Bull. Korean Math. Soc. **54** (2017), No. 5, pp. 1779–1801
https://doi.org/10.4134/BKMS.b160757
pISSN: 1015-8634 / eISSN: 2234-3016

MINIMAL DEL PEZZO SURFACES OF DEGREE 2 OVER FINITE FIELDS

Andrey Trepalin

Abstract. Let X be a minimal del Pezzo surface of degree 2 over a finite field $\mathbb{F}_q$. The image Γ of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ in the group $\mathrm{Aut}(\mathrm{Pic}(\overline{X}))$ is a cyclic subgroup of the Weyl group $W(E_7)$. There are 60 conjugacy classes of cyclic subgroups in $W(E_7)$ and 18 of them correspond to minimal del Pezzo surfaces. In this paper we study which possibilities of these subgroups for minimal del Pezzo surfaces of degree 2 can be achieved for given q.

1. Introduction

Let X be a del Pezzo surface of degree d over a finite field $\mathbb{F}_q$, and $\overline{X} = X \otimes \overline{\mathbb{F}}_q$. The image Γ of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ in the group $\mathrm{Aut}(\mathrm{Pic}(\overline{X}))$ is a cyclic group, which preserves the intersection form. There are finitely many conjugacy classes of cyclic subgroups in the subgroup $\mathrm{Aut}(\mathrm{Pic}(\overline{X}))$ preserving the intersection form. The natural question is which of these classes can realise the group Γ for given q.

A surface S is called *minimal* if any birational morphism $S \to S'$ is an isomorphism. The minimality of X can be described in terms of Γ-action on $\mathrm{Pic}(\overline{X})$. If X is not a minimal surface, then it is isomorphic to a blowup of surface Y at number of points. In this case the action of the group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on $\mathrm{Pic}(\overline{X})$ is prescribed by the action of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on $\mathrm{Pic}(\overline{Y})$ and the degrees of the points of blowup. Therefore the cases of Γ for which X is minimal are most interesting for us.

If X is a minimal geometrically rational surface, then either X admits a conic bundle structure or X is a del Pezzo surface with the Picard number $\rho(X) = \mathrm{rk}\,\mathrm{Pic}(X) = 1$ (see [3, Theorem 1]). In the paper [7] it is shown how the group Γ can act on the components of singular fibres of a minimal conic bundle, and for all possibilities of Γ corresponding minimal conic bundles

Received September 19, 2016; Accepted December 26, 2016.

2010 *Mathematics Subject Classification.* 14J26, 14G15, 14G10, 11G25.

Key words and phrases. del Pezzo surfaces, finite field, zeta function.

The research was carried out at the IITP RAS at the expense of the Russian Foundation for Sciences (project $N^{\underline{o}}$ 14-50-00150).

are constructed. Del Pezzo surfaces of degree greater than 4 are $\mathbb{F}_q$-rational (see [4, Chapter 4]). Therefore minimal del Pezzo surfaces of degree greater than 4 can be constructed by blowing up some points on $\mathbb{P}^2_{\mathbb{F}_q}$ and contracting some exceptional curves. All types of minimal del Pezzo surfaces of degree 4 are constructed in [7, Theorem 3.2]. One case of minimal cubic surfaces is constructed in [10] for any q. The other cases of minimal cubic surfaces are constructed in the paper [8] but there are some restrictions on q. In the paper [1] for del Pezzo surfaces of 3, 2 and 1 and any q it is shown how many $\mathbb{F}_q$-points can a surface have. Some results of the paper [1] give constructions of minimal surfaces for certain Γ. Also it is shown that for any Γ there exists the corresponding surface for any sufficiently big q (see [1, Theorem 1.7]).

The aim of this paper is to construct minimal del Pezzo surfaces of degree 2 with given cyclic group Γ. Note that in this case the group of automorphisms of $\operatorname{Pic}(\overline{X})$, preserving the intersection form, is the Weyl group $W(E_7)$. The conjugacy classes of elements in this group are well-known (see [2]). For convenience of the reader we give a table of these conjugacy classes and some of their properties in Appendix A. We have 18 conjugacy classes in $W(E_7)$ for which X is minimal. For 6 of those classes the invariant Picard number $\rho(\overline{X})^\Gamma = 2$ and X admits a conic bundle structure. For the other 12 conjugacy classes $\rho(\overline{X})^\Gamma = 1$ and X does not admit a structure of a conic bundle.

The considered problem is closely related to zeta-functions. Let N_d be the order of the set $X(\mathbb{F}_{q^d})$. The zeta-function of X is the formal power series

$$Z_X(t) = \exp\left(\sum_{d=1}^{\infty} \frac{N_d t^d}{d}\right).$$

For a rational surface X one has (see [5, IV.5])

$$Z_X(t) = \frac{1}{(1-t)P(t)(1-q^2t)},$$

where

$$P(t) = \det(1 - qt\mathbf{F} | \operatorname{Pic}(\overline{X}) \otimes \mathbb{Q}),$$

and $\mathbf{F}$ is a linear automorphism of $\operatorname{Pic}(\overline{X}) \otimes \mathbb{Q}$ induced by the Frobenius element. Therefore the zeta-function of a surface X is totally defined by the group Γ. Moreover, for each cyclic subgroup of $W(E_7)$ we can write down such function. But it is not known whether a given zeta-function corresponding to a subgroup of $W(E_7)$ can be realised by a del Pezzo surface of degree 2.

This paper gives an answer for this question for minimal del Pezzo surfaces of degree 2. In the notation of Table 1 these surfaces have types 31, 35, 40, 43–45 and 49–60. The main result of this paper is the following.

Theorem 1.1. *In the notation of Table 1 the following holds.*

(1) *A del Pezzo surface of degree* 2 *of type* 49 *does not exist for* $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$, $\mathbb{F}_5$, $\mathbb{F}_7$, $\mathbb{F}_8$, *and exists for the other finite fields.*

(2) *A del Pezzo surface of degree* 2 *of type* 31 *does not exist for* $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$, *and exists for the other finite fields.*
(3) *Del Pezzo surfaces of degree* 2 *of types* 40, 50, 53, 55, 60 *do not exist for* $\mathbb{F}_2$, *and exist for the other finite fields.*
(4) *Del Pezzo surfaces of degree* 2 *of types* 43, 44, 45, 52, 54, 57, 59 *exist for all finite fields.*
(5) *A del Pezzo surface of degree* 2 *of type* 35 *does not exist for* $\mathbb{F}_2$, *and exists for any* $\mathbb{F}_q$ *where* $q \geqslant 4$.
(6) *Del Pezzo surfaces of degree* 2 *of types* 51, 58 *exist for any* $\mathbb{F}_q$ *where* q *is odd.*
(7) *A del Pezzo surface of degree* 2 *of type* 56 *exists for any* $\mathbb{F}_q$ *where* $q = 6k + 1$.

Remark 1.2. The author does not know, how to construct a del Pezzo surface of degree 2 of type 35 over $\mathbb{F}_3$, or show that such surface does not exist. Maybe it is better to use a computer in this case. The existence of del Pezzo surfaces of degree 2 of types 51, 56, 58 is equivalent to the existence of minimal cubic surfaces of certain types (see Lemma 3.32). The restrictions on q come from the paper [8], where minimal cubic surfaces are considered. The complete answer in these cases is not known.

The plan of this paper is as follows.

In Section 2 we consider minimal del Pezzo surfaces of degree 2 which admit structure of conic bundles. For these cases we apply [7, Theorem 2.11] and get a minimal conic bundle with singular fibres over points of required degrees. Then, if it is possible, we construct some birational links from these bundles to minimal del Pezzo surfaces admitting structure of conic bundle.

In Section 3 we consider minimal del Pezzo surfaces of degree 2 such that the Picard number $\rho(X)$ is equal to 1. We define *Geiser twist* (see Definition 3.21) which gives us correspondence between these surfaces and non-minimal del Pezzo surfaces of degree 2 of certain types. Then we realise the obtained surfaces as the blowups of del Pezzo surfaces of higher degree at several points.

In Appendix A there is a table which gives the classification of cyclic subgroups of the Weyl group $W(E_7)$ and some properties of these subgroups.

The author is a Young Russian Mathematics award winner and would like to thank its sponsors and jury. Also the author is grateful to C. Shramov for many useful discussions which form the basis of this paper, to A. Duncan, S. Gorchinskiy and S. Rybakov for discussions about the theme of this work, and to B. Banwait, F. Fité, D. Loughran for introducing their results which are very useful and allow the author to avoid many technical problems.

2. The conic bundle case

In this section we construct minimal del Pezzo surfaces of degree 2 admitting structure of conic bundles. We use the following theorem.

Theorem 2.3 (cf. [7, Theorem 2.11])**.** *Let $x_1, \ldots, x_s$ be a set of points on $B = \mathbb{P}^1_{\mathbb{F}_q}$ of possibly different degrees. Then there exists a relatively minimal conic bundle $X \to B$ with degenerate fibres over points $x_1, \ldots, x_s$ if and only if s is even.*

For del Pezzo surfaces of degree 2 admitting a structure of a conic bundle there are exactly 6 degenerate geometric fibres. So by Theorem 2.3 there are six possibilities:

(31) the degenerate fibres are over six $\mathbb{F}_q$-points;
(35) the degenerate fibres are over two $\mathbb{F}_q$-points and two points of degree 2;
(40) the degenerate fibres are over three $\mathbb{F}_q$-points and a point of degree 3;
(43) the degenerate fibres are over an $\mathbb{F}_q$-point and a point of degree 5;
(44) the degenerate fibres are over a point of degree 2 and a point of degree 4;
(45) the degenerate fibres are over two points of degree 3.

The numeration of cases is taken from Table 1.

Remark 2.4. Case (31) cannot be achieved for $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$ since there are no six $\mathbb{F}_q$-points on $\mathbb{P}^1_{\mathbb{F}_2}$, $\mathbb{P}^1_{\mathbb{F}_3}$ and $\mathbb{P}^1_{\mathbb{F}_4}$. Case (35) cannot be achieved for $\mathbb{F}_2$ since there are no two points of degree 2 on $\mathbb{P}^1_{\mathbb{F}_2}$.

The main problem is that not any surface admitting a structure of a conic bundle with 6 degenerate fibres is a del Pezzo surface of degree 2. The following proposition is well-known (see e.g. [6, Chapter 8, Exercise 3]). We give a proof for convenience of the reader.

