Bull. Korean Math. Soc. **54** (2017), No. 5, pp. 1757–1771
https://doi.org/10.4134/BKMS.b160742
pISSN: 1015-8634 / eISSN: 2234-3016

HILBERT'S THEOREM 90 FOR NON-COMPACT GROUPS

Marat Rovinsky

Abstract. Let K be a field and G be a group of its automorphisms. It follows from Speiser's generalization of Hilbert's Theorem 90, [10] that any K-*semilinear* representation of the group G is isomorphic to a direct sum of copies of K, if G is finite.

In this note three examples of pairs (K, G) are presented such that certain irreducible K-semilinear representations of G admit a simple description: (i) with precompact G, (ii) K is a field of rational functions and G permutes the variables, (iii) K is a universal domain over field of characteristic zero and G its automorphism group. The example (iii) is new and it generalizes the principal result of [7].

1. Introduction

1.1. Motivation

This is a revised version of my talk, motivated by a problem on birational invariants of motivic nature, admitting a translation into representation-theoretic terms.

The problem on birational invariants, which is a consequence of the filtration conjecture due to S. Bloch and A. Beilinson, asks to show that if $H^i(X, \mathcal{O}_X) = 0$ for a smooth projective complex variety X, an integer $s \geq 0$ and all $i \geq s$ then there exists a smooth projective s-dimensional variety Z and a morphism $Z \to X$ inducing a surjection $CH_0(Z) \longrightarrow\!\!\!\!\rightarrow CH_0(X)$.

It is shown in [9, Corollary 3.2] that this problem can be reduced to a description of certain irreducible representations of certain topological groups. More precisely, the goal is to show that such representations are contained in an explicit representation $\Omega^\bullet_{F|k}$, cf. §3.

In fact, this explicit representation $\Omega^\bullet_{F|k}$ is a *semilinear* representation. And we are going to use semilinear representations as a tool to study usual representations.

The group here is an example of *permutation group.*

Received September 13, 2016; Accepted December 26, 2016.
2010 *Mathematics Subject Classification.* 14C15, 14F20, 14F43, 20B27, 20C32.
Key words and phrases. non-compact groups.

Definition 1.1. A *permutation group* is a Hausdorff topological group G admitting a base of open subsets consisting of the left and right shifts of subgroups.

If we denote by B a collection of open subgroups such that the finite intersections of conjugates of elements of B form a base of open neighbourhoods of 1 in G (e.g., the set of all open subgroups of G), then G acts faithfully on the set $\Psi := \coprod_{U \in B} G/U$, so (i) G becomes a *permutation group of* Ψ, (ii) the left or right translates of the pointwise stabilizers G_T of the finite subsets $T \subset \Psi$ form a base of the topology of G. Clearly, G is totally disconnected.

1.2. Semilinear representations

Let K be a field and G be a group of its automorphisms. Then G becomes a Hausdorff group when we take the left (equivalently, the right) translates of the pointwise stabilizers $G_T \subseteq G$ of finite subsets T of K as a base of open subsets of G.

For an abelian group A and a set S we denote by $A[S]$ the direct sum of copies of A indexed by S. In some cases, $A[S]$ will be endowed with an additional structure, e.g., of a module, a ring, etc.

A K-*semilinear representation* of G is a K-vector space V endowed with an additive semilinear G-action: $g(fv) = gf \cdot gv$ for any $g \in G$, $f \in K$, $v \in V$. This is the same as a left $K\langle G\rangle$-module, where $K\langle G\rangle$ denotes the unital associative subring in $\mathrm{End}_{\mathbb{Z}}(K[G])$ generated by the natural left action of K and the diagonal left action of G on $K[G]$. In other words, $K\langle G\rangle$ is the ring of K-valued measures on G with finite support. Then $K\langle G\rangle$ is a *central* algebra over the fixed field K^G.

More explicitly, the elements of $K\langle G\rangle$ are the finite formal sums $\sum_{i=1}^N a_i[g_i]$ for all integer $N \geq 0$, $a_i \in K$, $g_i \in G$. Addition is defined obviously; multiplication is a unique distributive one such that $(a[g])(b[h]) = ab^g[gh]$, where we write a^h for the result of applying of $h \in G$ to $a \in K$.

For an abelian group A and a set S we denote by $A[S]$ the direct sum of copies of A indexed by S. In some cases, $A[S]$ will be endowed with an additional structure, e.g., of a module, a ring, etc.

A G-action on a set is called *smooth* if this G-action is continuous when the set is endowed with the discrete topology (i.e., the stabilizers are open).

