are technically standard, we omit them here.

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