Proposition 2.5. *Let $\pi : X \to B$ be a minimal conic bundle over $\mathbb{P}^1_{\mathbb{F}_q}$ with* 6 *degenerate fibres. Then we have one of the following possibilities.*

(1) *The surface X is a del Pezzo surface of degree* 2 *admitting two structures of conic bundles.*
(2) *There is a geometrically irreducible* 2*-section D on X such that $D^2 = -2$.*
(3) *There are two geometrically irreducible sections C_1 and C_2 on X such that $C_1^2 = C_2^2 = -2$ and $C_1 \cdot C_2 = 1$.*
(4) *There are four geometrically irreducible disjoint sections C_1, C_2, C_3 and C_4 on X such that $C_1^2 = C_2^2 = C_3^3 = C_4^2 = -2$.*
(5) *There are two geometrically irreducible disjoint sections C_1 and C_2 on X such that $C_1^2 = C_2^2 = -3$.*

Proof. For a minimal conic bundle $X \to B$ the group $\mathrm{Pic}(X)$ is generated by $-K_X$ and F, where F is the class of fibre $X \to B$. In this basis one has $K_X^2 = 2$, $K_X \cdot F = -2$ and $F^2 = 0$.

Assume that the anticanonical linear system $|-K_X|$ is not nef. Then there exists a $\Bbbk$-irreducible reduced curve C such that $-K_X \cdot C < 0$. Thus the curve C has a class $-aK_X - bF$ and $a < b$, since $-K_X \cdot C < 0$. By Riemann–Roch theorem $\dim|-K_X| = 2$. Therefore $|-K_X| = |C+M|$, where $|M|$ is a moveable linear system of dimension 2. One has $M \sim (a-1)K_X + bF$. Hence

$$M^2 = 2(a-1)^2 - 2(a-1)b = 2(a-1)(a-1-b).$$

This number can be non-negative only if $a = 1$, and the linear system $|M| \sim |bF|$ has dimension 2 only if $b = 2$. Therefore $C \sim -K_X - 2F$. For the ariphmetic genus of C one has

$$2\mathrm{p}_a(C) - 2 = C \cdot (C + K_X) = -4.$$

Therefore C is geometrically reducible and consists of two disjoint geometrically irreducible sections with the selfintersection number -3. This is case (5) of Proposition 2.5.

Now assume that the anticanonical linear system $|-K_X|$ is nef but not ample. Then there exists a $\mathbb{k}$-irreducible reduced curve C such that $-K_X \cdot C = 0$. The curve C has class $-aK_X - bF$ and consists of geometrically irreducible curves with the selfintersection number -2. One has $a = b$ since $-K_X \cdot C = 0$. The number of geometrically irreducible components of C is no greater than $2a = C \cdot F$. Therefore one has $-2a^2 = C^2 \geqslant -4a$, and $a \leqslant 2$.

If $a = 2$, then $C^2 = -8$, and C consists of four disjoint geometrically irreducible sections with the selfintersection number -2. This is case (4) of Proposition 2.5.

If $a = 1$, then $C^2 = -2$. If C is geometrically reducible, then it consists of two disjoint geometrically irreducible sections C_1 and C_2 such that $C_1^2 = C_2^2 = -2$ and $C_1 \cdot C_2 = 1$. This is case (3) of Proposition 2.5.

If C is geometrically irreducible, then its selfintersection number is -2. This is case (2) of Proposition 2.5.

If $|-K_X|$ is ample, then X is a del Pezzo surface of degree 2 and the linear systems $|F|$ and $|-2K_X - F|$ give two conic bundle structures. This is case (1) of Proposition 2.5. □

To construct minimal del Pezzo surfaces of degree 2 admitting a conic bundle structure we apply Theorem 2.3 and then construct a sequence of Sarkisov links ending at a del Pezzo surface of degree 2. But it is not possible to construct such links in an arbitrary situation.

Example 2.6. A minimal del Pezzo surface of degree 4 over $\mathbb{F}_3$ admitting a structure of conic bundle with four degenerate fibres over $\mathbb{F}_3$-points does not exist, since such a surface should contain eight $\mathbb{F}_3$-points of intersection of (-1)-curves (three or more (-1)-curves can not meet each other at one point on a del Pezzo surface of degree 4). But there are only four $\mathbb{F}_3$-points on a minimal conic bundle over $\mathbb{P}^1_{\mathbb{F}_3}$ with four degenerate fibres over $\mathbb{F}_3$-points. Nevertheless by Theorem 2.3 there exists a conic bundle with four smooth fibres over four $\mathbb{F}_3$-points on $\mathbb{P}^1_{\mathbb{F}_3}$.

Example 2.6 improves results of [7, Theorem 3.2]. The complete result about minimal del Pezzo surfaces of degree 4 is the following.

Theorem 2.7 (cf. [7, Theorem 3.2]). *In the notation of* [7] *the following holds.*

(1) *Del Pezzo surfaces of degree* 4 *with the zeta-functions* Z_1, Z_2, Z_{10}, Z_{18} *exist for all finite fields.*
(2) *A del Pezzo surface of degree* 4 *with the zeta-function* Z_5 *does not exist for* $\mathbb{F}_2$, *and exists for the other finite fields.*
(3) *A del Pezzo surface of degree* 4 *with the zeta-function* Z_4 *does not exist for* $\mathbb{F}_2$ *and* $\mathbb{F}_3$, *and exists for the other finite fields.*

Proof. Let us remind (see the proof of [7, Theorem 3.2]) that the zeta-functions Z_2, Z_4 and Z_5 come from del Pezzo surfaces of degree 4 which admit a conic bundle structure:

- the case Z_2 corresponds to a conic bundle with singular fibres over an $\mathbb{F}_q$-point and a point of degree 3;
- the case Z_4 corresponds to a conic bundle with singular fibres over four $\mathbb{F}_q$-points;
- the case Z_5 corresponds to a conic bundle with singular fibres over two points of degree 2.

In [7, Theorem 3.2] it is proved that del Pezzo surfaces of degree 4 with zeta-functions Z_1, Z_{10} and Z_{18} exist for any $\mathbb{F}_q$, and del Pezzo surfaces of degree 4 with zeta-functions Z_2, Z_4 and Z_5 exist for any $\mathbb{F}_q$ where $q > 3$.

Del Pezzo surfaces of degree 4 with the zeta-functions Z_4 and Z_5 do not exist over $\mathbb{F}_2$ since there are no four $\mathbb{F}_2$-points and two points of degree 2 on $\mathbb{P}^1_{\mathbb{F}_2}$. From the proof of [7, Theorem 3.2] one can see that for the other possibilities of q there exists a del Pezzo surface of degree 4 with the zeta-function Z_2, Z_4 or Z_5 if there exists a smooth fibre over an $\mathbb{F}_q$-point. Therefore a del Pezzo surface of degree 4 with the zeta-function Z_2 exists for any $\mathbb{F}_q$, and a del Pezzo surface of degree 4 with the zeta-function Z_5 exists for any $\mathbb{F}_q$, where $q \geqslant 3$. Example 2.6 shows that a del Pezzo surface of degree 4 with the zeta-function Z_4 does not exist over $\mathbb{F}_3$. □

We want to know some facts about curves with negative selfintersection on conic bundles.

Proposition 2.8. *Assume that* $X \to \mathbb{P}^1_{\Bbbk}$ *is a minimal conic bundle over arbitrary field* $\Bbbk$ *with* $n > 0$ *degenerate geometric fibres. Then* $\overline{X}$ *is isomorphic to a blowup of* $\mathbb{P}^1_{\Bbbk} \times \mathbb{P}^1_{\Bbbk}$ *at set of points* $p_1, \ldots, p_n$.

Proof. The conic bundle $X \to \mathbb{P}^1_{\Bbbk}$ is minimal therefore there are at least two sections C_1 and C_2 with negative selfintersection $-k$ on $\overline{X}$, since otherwise there is a unique section with negative selfintersection number and one can contract over $\Bbbk$ all components of singular fibres meeting this section. If $k > n$, then we can contract n components of singular fibres on $\overline{X}$ and get a conic bundle $\overline{Y} \to \mathbb{P}^1_{\Bbbk}$ without singular fibres. But the images of C_1 and C_2 on $\overline{Y}$ are curves with negative selfintersection. It is impossible since any conic bundle without singular fibres is either $\mathbb{P}^1_{\Bbbk} \times \mathbb{P}^1_{\Bbbk}$ or a Hirzebruch surface $\mathbb{F}_m$ and there is at most one curve with negative selfintersection.

If $k \leqslant n$, then we can blow down any k components of singular fibres meeting with C_1 and for the other $n-k$ singular fibres blow down components not meeting C_1. Then we get a conic bundle $\overline{Y} \to \mathbb{P}^1_{\Bbbk}$ without singular fibres, and the image of C_1 on this bundle is a curve with selfintersection 0. But there are no curves with selfintersection 0 on Hirzebruch surfaces $\mathbb{F}_m$ for $m > 0$. Thus $\overline{Y}$ is $\mathbb{P}^1_{\Bbbk} \times \mathbb{P}^1_{\Bbbk}$. □

Corollary 2.9. *The Picard group* $\mathrm{Pic}(\overline{X})$ *of a minimal conic bundle* $X \to \mathbb{P}^1_{\Bbbk}$ *is generated by the class of fibre* F*, a class of section* C *such that* $C^2 = 0$*, and the classes of exceptional divisors* $E_1, \ldots, E_n$*. One has*

$$C \cdot F = 1, \qquad C \cdot E_i = 0, \qquad F \cdot E_i = 0.$$

Note that any group acting on $\mathrm{Pic}(\overline{X})$ and preserving the conic bundle structure should preserve F and $K_X = -2C - 2F + \sum_{i=1}^n E_i$. Thus this group acts on the subspace $K_X^{\perp} \cap F^{\perp}$ of $\mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}$, that is the subspace of classes $H \in \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}$ such that $H \cdot K_X = H \cdot F = 0$. This subspace is generated by

$$F - E_1 - E_2, \quad E_1 - E_2, \quad \ldots, \quad E_{n-1} - E_n.$$

Those generators form a root system of type D_n. The Weyl group $W(D_n)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$, where S_n is a symmetric group of degree n. A subgroup S_n permutes E_i and the normal group $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ is generated by involutions ι_{ij} such that

$$\iota_{ij}(C) = C + F - E_i - E_j, \quad \iota_{ij}(F) = F, \quad \iota_{ij}(E_i) = F - E_i,$$
$$\iota_{ij}(E_j) = F - E_j, \quad \iota_{ij}(E_k) = E_k$$

for $k \neq i$, $k \neq j$.