Lemma 1.2. *Let G be a group of automorphisms of a field K. Then the category of smooth K-semilinear representations of the group G is "simple" in the sense that all nonzero subcategories in it, closed under direct products and subquotients, are equivalent.*

Proof. Let V be a nonzero smooth K-semilinear representation of G. The semilinear representations $K[G/G_T]$ for all finite subsets $T \subseteq K$ such that $V^{G_T} \neq 0$ form a system of generators of the category of smooth K-semilinear representations of the group G, so it suffices to show that there is an embedding of $K[G/G_T]$ into a direct product of copies of V, or equivalently, that there is a family of morphisms $K[G/G_T] \to V$ with the vanishing common kernel.

Fix a nonzero morphism $\alpha \in \mathrm{Hom}_{K\langle G\rangle}(K[G/G_T], V)$ and consider, for all $t \in K^{G_T}$, the morphisms $t\alpha : K[G/G_T] \to V$, $[\sigma] \mapsto \sigma t \cdot \alpha([\sigma])$. Let $\xi = \sum_{i=1}^{N} a_i\sigma_i \in K[G/G_T]$ be an element in the common kernel of the morphisms $t\alpha$. The elements of the set G/G_T can be considered as (pairwise distinct) $K^{\times}$-valued characters of the group $(K^{G_T})^{\times}$, since $G_{K^{G_T}} = G_T$, so the element ξ can be considered as a K-linear relation between characters. Due to the linear independence of such characters, one has $a_1 = \cdots = a_N = 0$, i.e., $\bigcap_{t\in(K^{G_T})^{\times}} \ker(K[G/G_T] \xrightarrow{t\alpha} V) = 0$. □

1.3. Examples considered in this note

I am going to present two examples and a half of groups G admitting such fields K endowed with a smooth G-action (called G-fields for brevity) that the *smooth irreducible* K-semilinear representations of G can be described explicitly.

The first example of G is an arbitrary precompact group (i.e., any open subgroup of G is of finite index). The second example of G is the infinite symmetric groups acting on rational functions by permuting the variables. In all these examples, the isomorphism classes of the irreducible objects turn out to be one-dimensional as K-vector spaces.

In the remaining half-example, the group is as it is supposed to be in the original geometric problem of §1.1, but the considered semilinear representations satisfy an extra restriction (an appropriate replacement of the finite dimensionality).

2. Speiser's and further generalizations of Hilbert's theorem 90

Speiser's generalization of Hilbert's theorem 90 ([10, Satz 1]) can be generalized further as follows.

Theorem 2.1. *Let G be a permutation group and K be a field endowed with a smooth G-action. Then the object K is a generator of the category of smooth K-semilinear representations of G if and only if G is precompact and the G-action on K is faithful.*

Proof. If G is a precompact automorphism group of K, V is a smooth K-semilinear representations of G and $v \in V$ then the intersection H of all conjugates of the stabilizer of v in G is of finite index. Thus, v is contained in the K^H-semilinear representation V^H of the finite group $\bar{G} := G/H$. It is [10, Satz 1], appropriately reformulated, that any K^H-semilinear representation of $\bar{G}$ is isomorphic to a direct sum of copies of K^H. Namely, the natural $\bar{G}$-action on K^H gives rise to a K^G-algebra homomorphism $K^H\langle\bar{G}\rangle \to \mathrm{End}_{K^G}(K^H)$, which is (a) surjective by Jacobson's density theorem and (b) injective by independence of characters. Then the field extension $K^H|K^G$ is finite and any

$K^H\langle\bar{G}\rangle$-module is isomorphic to a direct sum of copies of K^H. As G/H is finite, $V^H = (V^H)^{G/H} \otimes_{(K^H)^{G/H}} K^H = V^G \otimes_{K^G} K^H$, i.e., v is contained in a subrepresentation isomorphic to a direct sum of copies of K.

If G is not precompact then it admits an open subgroup $U \subset G$ of infinite index, while the representation $K[G/U]$ of G has no nonzero vectors fixed by G, so $K[G/U]$ is not a direct sum of copies of K. (For a G-set S we consider $K[S]$ as a K-vector space with the diagonal G-action.) For any non-identical $g \in G$ there is an open subgroup $U \not\ni g$. Then g acts non-trivially on $K[G/U]$, so it acts non-trivially on K if $K[G/U]$ is isomorphic to a direct sum of copies of K. □

Fix now a field k and a permutation group G. As it follows from Theorem 2.1, for any field K endowed with a smooth G-action the object K is not a generator of the category of smooth K-semilinear representations of G if G is not precompact. One may ask, however, the following:

Question 2.2. Does there exist a G-field extension $K|k$ (a 'smooth period field' over k) such that K is a *cogenerator* of the category of smooth K-semilinear representations of G and $K^G = k$?

Remark 2.3. Let $K' \subseteq K$ be a G-invariant subfield and U be an open subgroup of G. Then the forgetful functor from the category of smooth K-semilinear representations of G to the category of smooth K'-semilinear representations of U (i) admits a left adjoint $K\langle G\rangle \otimes_{K'\langle U\rangle} (-)$, (ii) preserves cogenerators and injectives.