Any element of $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$ has form $\iota_{i_1 \cdots i_{2k}} \cdot \sigma$, where $\sigma \in S_n$ is a permutation permuting E_i, and $\iota_{i_1 \cdots i_{2k}} \in (\mathbb{Z}/2\mathbb{Z})^{n-1}$ switches components of singular fibres over even number of points $p_{i_1}, \ldots, p_{i_{2k}}$ on the base $\mathbb{P}^1_{\Bbbk}$.

In this notation for cases (31), (35), (40), (43), (44), (45) of minimal conic bundles with 6 degenerate geometric fibres the group Γ is generated by an element conjugate to ι_{123456}, $\iota_{1235}(34)(56)$, $\iota_{123456}(456)$, $\iota_{123456}(23456)$, $\iota_{13}(12)(3456)$ or $\iota_{123456}(123)(456)$ respectively.

Now we start constructing Sarkisov links of minimal conic bundles.

Lemma 2.10. *Let* $\pi : X \to B \cong \mathbb{P}^1_{\mathbb{F}_q}$ *be a minimal conic bundle of type* (5) *of Proposition 2.5. There exists a birational map* $f : X \dashrightarrow Y$ *such that* $\pi f^{-1} : Y \to B$ *is a conic bundle that does not have sections with selfintersection number* -3.

Proof. Applying Proposition 2.8 we may assume that the two sections C_1 and C_2 with selfintersection -3 have classes $C - E_1 - E_2 - E_3$ and $C - E_4 - E_5 - E_6$ respectively.

Let us show that there are no other sections with negative selfintersection number. Any section D has class $C + aF - \sum_{i=1}^{6} b_i E_i$, where $a \geqslant 0$ and each b_i is 0 or 1. One has

$$D^2 = 2a - \sum_{i=1}^{6} b_i, \qquad C_1 \cdot D = a - b_1 - b_2 - b_3, \qquad C_2 \cdot D = a - b_4 - b_5 - b_6,$$

therefore $D^2 = C_1 \cdot D + C_2 \cdot D \geqslant 0$.

The curves C_1 and C_2 are not defined over $\mathbb{F}_q$. Therefore if there exists an $\mathbb{F}_q$-point $P \in B$ such that $\pi^{-1}(P)$ is a smooth fibre, then any $\mathbb{F}_q$-point on the fibre $\pi^{-1}(P)$ does not lie on a section with negative selfintersection number. So we can blow up such a point, contract the transform of $\pi^{-1}(P)$ and get a minimal conic bundle $Y \to B$ of type (3).

Such $\mathbb{F}_q$-point P exists for all cases except $\Gamma = \langle \iota_{123456}(456) \rangle$ and $q = 2$, or $\Gamma = \langle \iota_{123456} \rangle$ and $q = 5$. In these cases we can find on B a point Q of degree 2, and choose a point of degree 2 on $\pi^{-1}(Q)$ that does not lie on a section with negative selfintersection number. We can blow up such a point, contract the transform of $\pi^{-1}(Q)$ and get a minimal conic bundle $Y \to B$ without sections with selfintersection less than -2. □

Lemma 2.11. *Let $\pi : X \to B \cong \mathbb{P}^1_{\mathbb{F}_q}$ be a minimal conic bundle of type* (4) *of Proposition 2.5. There exists a birational map $f : X \dashrightarrow Y$ such that $\pi f^{-1} : Y \to B$ is a conic bundle of type* (1) *or* (2) *of Proposition 2.5.*

Proof. Applying Proposition 2.8 we may assume that the four sections C_1, C_2, C_3 and C_4 with selfintersection -2 have classes $C - E_1 - E_2$, $C - E_3 - E_4$, $C - E_5 - E_6$ and $C + 2F - \sum_{i=1}^{6} E_i$ respectively.

Let us find other sections with negative selfintersection number. Any section D has class $C + aF - \sum_{i=1}^{6} b_i E_i$, where $a \geqslant 0$ and each b_i is 0 or 1. One has $D^2 = 2a - \sum_{i=1}^{6} b_i$, therefore $a \leqslant 2$. One can check that there are 8 sections with selfintersection -1 on X. Their classes are $C + F - E_i - E_j - E_k$ where $i \in \{1, 2\}$, $j \in \{3, 4\}$, $k \in \{5, 6\}$.

Note that the element ι_{123456} maps C_1 to $C + F - E_3 - E_4 - E_5 - E_6$ that is not an effective divisor. Therefore Γ is conjugate to $\iota_{1235}(34)(56)$ or $\iota_{13}(12)(3456)$. In these cases there are at most two orbits of the sections with selfintersection -1, and each of these orbits contains at most one $\mathbb{F}_q$-point.

The curves C_1, C_2, C_3 and C_4 are not defined over $\mathbb{F}_q$. Therefore on a smooth fibre over an $\mathbb{F}_q$-point P there is an $\mathbb{F}_q$-point which does not lie on a section with negative selfintersection. So we can blow up such a point, contract the transform of $\pi^{-1}(P)$ and get a minimal conic bundle $Y \to B$ without sections with selfintersection less than -1. Such a conic bundle has type (1) or (2). □

To construct links of minimal conic bundles for the cases (3) and (2) we need the following lemma.

Lemma 2.12. *Let $\pi : X \to B \cong \mathbb{P}^1_{\mathbb{F}_q}$ be a minimal conic bundle of type* (3) *or* (2) *of Proposition 2.5. The singular points of anticanonical curves lie on a divisor C_1+C_2+R or $D+R$ respectively, where R has class $2C+3F-\sum_{i=1}^{6} E_i$. In particular, there are at most* 4 *singular points of anticanonical curves on a fibre of π.*

Proof. Note that both divisors D in the case (2) and $C_1 + C_2$ in the case (3) have the class $2C + F - \sum_{i=1}^{6} E_i$. In both cases we denote these divisors by W.

Note that each point of W is a singular point of an anticanonical curve of form $W + F$.

The surface X is a weak del Pezzo surface, and the anticanonical linear system defines a separable map $f : X \to \mathbb{P}^2_{\mathbb{F}_q}$ of degree 2 that contract W to a point P. Anticanonical curves map to lines on $\mathbb{P}^2_{\mathbb{F}_q}$, and singular points on anticanonical curves come from points of intersection of these lines and the branch divisor of f. Therefore any singular point of an anticanonical curve lie on the ramification divisor of f. This divisor has class $-2K_X$ and consists of W and $R \sim 2C + 3F - \sum_{i=1}^{6} E_i$. □

Lemma 2.13. *Let $\pi : X \to B \cong \mathbb{P}^1_{\mathbb{F}_q}$ be a minimal conic bundle of type* (3) *of Proposition 2.5. Then there exists a birational map $f : X \dashrightarrow Y$ such that $\pi f^{-1} : Y \to B$ is a conic bundle of type* (1) *or* (2) *of Proposition 2.5.*

Proof. Applying Proposition 2.8 we may assume that the two sections C_1 and C_2 with selfintersection -2 have classes $C-E_1-E_2$ and $C+F-E_3-E_4-E_5-E_6$ respectively.

Let us find other sections with negative selfintersection number. Any section D has class $C + aF - \sum_{i=1}^{6} b_iE_i$, where $a \geqslant 0$ and each b_i is 0 or 1. One has $D^2 = 2a - \sum_{i=1}^{6} b_i$, therefore $a \leqslant 2$. One can check that there are 20 sections with selfintersection -1 on X. Their classes are $C - E_i$ where $i \in \{3, 4, 5, 6\}$, $C + F - E_i - E_j - E_k$ where $i \in \{1, 2\}$, $j \in \{3, 4, 5, 6\}$, $k \in \{3, 4, 5, 6\}$, and $C + 2F - \sum_{i=1}^{6} E_i + E_j$ where $j \in \{3, 4, 5, 6\}$.

If we find an $\mathbb{F}_q$-point P on a smooth fibre which does not lie on any section with negative selfintersection, then we can blowup X at P, blow down the strict transform of fibre containing P and get a minimal conic bundle $Y \to B$ without sections with selfintersection less than -1. Such a conic bundle has type (1) or (2).

Let us find such a point for each possibility of Γ.

If $\operatorname{ord}\Gamma = 4$ or $\operatorname{ord}\Gamma = 8$, then Γ contains an element conjugate to ι_{1234}. But such an element can not map C_1 to C_2. Therefore these cases are impossible.

In the other cases Γ contains the element ι_{123456} that maps any section T with selfintersection -1 to $-K_X - T$. Thus any $\mathbb{F}_q$-point on any section with selfintersection -1 is a singular point of an anticanonical curve and lie on R (see Lemma 2.12).

If $q > 2$, then on the smooth fibre containing the point of intersection of C_1 and C_2 there are at least 3 other $\mathbb{F}_q$-points. At most 2 of these points lie on R. Therefore there is an $\mathbb{F}_q$-point P on this fibre which does not lie on any section with negative selfintersection, and we are done.

If $q = 2$, then $\Gamma = \langle \iota_{1456}(23456) \rangle$ or $\Gamma = \langle \iota_{2456}(123)(456) \rangle$ since in the other two remaining cases there are at least four fibres over $\mathbb{F}_q$-points that is impossible. Therefore there is a fibre over an $\mathbb{F}_q$-point that does not contain the point of intersection of C_1 and C_2. The divisor $C_1 + C_2$ intersects this fibre at a point of degree 2. Therefore by Lemma 2.12 there is an $\mathbb{F}_q$-point P on this fibre which does not lie on R, and we are done. □

Proposition 2.14. *Let $\pi : X \to B \cong \mathbb{P}^1_{\mathbb{F}_q}$ be a minimal conic bundle of type* (2) *of Proposition 2.5. Then there exists a birational map $f : X \dashrightarrow Y$ such that $\pi f^{-1} : Y \to B$ is a conic bundle of type* (1) *of Proposition 2.5 in all possible cases except the following cases: the group Γ is conjugate to $\langle \iota_{123456}(456) \rangle$ and $q = 2$; the group Γ is conjugate to $\langle \iota_{1235}(34)(56) \rangle$ and q is* 3 *or* 4*; the group Γ is conjugate to $\langle \iota_{123456}(23456) \rangle$ and $q = 2$.*

To prove this proposition we need several lemmas.

Lemma 2.15. *Let $\pi : X \to B \cong \mathbb{P}^1_{\mathbb{F}_q}$ be a minimal conic bundle of type* (2) *of Proposition 2.5. If there is an $\mathbb{F}_q$-point P on a smooth fibre that is not a singular point of an anticanonical curve and does not lie on any section with negative selfintersection. Then there exists a birational map $f : X \dashrightarrow Y$ such that $\pi f^{-1} : Y \to B$ is a conic bundle of type* (1) *of Proposition 2.5.*

Proof. Applying Proposition 2.8 we may assume that the 2-section D with selfintersection -2 has the class $2C + F - \sum_{i=1}^{6} E_i$.

One can check that there are 32 sections with selfintersection -1 on X. Their classes are $C - E_i$, $C + F - E_i - E_j - E_k$ and $C + 2F - \sum_{i=1}^{6} E_i + E_j$.

Let us show that there are no other 2-sections with negative selfintersection number. Any 2-section H has class $2C + aF - \sum_{i=1}^{6} b_i E_i$, where $a \geqslant 1$ and each b_i is 0, 1 or 2. We can assume that $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_6$. Note that

$$H \cdot \left(C + 2F - \sum_{i=1}^{5} E_i \right) = a + 4 - \sum_{i=1}^{5} b_i \geqslant 0,$$
$$H \cdot (C + F - E_1 - E_2 - E_3) = a + 2 - b_1 - b_2 - b_3 \geqslant 0,$$
$$H \cdot (C - E_1) = a - b_1 \geqslant 0.$$

Therefore if $a = 5$, then $b_5 \neq 2$, if $a = 4$, then either $b_4 \neq 2$ or $b_5 = 0$, if $a = 3$, then $b_3 \neq 2$. But for $a \geqslant 6$ and in these cases one has $H^2 = 4a - \sum_{i=1}^{6} b_i^2 \geqslant 0$. Therefore $a = 1$ or $a = 2$.

If $a = 1$, then $b_1 = 1$ and $H^2 = D \cdot H \geqslant 0$. If $a = 2$, then $b_2 = 1$ or $b_3 = 0$. One has $H^2 < 0$ only for $b_1 = 2$ and $b_2 = b_3 = \cdots = b_6 = 1$. But in this case $D \cdot H = -1$ that is impossible. Thus D is the only 2-section with negative selfintersection number.

Assume that an $\mathbb{F}_q$-point P on a smooth fibre does not lie on any section with negative selfintersection and not a singular point of an anticanonical curve. If we blow up X at P and blow down the strict transform of fibre containing P, then we get a minimal conic bundle $Y \to B$ without sections with selfintersection less than -1.

Assume that there is a 2-section with selfintersection less than -1 on Y. Let $H \subset X$ be the preimage of such a 2-section. Then H intersects each component of each degenerate fibre at a point since Y has type (2). Therefore H has class $2C + aF - \sum_{i=1}^{6} E_i$. If $a = 1$, then $H \cdot D = -2$. It means that $H = D$ and P lies on D, but each point of D is a singular point of an anticanonical curve $D + F$, so this is impossible.

Note that the multiciplicity of P on H is no greater than 2. Therefore if $a > 2$, then $H^2 > 2$ and selfintersection of the transform of H on Y is greater than -2. Thus $a = 2$, and H is an anticanonical curve with singularity at P. But P is not a singular point of anticanonical curve. So we have contradiction, and there are no 2-sections with selfintersection less than -1 on Y. □

Lemma 2.16. *There is no a minimal conic bundle* $\pi : X \to B \cong \mathbb{P}^1_{\mathbb{F}_q}$ *of type* (2) *with* $\operatorname{ord}\Gamma = 2$ *over* $\mathbb{F}_5$.

Proof. If $\operatorname{ord}\Gamma = 2$ and $q = 5$, then D contains six $\mathbb{F}_q$-points. But there are only six $\mathbb{F}_q$-points on X which lie on the six singular fibres. Therefore the map $D \to B$ is a double cover branched at six points. This is impossible, since D is a smooth rational curve. Thus this case does not occur. □

Lemma 2.17. *Let* $\pi : X \to B \cong \mathbb{P}^1_{\mathbb{F}_q}$ *be a minimal conic bundle of type* (2) *of Proposition 2.5. There exists an* $\mathbb{F}_q$*-point* P *on a smooth fibre that is not a singular point of an anticanonical curve and does not lie on any section with negative selfintersection in all possible cases except the following cases: the group* Γ *is conjugate to* $\langle \iota_{123456}(456) \rangle$ *and* q *is* 2 *or* 3; *the group* Γ *is conjugate to* $\langle \iota_{1235}(34)(56) \rangle$ *and* q *is* 3 *or* 4; *the group* Γ *is conjugate to* $\langle \iota_{123456}(23456) \rangle$ *and* $q = 2$.

Proof. Let us find such a point for each possibility of Γ. The group Γ acts on the set of sections with selfintersection -1. Any $\mathbb{F}_q$-point on an Γ-orbit of such curve is an intersection point of all curves in this orbit. Therefore an orbit of length 2 contains at most two $\mathbb{F}_q$-points, an orbit of length 4 contains at most one $\mathbb{F}_q$-point, an orbit of length 6 or more does not contain $\mathbb{F}_q$-points.

Note that the element ι_{123456} maps any section T with selfintersection -1 to $-K_X - T$. Thus if $\iota_{123456} \in \Gamma$, then all $\mathbb{F}_q$-points on sections with negative selfintersection are singular points of anticanonical curves. If $\operatorname{ord}\Gamma = 8$, then the orbits of sections with selfintersection -1 consist of 8 curves and do not contain any $\mathbb{F}_q$-points. If $\operatorname{ord}\Gamma = 4$, then the orbits of sections with selfintersection -1 consist of 4 meeting each other curves. So this is the only case when an $\mathbb{F}_q$-point on a section with negative selfintersection can be not a singular point of an anticanonical curve, and there are at most 8 such points, since there are 32 sections with selfintersection -1.

Assume that $\operatorname{ord}\Gamma \neq 4$, there is a smooth fibre over an $\mathbb{F}_q$-point, and $q \geqslant 4$. Then there are 5 or more $\mathbb{F}_q$-points on this fibre. At most four of these points are singular points of anticanonical curves by Lemma 2.12. Thus there is an $\mathbb{F}_q$-point P on this fibre that is not a singular point of an anticanonical curve and does not lie on any section with negative selfintersection, and we are done.

The latest assumption does not hold if $\operatorname{ord}\Gamma = 2$ and $q = 5$, $\operatorname{ord}\Gamma \geqslant 6$ and $q \leqslant 3$, or $\operatorname{ord}\Gamma = 4$. The case $\operatorname{ord}\Gamma = 2$ and $q = 5$ does not occur by Lemma 2.16.

If $q = 3$ and Γ is conjugate to $\langle \iota_{123456}(23456)\rangle$, $\langle \iota_{123456}(123)(456)\rangle$ or $\langle \iota_{13}(12)(3456)\rangle$, then there are at least 3 smooth fibres over $\mathbb{F}_3$-points. On these fibres there are at least twelve $\mathbb{F}_3$-points, at most 6 of them lie on R (see Lemma 2.12) and at most 4 lie on D. Thus there is an $\mathbb{F}_3$-point on a smooth fibre that is not a singular point of an anticanonical curve and does not lie on any section with negative selfintersection, and we are done.

If $q = 2$ and Γ is conjugate to $\langle \iota_{123456}(123)(456)\rangle$ or $\langle \iota_{13}(12)(3456)\rangle$, then there are 3 smooth fibres over $\mathbb{F}_2$-points and nine $\mathbb{F}_2$-points on X. The anticanonical linear system $|-K_X|$ contains 7 elements, and three of these elements have form $D + F$. The other four elements have at most five singular points, since on the set of negative sections there is one Γ-orbit of length 2 for $\Gamma = \langle \iota_{123456}(123)(456)\rangle$ and no such orbits for $\Gamma = \langle \iota_{13}(12)(3456)\rangle$, and irreducible anticanonical curves have at most one singular point. The curve D contains three $\mathbb{F}_2$-points. Thus there is an $\mathbb{F}_2$-point on a smooth fibre that is not a singular point of an anticanonical curve and does not lie on any section with negative selfintersection, and we are done.

Now assume that $\operatorname{ord}\Gamma = 4$. If $q \geqslant 5$, then there are $q - 1$ smooth fibres over $\mathbb{F}_q$-points. On these fibres there are at least $q^2 - 1$ points and at least $(q-1)^2$ of them do not lie on R (see Lemma 2.12). The curve D contains $q+1$ points defined over $\mathbb{F}_q$ and at most eight $\mathbb{F}_q$-points lie on sections with negative selfintersection. One has $(q-1)^2 > q + 9$ for $q \geqslant 5$. Thus there is an $\mathbb{F}_q$-point on a smooth fibre that is not a singular point of anticanonical curve and does not lie on any section with negative selfintersection, and we are done. □

Lemma 2.18. *Let $\pi : X \to B \cong \mathbb{P}^1_{\mathbb{F}_3}$ be a minimal conic bundle of type (2) of Proposition 2.5 and Γ is conjugate to $\langle \iota_{123456}(456)\rangle$ over $\mathbb{F}_3$. There exists an $\mathbb{F}_3$-point P on a smooth fibre that is not a singular point of an anticanonical curve and does not lie on any section with negative selfintersection.*

Proof. There are 3 singular fibres over $\mathbb{F}_3$-points and one smooth fibre over an $\mathbb{F}_3$-point. The curve D contains four $\mathbb{F}_3$-points, and at least two of these points lie on the singular fibres. If there is less than four singular anticanonical curves with singular $\mathbb{F}_3$-points on the smooth fibre, then there is an $\mathbb{F}_3$-point on this fibre that is not a singular point of an anticanonical curve, and does not lie on any section with negative selfintersection, and we are done. Let us show that there can not be four singular $\mathbb{F}_3$-points of anticanonical curves on the smooth fibre.

Assume that there exists an irreducible singular anticanonical curve A defined over $\mathbb{F}_3$. Then A contains at least three $\mathbb{F}_3$-points and at least two of these points lie on the singular fibres. Thus A and D have a common point and we have contradiction, since $-K_X \cdot D = 0$. Therefore only reducible anticanonical curves can have singular $\mathbb{F}_3$-points. If such a curve consisting of two sections with selfintersection -1 contains two $\mathbb{F}_3$-points, then one of these points lies on a singular fibre over an $\mathbb{F}_3$-point. But $-K_X \cdot F = 2$, so it is impossible. Thus two $\mathbb{F}_3$-points on the smooth fibre not lying on D can be singular points of an anticanonical curves only if these curves A_1 and A_2 are reducible curves, consisting of sections with selfintersection -1 which are tangent.

In this case let us consider the anticanonical map $\varphi : X \to \mathbb{P}^2_{\mathbb{F}_3}$. This map has degree 2 and the branch divisor is a singular plane quartic curve B' with an ordinary double point $\varphi(D)$. The images $\varphi(A_1)$ and $\varphi(A_2)$ are lines each of which intersects B' at a point with multiplicity 4. One can choose coordinates on $\mathbb{P}^2_{\mathbb{F}_3}$ such that the points $A_1 \cap A_2$, $A_1 \cap B'$, $A_2 \cap B'$ and $\varphi(D)$ have coordinates $(0 : 0 : 1)$, $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(1 : 1 : 0)$ respectively, since the points $A_1 \cap B'$, $A_2 \cap B'$ and $\varphi(D)$ lie on a line which is the image of the smooth fibre. In these coordinates B' is given by the equation

$$xy\left(U(x-y)^2 + V(x-y)z + Wz^2\right) - z^4 = 0.$$

The preimages of lines $x = y$, $z = x - y$ and $z = y - x$ on X are singular fibres. Therefore each of these lines is either bitangent to B' or passes through $\varphi(D)$ with multiplicity greater than 2. Thus we have $V = W = 0$, $U = -1$. But the singular point $(1 : 1 : 0)$ on the curve $xy(x-y)^2 + z^4 = 0$ is not a node. This contradiction finishes the proof. □

Now we can prove Proposition 2.14.

Proof of Proposition 2.14. By Lemmas 2.15 and 2.17 the map $X \dashrightarrow Y$, where Y is a conic bundle of type (1) of Proposition 2.5, exists for all possible cases except the following cases: the group Γ is conjugate to $\langle \iota_{123456}(456) \rangle$ and q is 2 or 3; the group Γ is conjugate to $\langle \iota_{1235}(34)(56) \rangle$ and q is 3 or 4; the group Γ is conjugate to $\langle \iota_{123456}(23456) \rangle$ and $q = 2$.

By Lemmas 2.15 and 2.18 such a map exists for Γ conjugate to $\langle \iota_{123456}(456) \rangle$ over $\mathbb{F}_3$. □

Now we collect results of this section in the following proposition.

Proposition 2.19. *In the notation of Table 1 the following holds.*

(i) *A del Pezzo surface of degree* 2 *of type* 31 *does not exist for* $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$, *and exists for the other finite fields.*

(ii) *A del Pezzo surface of degree* 2 *of type* 35 *does not exist for* $\mathbb{F}_2$, *and exists for any* $\mathbb{F}_q$ *where* $q \geqslant 4$.

(iii) *Del Pezzo surfaces of degree* 2 *of types* 40, 43 *exist for any* $\mathbb{F}_q$ *where* $q \geqslant 3$.

(iv) *Del Pezzo surfaces of degree* 2 *of types* 44, 45 *exist for all finite fields.*

Proof. The surfaces of type 31 do not exist for $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$ and the surfaces of type 35 do not exist for $\mathbb{F}_2$ by Remark 2.4.

We apply Theorem 2.3 for each remaining case and then consequently apply Lemmas 2.10, 2.11, 2.13 and Proposition 2.14. Then we get a minimal del Pezzo surface X of degree 2 in all cases except the following: the surface X has type 40 and $q = 2$; the surface X has type 35 and q is 3 or 4; the surface X has type 43 and $q = 2$.

The surface of type 35 exists over $\mathbb{F}_4$ since the surface of type 44 exists over $\mathbb{F}_2$, and the other cases are excluded by the conditions of this proposition. □

Del Pezzo surfaces of types 40 and 43 over $\mathbb{F}_2$ are considered in Section 3.

3. The case $\rho(\overline{X})^{\Gamma} = 1$

In this section we construct minimal del Pezzo surfaces of degree 2 with the Picard number 1. In this case a del Pezzo surface X is not a blow up of del Pezzo surface of higher degree and does not admit a structure of conic bundle.

For a del Pezzo surface X of degree 2 the linear system $|-K_X|$ gives a double cover of $\mathbb{P}^2_{\mathbb{F}_q}$. This cover defines an involution γ on X which is called *Geiser involution.* Therefore we can apply the following proposition.

Proposition 3.20 ([8, Proposition 4.4])**.** *Let* X_1 *be a smooth algebraic variety over finite field* $\mathbb{F}_q$ *such that a cyclic group* G *of order* n *acts on* X_1 *and this action induces a faithful action of* G *on the group* $\mathrm{Pic}(\overline{X}_1)$. *Let* Γ_1 *be the image of the Galois group* $\mathrm{Gal}\left(\overline{\mathbb{F}}_q/\mathbb{F}_q\right)$ *in the group* $\mathrm{Aut}\left(\mathrm{Pic}(\overline{X}_1)\right)$. *Let* h *and* g *be the generators of* Γ_1 *and* G *respectively.*

Then there exists a variety X_2 *such that the image* Γ_2 *of the Galois group* $\mathrm{Gal}\left(\overline{\mathbb{F}}_q/\mathbb{F}_q\right)$ *in the group* $\mathrm{Aut}\left(\mathrm{Pic}(\overline{X}_2)\right) \cong \mathrm{Aut}\left(\mathrm{Pic}(\overline{X}_1)\right)$ *is generated by the element* gh.

Note that in Proposition 3.20 one has $\overline{X}_1 \cong \overline{X}_2$. Therefore if X_1 is a del Pezzo surface, then X_2 is a del Pezzo surface of the same degree.

Definition 3.21. Let X_1 be a del Pezzo surface of degree 2, such that the image Γ_1 of the Galois group $\mathrm{Gal}\left(\overline{\mathbb{F}}_q/\mathbb{F}_q\right)$ in the group $\mathrm{Aut}\left(\mathrm{Pic}(\overline{X}_1)\right)$ is generated by an element h. Then by Proposition 3.20 there exists a del Pezzo surface X_2 of degree 2, such that the image Γ_2 of the Galois group $\mathrm{Gal}\left(\overline{\mathbb{F}}_q/\mathbb{F}_q\right)$ in the group $\mathrm{Aut}\left(\mathrm{Pic}(\overline{X}_2)\right)$ is generated by an element γh. We say that the surface X_2 is a *Geiser twist* of the surface X_1.

Note that Geiser twists are also used in the paper [1, see 4.1.2].

Remark 3.22. Note that the Geiser involution γ acts on $K_X^{\perp}$ by multiplying all elements by -1. Therefore the eigenvalues of the group Γ_2 are the eigenvalues of the group Γ_1 multiplied by -1. Thus for each type of the group Γ_1 it is easy to find the type of the corresponding group Γ_2 (see Table 1), except the cases where two types of Γ_2 have the same collections of eigenvalues. These cases do not appear in this paper.

Now we consider the remaining cases of Section 2.

Lemma 3.23. *A del Pezzo surface of degree* 2 *of type* 43 *exists for any field* $\mathbb{F}_q$.

Proof. By [1, Section 3, case $a = 0$] for any q there exists a del Pezzo surface X_1 of degree 2 that is the blowup of a point of degree 2 and a point of degree 5 on $\mathbb{P}^2_{\mathbb{F}_q}$. The surface X_1 has type 24 since the generator of the group Γ_1 has eigenvalues 1, -1, 1, ξ_5, ξ_5^2, ξ_5^3, ξ_5^4 on $K_X^{\perp} \subset \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}$, where ξ_5 is a fifth root of unity. By Remark 3.22 for the Geiser twist X_2 of X_1 the generator of the group Γ_2 has eigenvalues -1, 1, -1, $-\xi_5$, $-\xi_5^2$, $-\xi_5^3$, $-\xi_5^4$ and X_2 has type 43 (see Table 1). □

Lemma 3.24. *A del Pezzo surface of degree* 2 *of type* 40 *does not exist for* $\mathbb{F}_2$.

Proof. Assume that a del Pezzo surface X_1 of degree 2 of type 40 exists for $\mathbb{F}_2$. Then the generator of the group Γ_1 has eigenvalues 1, -1, -1, -1, -1, $-\omega$, $-\omega^2$ on $K_X^{\perp} \subset \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}$, where ω is a third root of unity. By Remark 3.22 for the Geiser twist X_2 of X_1 the generator of the group Γ_2 has eigenvalues -1, 1, 1, 1, 1, ω, ω^2 and X_2 has type 7 (see Table 1). Thus X_2 is the blowup of $\mathbb{P}^2_{\mathbb{F}_2}$ at two $\mathbb{F}_q$-points p_1 and p_2, a point p_3 of degree 2, and a point p_4 of degree 3. The line L passing through p_3 and the conic C passing through p_3 and p_4 are defined over $\mathbb{F}_2$ and do not have common $\mathbb{F}_2$-points. Therefore there are eight different $\mathbb{F}_2$-points on $\mathbb{P}^2_{\mathbb{F}_2}$: three $\mathbb{F}_2$-points on C, three $\mathbb{F}_2$-points on L, p_1 and p_2. That is impossible. □

Now let us consider Geiser twists of del Pezzo surfaces X with $\rho(\overline{X})^{\Gamma} = 1$.

Proposition 3.25. *Let X be a del Pezzo surface of degree* 2 *over $\mathbb{F}_q$ such that $\rho(\overline{X})^{\Gamma} = 1$. Then X exists if and only if there exists a del Pezzo surface X' of degree* 2 *such that* (*we use the notation Table 1*) *the following holds*:

1. *if X has type* 60, *then X' has type* 32. *Therefore X' is a blowup of a cubic surface of type* (c_{11}) (*see* [9]) *at an $\mathbb{F}_q$-point*;
2. *if X has type* 59, *then X' has type* 36. *Therefore X' is a blowup of a minimal del Pezzo surface of degree* 5 *at a point of degree* 3;
3. *if X has type* 58, *then X' has type* 46. *Therefore X' is a blowup of a cubic surface of type* (c_{13}) (*see* [9]) *at an $\mathbb{F}_q$-point*;
4. *if X has type* 57, *then X' has type* 39. *Therefore X' is a blowup of $\mathbb{P}^2_{\mathbb{F}_q}$ at a point of degree* 7;
5. *if X has type* 56, *then X' has type* 47. *Therefore X' is a blowup of a cubic surface of type* (c_{14}) (*see* [9]) *at an $\mathbb{F}_q$-point*;

(6) *if X has type 55, then X' has type 12. Therefore X' is a blowup of $\mathbb{P}^2_{\mathbb{F}_q}$ at two points of degree 3 and an $\mathbb{F}_q$-point*;

(7) *if X has type 54, then X' has type 15. Therefore X' is a blowup of $\mathbb{P}^2_{\mathbb{F}_q}$ at a point of degree 5 and an $\mathbb{F}_q$-point*;

(8) *if X has type 53, then X' has type 4. Therefore X' is a blowup of $\mathbb{P}^2_{\mathbb{F}_q}$ at a point of degree 3 and four $\mathbb{F}_q$-points*;

(9) *if X has type 52, then X' has type 44. Therefore X' is a minimal del Pezzo surface of degree 2 admitting a conic bundle structure with degenerate fibres over points of degree 2 and 4*;

(10) *if X has type 51, then X' has type 48. Therefore X' is a blowup of a cubic surface of type (c_{12}) (see [9]) at an $\mathbb{F}_q$-point*;

(11) *if X has type 50, then X' has type 17. Therefore X' is a blowup at two $\mathbb{F}_q$-points of a minimal del Pezzo surface of degree 4 admitting a conic bundle structure with degenerate fibres over two points of degree 2*;

(12) *if X has type 49, then X' has type 1. Therefore X' is a blowup of $\mathbb{P}^2_{\mathbb{F}_q}$ at seven $\mathbb{F}_q$-points.*

Proof. By Remark 3.22 for each type of the group Γ it is easy to find the type of the corresponding group Γ' (see Table 1).

Note that each considered type of the group Γ has unique collection of eigenvalues of action on $K_X^{\perp} \subset \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}$. Moreover, a blowup of a del Pezzo surface at a point of degree d adds $\xi_d, \xi_d^2, \ldots, \xi_d^{d-1}, 1$ to the collection of eigenvalues of the group Γ. Therefore each minded nonminimal del Pezzo surface X' can be realised as a blowup of a del Pezzo surface at number of points of certain degrees. □

A del Pezzo surface of degree 2 of type 44 was constructed in Proposition 2.19(iv).

To construct the other types of del Pezzo surfaces of degree 2, such that $\rho(X) = 1$, it is sufficient to blow up a number of points of certain degrees on del Pezzo surfaces of higher degree. We apply the following well-known theorem.

Theorem 3.26 (cf. [5, Theorem 2.5])**.** *Let $1 \leqslant d \leqslant 9$, and $p_1, \ldots, p_{9-d}$ be $9-d$ geometric points on the projective plane $\mathbb{P}^2_{\overline{\Bbbk}}$ such that*

- *no three lie on a line*;
- *no six lie on a conic*;
- *for $d = 1$ the points are not on a singular cubic curve with singularity at one of these points.*

Then the blowup of $\mathbb{P}^2_{\overline{\Bbbk}}$ at $p_1, \ldots, p_{9-d}$ is a del Pezzo surface of degree d.

Moreover, any del Pezzo surface $\overline{X}$ of degree $1 \leqslant d \leqslant 7$ over algebraically closed field $\overline{\Bbbk}$ is the blowup of such set of points.

Definition 3.27. If for $1 \leqslant d \leqslant 9$ geometric points $p_1, \ldots, p_{9-d}$ on $\mathbb{P}^2_{\mathbb{F}_q}$ satisfy the conditions of Theorem 3.26, then we say that the points $p_1, \ldots, p_{9-d}$ are in a *general position.*

Corollary 3.28. *Let $\overline{X}$ be a del Pezzo surface of degree $3 \leqslant d \leqslant 7$ and p be a geometric point which does not lie on (-1)-curves. Then the blowup of $\overline{X}$ at p is a del Pezzo surface of degree $d-1$.*

Proof. By Theorem 3.26 the surface $\overline{X}$ is the blowup $f : \overline{X} \to \mathbb{P}^2_{\overline{\Bbbk}}$ of points $p_1, \ldots, p_{9-d}$ on $\mathbb{P}^2_{\overline{\Bbbk}}$. Moreover, no three points in the set $f(p), p_1, \ldots, p_{9-d}$ lie on a line, and no six points in this set lie on a conic, since p does not lie on (-1)-curves. Thus the blowup of the points $f(p), p_1, \ldots, p_{9-d}$ is a del Pezzo surface of degree $d-1$ by Theorem 3.26. □

Now we construct del Pezzo surfaces of degree 2 of types 1, 4, 12, 15, 17, 32, 36, 39, 46, 47, 48 which Geiser twists are minimal surfaces with $\rho(X)^{\Gamma} = 1$.

Lemma 3.29.
- *A del Pezzo surface of degree 2 of type 1 does not exist for $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$, $\mathbb{F}_5$, $\mathbb{F}_7$, $\mathbb{F}_8$, and exists for the other finite fields.*
- *Del Pezzo surfaces of degree 2 of types 4, 12 do not exist for $\mathbb{F}_2$, and exist for the other finite fields.*
- *Del Pezzo surfaces of degree 2 of types 15, 39 exist for all finite fields.*

Proof. Considered types of del Pezzo surfaces of degree 2 are blowups of $\mathbb{P}^2_{\mathbb{F}_q}$ at sets of points of certain degrees in a general position. Types 1, 4 and 15 were considered in [1, Subsection 4.2] in cases $a = 8$, $a = 5$ and $a = 3$ respectively.

A del Pezzo surface of degree 2 of type 12 is the blowup of $\mathbb{P}^2_{\mathbb{F}_q}$ at two points of degree 3 and an $\mathbb{F}_q$-point. Such configuration of points in a general position does not exist for $\mathbb{F}_2$ since one cannot blow up seven $\mathbb{F}_8$-points on $\mathbb{P}^2_{\mathbb{F}_8}$ in a general position (see [1, Subsection 4.2, case $a = 8$]).

For $q \geqslant 3$ one can construct a cubic surface S which is the blowup of $\mathbb{P}^2_{\mathbb{F}_q}$ at two points of degree 3 in a general position (see [8, Proposition 6.2]). The 27 lines on S form 9 triples defined over $\mathbb{F}_q$, and only 3 of these triples consist of meeting each other lines. Therefore at most three $\mathbb{F}_q$-points on S lie on the lines. So one can find an $\mathbb{F}_q$-point on S which does not lie on the lines, blow up this point, and get a del Pezzo surface of type 12 by Corollary 3.28.

A del Pezzo surface of degree 2 of type 39 is the blowup of $\mathbb{P}^2_{\mathbb{F}_q}$ at a point of degree 7. Let $p_1 = \left(a^3 : a : 1\right)$, where $a \in \mathbb{F}_{q^7} \setminus \mathbb{F}_q$, and $p_2, \ldots, p_7$ be the conjugates of p_1. If six points from the set $p_1, \ldots, p_7$ lie on a conic, then all these points lie on a conic defined over $\mathbb{F}_q$. But it is impossible, since for the conic given by $Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0$ the equality $Aa^6+Ba^4+Ca^2+Da^3+Ea+F = 0$ holds only if $A = B = C = D = E = F = 0$.

Note that three points $\left(x^3 : x : 1\right)$, $\left(y^3 : y : 1\right)$, $\left(z^3 : z : 1\right)$ lie on a line if and only if $x = y$, $x = z$, $y = z$ or $x+y+z = 0$. Therefore if three points in the set $p_1, \ldots, p_7$ lie on a line, then $a^{q^i} + a^{q^j} + a^{q^k} = 0$. One may assume that $i = 0$. Up to symmetries there are four possibilities:

- $j = 1,\ k = 2$;
- $j = 1,\ k = 3$;
- $j = 1,\ k = 4$;
- $j = 2,\ k = 4$.

Note that $a + a^q + \cdots + a^{q^6} \in \mathbb{F}_q$.

If $j = 1,\ k = 2$, then $a + a^q + a^{q^2} = a^{q^3} + a^{q^4} + a^{q^5} = 0$, and $a^{q^6} \in \mathbb{F}_q$ that is impossible.

If $j = 1,\ k = 4$, then $a + a^q + a^{q^4} = a^{q^2} + a^{q^3} + a^{q^6} = 0$, and $a^{q^5} \in \mathbb{F}_q$ that is impossible.

If $j = 2,\ k = 4$, then $a + a^{q^2} + a^{q^4} = a^q + a^{q^3} + a^{q^5} = 0$, and $a^{q^6} \in \mathbb{F}_q$ that is impossible.

If $j = 1,\ k = 3$, then $a + a^q + a^{q^3} = a^q + a^{q^2} + a^{q^4} = a^{q^5} + a^{q^6} + a^q = 0$, and $2a^q \in \mathbb{F}_q$ that is possible only for even q. But in this case $a + a^q + \cdots + a^{q^6} = 0$. One can put $a' = a + 1$, and have $a' + a'^q + \cdots + a'^{q^6} = 1$. Now $a'^{q^i} + a'^{q^j} + a'^{q^k} \neq 0$ for any i, j and k.

Therefore for any $\mathbb{F}_q$ we can find a point of degree 7 on $\mathbb{P}^2_{\mathbb{F}_q}$ in a general position, blow up this point, and get a del Pezzo surface of type 39 by Theorem 3.26. □

Lemma 3.30. *A del Pezzo surface of degree* 2 *of type* 36 *exists for all finite fields.*

Proof. A del Pezzo surface of degree 2 of type 36 is the blowup of a minimal del Pezzo surface of degree 5 at a point of degree 3. One can blow up a point of degree 5 lying on a conic in $\mathbb{P}^2_{\mathbb{F}_q}$, contract the transform of this conic and get a minimal del Pezzo surface of degree 5.

Let P and Q be two conics on $\mathbb{P}^2_{\mathbb{F}_q}$ defined over $\mathbb{F}_q$; the point p_1 be a geometric point on P defined over $\mathbb{F}_{q^5}$; the points $p_2, \ldots, p_5$ be the conjugates of p_1; the point q_1 be a geometric point on Q defined over $\mathbb{F}_{q^3}$ which does not lie on P; and q_2 and q_3 be the conjugates of q_1. Let F be the Frobenius automorphism of $\mathbb{P}^2_{\mathbb{F}_q}$:

$$F(x : y : z) = (x^q : y^q : z^q).$$

Assume that points p_i, p_j and q_k lie on a line. Then the points $F^5 p_i = p_i$, $F^5 p_j = p_j$ and $F^5 q_k$ lie on the same line. Therefore the points q_1, q_2 and q_3 lie on a line. But this is impossible since any line meets Q at 2 or 1 point. The same arguments show that three points p_i, q_j and q_k can not lie on a line.

If a conic passes through six points p_i, p_j, p_k, q_1, q_2 and q_3, then this conic passes through the points $p_1, \ldots, p_5$ since either the set $\{Fp_i, Fp_j, Fp_k\}$ or the set $\{F^2 p_i, F^2 p_j, F^2 p_k\}$ has two common points with the set $\{p_i, p_j, p_k\}$. If a conic C passes through four points from the set $\{p_1, \ldots, p_5\}$ and two points from the set $\{q_1, q_2, q_3\}$, then it passes through all points from these sets since it has 5 common points with the conics $F^3 C$ and $F^5 C$. All these cases are impossible since the points q_1, q_2 and q_3 do not lie on P.

If an irreducible plane cubic curve C passes through the eight points $p_1, \ldots,$ p_5, q_1, q_2, q_3 and has a singularity at one of these points, then it has at least three singular points since $C \cdot FC \geqslant 10$ and $C \cdot F^2C \geqslant 10$. That is impossible.

Thus the points $p_1, \ldots, p_5, q_1, q_2, q_3$ lie in a general position. The blowup of $\mathbb{P}^2_{\mathbb{F}_q}$ at these points is a del Pezzo surface of degree 1 by Theorem 3.26. One can contract the transform of P, and get a del Pezzo surface of type 36. □

Lemma 3.31. *A del Pezzo surface of degree* 2 *of type* 17 *does not exist for* $\mathbb{F}_2$*, and exists for the other finite fields.*

Proof. A del Pezzo surface of degree 2 of type 17 is the blowup at two $\mathbb{F}_q$-points of a minimal del Pezzo surface S of degree 4 admitting a conic bundle structure with degenerate fibres over two points of degree 2. Such del Pezzo surface does not exist over $\mathbb{F}_2$ and exists for the other finite fields by [7, Theorem 3.2]. Assume that $q \geqslant 3$. The surface S admits two structures of conic bundles and each of 16 lines is a component of a singular fibre of one of these conic bundles. These lines form four $\mathrm{Gal}\left(\overline{\mathbb{F}}_q/\mathbb{F}_q\right)$-orbits, each consisting of 4 curves. Therefore there are no $\mathbb{F}_q$-points on the (-1)-curves. But there are $(q+1)^2$ points defined over $\mathbb{F}_q$ on the smooth fibres. Let $f : \widetilde{S} \to S$ be the blowup of S at an $\mathbb{F}_q$-point P. By Corollary 3.28 the surface $\widetilde{S}$ is a cubic surface. There are three lines on $\widetilde{S}$ defined over $\mathbb{F}_q$: the exceptional divisor $E = f^{-1}(P)$, and the proper transforms C_1 and C_2 of fibres of two conic bundles structures passing through P.

Let F be the Frobenius automorphism. We show that all other F-orbits of lines consist of 4 lines. Let L be a line on $\widetilde{S}$ that differs from E, C_1 and C_2. If $L \cdot E = 0$, then $f(L)$ is a (-1)-curve and the orbit of this curve consists of 4 curves. Assume that $E \cdot L = 1$. Then $C_1 \cdot L = C_2 \cdot L = 0$ since $E + C_1 + C_2 \sim -K_{\widetilde{S}}$. It means that $f(L)$ is a section of any conic bundle on S. For any singular fibre this section must meet one component D_1 of this fibre at a point, and for the other component D_2 of this fibre $f(L) \cdot D_2 = 0$. But we have $F^2D_1 = D_2$, therefore $F^2f(L) \cdot D_2 = f(L) \cdot D_1 = 1$. Thus $F^2f(L) \neq f(L)$ and the orbit of L consists of 4 lines.

Four lines on a cubic surface can not have a common point. Therefore all $\mathbb{F}_q$ points on lines on $\widetilde{S}$ are contained in E, C_1 and C_2. Thus there are q^2 points defined over $\mathbb{F}_q$ not lying on a lines on $\widetilde{S}$. One can blow up one of these points, and get a del Pezzo surface of type 17 by Corollary 3.28. □

Lemma 3.32.
- *A del Pezzo surface of degree* 2 *of type* 32 *does not exist for* $\mathbb{F}_2$*, and exists for the other finite fields.*
- *Del Pezzo surfaces of degree* 2 *of types* 46*,* 47*,* 48 *exist only for those finite fields for which exist minimal cubic surfaces of types* c_{13}*,* c_{14} *and* c_{12} *respectively* (*see e.g.* [5, IV.9, Table 1]).

Proof. Considered types of del Pezzo surfaces of degree 2 are blowups of minimal cubic surfaces at an $\mathbb{F}_q$-point. Note that the $\mathrm{Gal}\left(\overline{\mathbb{F}}_q/\mathbb{F}_q\right)$-orbits of lines on a cubic surface with length greater than 3 do not contain $\mathbb{F}_q$-points and any orbit of length 3 can contain at most one $\mathbb{F}_q$-point.

Del Pezzo surfaces of degree 2 of types 46, 47, 48 are blowups at an $\mathbb{F}_q$-point of minimal cubic surfaces of types c_{13}, c_{14} and c_{12} respectively. For these cubic surfaces all orbits of lines have length 3 or greater, moreover, there is at most one orbit of length 3 (see [5, IV.9, Table 1]). It means that there is at most one $\mathbb{F}_q$-point lying on the lines on such cubic surfaces. But there are q^2+1, q^2+q+1 and q^2+2q+1 points defined over $\mathbb{F}_q$ on such cubic surfaces respectively. Thus in each of those cases there is an $\mathbb{F}_q$-point not lying on the lines. One can blow up this point, and get a del Pezzo surface of type 46, 47 or 48 respectively by Corollary 3.28.

A Del Pezzo surface of degree 2 of type 32 is the blowup at an $\mathbb{F}_q$-point of a minimal cubic surface S of type c_{11}. This type of cubic surface was constructed in [10] for any finite field. Each $\mathrm{Gal}\left(\overline{\mathbb{F}}_q/\mathbb{F}_q\right)$-orbit of lines consist of three lines (see [5, IV.9, Table 1]. Therefore there are at most nine $\mathbb{F}_q$-points lying on the lines, and all these points are Eckardt points.

There are q^2-2q+1 points defined over $\mathbb{F}_q$ on S. Moreover, each nonsingular cubic surface with q^2-2q+1 points defined over $\mathbb{F}_q$ has type c_{11}. If $q>4$, then we can find an $\mathbb{F}_q$-point on S which does not lie on the lines, blow up this point, and get a del Pezzo surface of type 32 by Corollary 3.28.

If $q=2$, then S contains a unique $\mathbb{F}_2$-point. By direct computation one can check that any cubic surface containing a unique $\mathbb{F}_2$-point is isomorphic to the surface given by the following equation:

$$x^3+y^3+z^3+x^2y+y^2z+z^2x+xyz+z^2t+zt^2=0. \tag{3.33}$$

The $\mathbb{F}_2$-point $(0:0:0:1)$ is an Eckardt point. Therefore all $\mathbb{F}_2$-points on S are contained in the lines, and a del Pezzo surface of type 32 does not exist over $\mathbb{F}_2$.

The cubic given by equation (3.33) considered over $\mathbb{F}_4$ has type c_{11} and contains $\mathbb{F}_4$-point $(0:\omega:\omega:1)$, where $\omega^3=1$. This point is not an Eckardt point. Therefore one can blow up this point, and get a del Pezzo surface of type 32 by Corollary 3.28.

For $q=3$ the cubic surface given by the equation

$$x^2y+xy^2+x^2z+xyz+y^2z-xyt-xzt-yzt-z^2t-zt^2+t^3=0$$

contains exactly four $\mathbb{F}_3$-points: $(1:0:0:0)$, $(0:1:0:0)$, $(1:-1:0:0)$, $(0:0:1:0)$. Thus this surface has type c_{11}. One can check that the point $(0:0:1:0)$ is not an Eckardt point. Therefore one can blow up this point, and get a del Pezzo surface of type 32 by Corollary 3.28. □

Now we collect results about del Pezzo surfaces of degree 2 with $\rho(\overline{X})^{\Gamma}=1$.

Proposition 3.34. *In the notation of Table 1 the following holds.*

(i) *A del Pezzo surface of degree* 2 *of type* 49 *does not exist for* $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$, $\mathbb{F}_5$, $\mathbb{F}_7$, $\mathbb{F}_8$, *and exists for the other finite fields.*

(ii) *Del Pezzo surfaces of degree* 2 *of types* 50, 53, 55, 60 *do not exist for* $\mathbb{F}_2$, *and exist for the other finite fields.*
(iii) *Del Pezzo surfaces of degree* 2 *of types* 52, 54, 57, 59 *exist for all finite fields.*
(iv) *Del Pezzo surfaces of degree* 2 *of types* 51, 58 *exist for any* $\mathbb{F}_q$ *where* q *is odd.*
(v) *A del Pezzo surface of degree* 2 *of type* 56 *exists for any* $\mathbb{F}_q$ *where* $q = 6k + 1$.

Proof. We apply Proposition 3.25 and then apply Lemma 3.29 for types 49, 53, 54, 55 and 57; Lemma 3.30 for type 59; Lemma 3.31 for type 50; Lemma 3.32 for types 51, 56, 58, 60; and Proposition 2.19(iv) for type 52.

By [8, Theorem 1.2] cubic surfaces of types c_{12} and c_{13} exist for odd q, and cubic surfaces of type c_{14} exist for $q = 6k + 1$. These types of cubic surfaces correspond to types 51, 58 and 56 of del Pezzo surfaces of degree 2 respectively by Proposition 3.25 and Lemma 3.32. □

Now we prove Theorem 1.1.

Proof. Case (1) is Proposition 3.34(i); case (2) is Proposition 2.19(i); case (3) follows from Proposition 2.19(iii), Lemma 3.24 and Proposition 3.34(ii); case (4) follows from Proposition 2.19(iii), Lemma 3.23, Proposition 2.19(iv) and Proposition 3.34(iii); case (5) is Proposition 2.19(ii); case (6) is Proposition 3.34(iv); and case (7) is Proposition 3.34(v). □

Appendix A. Conjugacy classes of elements in $W(E_7)$

In the following table we collect some facts about conjugacy classes of elements in the Weyl group $W(E_7)$. This table is based on [2, Table 10]. The first column is a number of a conjugacy class in order of their appearance. Throughout the paper this number is called a *type* of del Pezzo surface. The second column is a Carter graph corresponding to the conjugacy class (see [2]). The third column is the order of an element. The fourth column is the collection of eigenvalues of the action of an element on $K_X^{\perp} \subset \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}$. The fifth column is the invariant Picard number $\rho(\overline{X})^{\Gamma}$. The last column is a number of the corresponding conjugacy class after the Geiser twist (see Definition 3.21). We denote by ω a third root of unity and by ξ_d a d-th root of unity.

Table 1: Conjugacy classes of elements in $W(E_7)$

Number	Graph	Order	Eigenvalues	$\rho(\overline{X})^{\Gamma}$	Geiser
1.	$\varnothing$	1	1, 1, 1, 1, 1, 1, 1	8	49.
2.	A_1	2	1, 1, 1, 1, 1, 1, −1	7	31.
3.	A_1^2	2	1, 1, 1, 1, 1, −1, −1	6	18.
4.	A_2	3	1, 1, 1, 1, 1, ω, ω^2	6	53.
5.	A_1^3	2	1, 1, 1, 1, −1, −1, −1	5	9.
6.	A_1^3	2	1, 1, 1, 1, −1, −1, −1	5	10.

7.	$A_2 \times A_1$	6	$1,\ 1,\ 1,\ 1,\ -1,\ \omega,\ \omega^2$	5	40.
8.	A_3	4	$1,\ 1,\ 1,\ 1,\ i,\ -1,\ -i$	5	33.
9.	A_1^4	2	$1,\ 1,\ 1,\ -1,\ -1,\ -1,\ -1$	4	5.
10.	A_1^4	2	$1,\ 1,\ 1,\ -1,\ -1,\ -1,\ -1$	4	6.
11.	$A_2 \times A_1^2$	6	$1,\ 1,\ 1,\ -1,\ -1,\ \omega,\ \omega^2$	4	27.
12.	A_2^2	3	$1,\ 1,\ 1,\ \omega,\ \omega^2,\ \omega,\ \omega^2$	4	55.
13.	$A_3 \times A_1$	4	$1,\ 1,\ 1,\ i,\ -1,\ -i,\ -1$	4	21.
14.	$A_3 \times A_1$	4	$1,\ 1,\ 1,\ i,\ -1,\ -i,\ -1$	4	22.
15.	A_4	5	$1,\ 1,\ 1,\ \xi_5,\ \xi_5^2,\ \xi_5^3,\ \xi_5^4$	4	54.
16.	D_4	6	$1,\ 1,\ 1,\ -1,\ -\omega^2,\ -1,\ -\omega$	4	19.
17.	$D_4(a_1)$	4	$1,\ 1,\ 1,\ i,\ -i,\ i,\ -i$	4	50.
18.	A_1^5	2	$1,\ 1,\ -1,\ -1,\ -1,\ -1,\ -1$	3	3.
19.	$A_2 \times A_1^3$	6	$1,\ 1,\ -1,\ -1,\ -1,\ \omega,\ \omega^2$	3	16.
20.	$A_2^2 \times A_1$	6	$1,\ 1,\ -1,\ \omega,\ \omega^2,\ \omega,\ \omega^2$	3	45.
21.	$A_3 \times A_1^2$	4	$1,\ 1,\ i,\ -1,\ -i,\ -1,\ -1$	3	13.
22.	$A_3 \times A_1^2$	4	$1,\ 1,\ i,\ -1,\ -i,\ -1,\ -1$	3	14.
23.	$A_3 \times A_2$	12	$1,\ 1,\ i,\ -1,\ -i,\ \omega,\ \omega^2$	3	42.
24.	$A_4 \times A_1$	10	$1,\ 1,\ \xi_5,\ \xi_5^2,\ \xi_5^3,\ \xi_5^4,\ -1$	3	43.
25.	A_5	6	$1,\ 1,\ -\omega^2,\ \omega,\ -1,\ \omega^2,\ -\omega$	3	37.
26.	A_5	6	$1,\ 1,\ -\omega^2,\ \omega,\ -1,\ \omega^2,\ -\omega$	3	38.
27.	$D_4 \times A_1$	6	$1,\ 1,\ -1,\ -\omega^2,\ -1,\ -\omega,\ -1$	3	11.
28.	$D_4(a_1) \times A_1$	4	$1,\ 1,\ i,\ -i,\ i,\ -i,\ -1$	3	35.
29.	D_5	8	$1,\ 1,\ -1,\ \xi_8,\ \xi_8^3,\ \xi_8^5,\ \xi_8^7$	3	41.
30.	$D_5(a_1)$	12	$1,\ 1,\ i,\ -i,\ -\omega^2,\ -1,\ -\omega$	3	34.
31.	A_1^6	2	$1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1$	2	2.
32.	A_2^3	3	$1,\ \omega,\ \omega^2,\ \omega,\ \omega^2,\ \omega,\ \omega^2$	2	60.
33.	$A_3 \times A_1^3$	4	$1,\ i,\ -1,\ -i,\ -1,\ -1,\ -1$	2	8.
34.	$A_3 \times A_2 \times A_1$	12	$1,\ i,\ -1,\ -i,\ \omega,\ \omega^2,\ -1$	2	30.
35.	A_3^2	4	$1,\ i,\ -1,\ -i,\ i,\ -1,\ -i$	2	28.
36.	$A_4 \times A_2$	15	$1,\ \xi_5,\ \xi_5^2,\ \xi_5^3,\ \xi_5^4,\ \omega,\ \omega^2$	2	59.
37.	$A_5 \times A_1$	6	$1,\ -\omega^2,\ \omega,\ -1,\ \omega^2,\ -\omega,\ -1$	2	25.
38.	$A_5 \times A_1$	6	$1,\ -\omega^2,\ \omega,\ -1,\ \omega^2,\ -\omega,\ -1$	2	26.
39.	A_6	7	$1,\ \xi_7,\ \xi_7^2,\ \xi_7^3,\ \xi_7^4,\ \xi_7^5,\ \xi_7^6$	2	57.
40.	$D_4 \times A_1^2$	6	$1,\ -1,\ -\omega^2,\ -1,\ -\omega,\ -1,\ -1$	2	7.
41.	$D_5 \times A_1$	8	$1,\ -1,\ \xi_8,\ \xi_8^3,\ \xi_8^5,\ \xi_8^7,\ -1$	2	29.
42.	$D_5(a_1) \times A_1$	12	$1,\ i,\ -i,\ -\omega^2,\ -1,\ -\omega,\ -1$	2	23.
43.	D_6	10	$1,\ -1,\ -\xi_5^3,\ -\xi_5^4,\ -1,\ -\xi_5,\ -\xi_5^2$	2	24.
44.	$D_6(a_1)$	8	$1,\ i,\ -i,\ \xi_8,\ \xi_8^3,\ \xi_8^5,\ \xi_8^7$	2	52.
45.	$D_6(a_2)$	6	$1,\ -\omega^2,\ -1,\ -\omega,\ -\omega^2,\ -1,\ -\omega$	2	20.
46.	E_6	12	$1,\ \omega,\ \omega^2,\ -i\omega,\ -i\omega^2,\ i\omega,\ i\omega^2$	2	58.
47.	$E_6(a_1)$	9	$1,\ \xi_9,\ \xi_9^2,\ \xi_9^4,\ \xi_9^5,\ \xi_9^7,\ \xi_9^8$	2	56.
48.	$E_6(a_2)$	6	$1,\ \omega,\ \omega^2,\ -\omega^2,\ -\omega,\ -\omega^2,\ -\omega$	2	51.
49.	A_1^7	2	$-1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1$	1	1.
50.	$A_3^2 \times A_1$	2	$-1,\ i,\ -1,\ -i,\ i,\ -1,\ -i$	1	17.
51.	$A_5 \times A_2$	6	$\omega,\ \omega^2,\ -\omega^2,\ \omega,\ -1,\ \omega^2,\ -\omega$	1	48.
52.	A_7	8	$\xi_8,\ i,\ \xi_8^3,\ -1,\ \xi_8^5,\ -i,\ \xi_8^7$	1	44.
53.	$D_4 \times A_1^3$	6	$-1,\ -1,\ -1,\ -1,\ -\omega^2,\ -1,\ -\omega$	1	4.
54.	$D_6 \times A_1$	10	$-1,\ -1,\ -1,\ -\xi_5^3,\ -\xi_5^4,\ -\xi_5,\ -\xi_5^2$	1	15.
55.	$D_6(a_2) \times A_1$	6	$-1,\ -\omega^2,\ -1,\ -\omega,\ -\omega^2,\ -1,\ -\omega$	1	12.

56.	E_7	18	$-1, -\xi_9^5, -\xi_9^7, -\xi_9^8, -\xi_9, -\xi_8^2, -\xi_9^4$	1	47.
57.	$E_7(a_1)$	14	$-\xi_7^4, -\xi_7^5, -\xi_7^6, -1, -\xi_7, -\xi_7^2, -\xi_7^3$	1	39.
58.	$E_7(a_2)$	12	$-\omega^2, -1, -\omega, -i\omega, -i\omega^2, i\omega, i\omega^2$	1	46.
59.	$E_7(a_3)$	30	$-\omega^2, -1, -\omega, -\xi_5^3, -\xi_5^4, -\xi_5, -\xi_5^2$	1	36.
60.	$E_7(a_4)$	6	$-\omega^2, -1, -\omega, -\omega^2, -\omega, -\omega^2, -\omega$	1	32.

References

[1] B. Banwait, F. Fité, and D. Loughran, *Del Pezzo surfaces over finite fields and their Frobenius traces*, preprint, see http://arxiv.org/abs/1606.00300.

[2] R. W. Carter, *Conjugacy classes in the weyl group*, Compositio Math. **25** (1972), 1–59.

[3] V. A. Iskovskikh, *Minimal models of rational surfaces over arbitrary field*, Math. USSR Izv. **43** (1979), 19–43. (in Russian)

[4] ———, *Factorization of birational mappings of rational surfaces from the point of view of Mori theory*, Uspekhi Mat. Nauk **51** (1996), 3–72 (in Russian); translation in Russian Math. Surveys **51** (1996), 585–652.

[5] Yu. I. Manin, *Cubic forms*: *algebra, geometry, arithmetic*, In: North-Holland Mathematical Library, Vol. 4, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1974.

[6] Yu. G Prokhorov, *Rational surfaces*, Lection courses of SEC, 24, pp. 3–76, 2015. (in Russian).

[7] S. Rybakov, *Zeta-functions of conic bundles and del Pezzo surfaces of degree 4 over finite fields*, Mosc. Math. J. **5** (2005), no. 4, 919–926.

[8] S. Rybakov and A. Trepalin, *Minimal cubic surfaces over finite fields*, Sb. Math. **208**: 9 (2017), DOI:10.1070/SM8880.

[9] H. P. F. Swinnerton-Dyer, *The zeta function of a cubic surface over a finite field*, Proceedings of the Cambridge Philosophical Soc. **63** (1967), 55–71.

[10] ———, *Cubic surfaces over finite fields*, Math. Proceedings of the Cambridge Philosophical Society **149** (2010), 385–388.

Andrey Trepalin
Institute for Information Transmission Problems
19 Bolshoy Karetnyi side-str., Moscow 127994, Russia
E-mail address: trepalin@mccme.ru