Lemma 2.4 provides an evident necessary condition on the field K in question 2.2, while Lemma 2.5 constructs fields satisfying the condition of Lemma 2.4 for arbitrary permutation groups G.

Lemma 2.4. *Let G be a permutation group and K be a field endowed with a smooth G-action. Suppose that K is a cogenerator of the category of smooth K-semilinear representations of G. Then the G-action on K is faithful and for any open subgroup $U \subseteq G$ one has $U = G_{K^U}$.*

Proof. For any open subgroup $U \subseteq G$ the $K\langle G\rangle$-module $K[G/U]$ can be embedded into a product of copies of K, so stabilizer U of the element of [1] is an intersection of stabilizers of some elements of K, i.e., the pointwise stabilizer of a subfield of K. □

For a field k and a set S, denote by $k(S)$ the field of rational functions over k in the variables enumerated by the set S.

The next result is well-known for profinite groups G.

Lemma 2.5. *For any field k and any permutation group G there exists a field extension $K|k$ endowed with a smooth G-action such that for any open subgroup $U \subseteq G$ one has $U = G_{K^U}$.*

Proof. Let $\{H_i \mid i \in I\}$ be such a set of open subgroups of G that any open subgroup of G is conjugated to some H_i. Set $S := \coprod_{i \in I} G/H_i$ and $K := k(S)$. For any subgroup $U \subseteq G$ one has $G_{K^U} \supseteq U$ and $S^U \subset K^U$. On the other hand, any open subgroup $U \subseteq G$ coincides with hH_ih^{-1} for some $i \in I$ and $h \in G$, so $[h]_i := h \bmod H_i \in S^U$, and therefore, if $g \in G_{K^U}$, then $g[h]_i = [h]_i$, which is possible if and only if $gUh = Uh$, i.e., if and only if $g \in U$. □

2.1. Hilbert's theorem 90 for symmetric groups

Denote by $\mathfrak{S}_\Psi$ the group of all permutations of an infinite set Ψ. In this section we give an affirmative answer to the question 2.2 for $G = \mathfrak{S}_\Psi$.

As $\mathfrak{S}_\Psi$ is topologically simple, any non-trivial smooth $\mathfrak{S}_\Psi$-action on a field K is faithful and the fixed field $k := K^{\mathfrak{S}_\Psi}$ is algebraically closed in K.

The group $\mathfrak{S}_\Psi$ acts on the field of rational functions $k(\Psi)$ by permuting the variables.

For each $d \in \mathbb{Z}$, let $V_d \subseteq k(\Psi)$ be the subset of homogeneous rational functions of degree d, so V_0 is an $\mathfrak{S}_\Psi$-invariant subfield and $V_d \subseteq k(\Psi)$ is an $\mathfrak{S}_\Psi$-invariant one-dimensional V_0-vector subspace.

Theorem 2.6 ([8])**.** *Let Ψ be a set.*

(1) *Let k be a field and let $K \subseteq k(\Psi)$ be an $\mathfrak{S}_\Psi$-invariant subfield. Then the object $k(\Psi)$ is an injective cogenerator of the category of smooth K-semilinear representations of $\mathfrak{S}_\Psi$.*

In particular, (i) *any smooth K-semilinear representation of $\mathfrak{S}_\Psi$ can be embedded into a direct product of copies of $k(\Psi)$*; (ii) *any smooth $k(\Psi)$-semilinear representation of $\mathfrak{S}_\Psi$ of finite length is isomorphic to a direct sum of copies of $k(\Psi)$.*

(2) *The objects V_d for $d \in \mathbb{Z}$ form a system of injective cogenerators of the category of smooth V_0-semilinear representations of $\mathfrak{S}_\Psi$, i.e., any smooth V_0-semilinear representation V of $\mathfrak{S}_\Psi$ can be embedded into a direct product of cartesian powers of V_d. In particular, any smooth V_0-semilinear representation of $\mathfrak{S}_\Psi$ of finite length is isomorphic to $\bigoplus_{d \in \mathbb{Z}} V_d^{m(d)}$ for a unique function $m : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ with finite support.*

(3) *Let $K \subset k(\Psi)$ be the subfield generated over k by the rational functions $x - y$ for all $x, y \in \Psi$, so the group $\mathfrak{S}_\Psi$ acts naturally on the fields $k(\Psi)$ and K. Then for each integer $N \geq 1$ there exists a unique isomorphism class of smooth K-semilinear indecomposable representations of $\mathfrak{S}_\Psi$ of length N. In particular, any smooth K-semilinear irreducible representation of $\mathfrak{S}_\Psi$ is isomorphic to K.*

Remark 2.7. In fact, 'smooth period fields' (from Question 2.2) are not unique. For instance, Theorem 2.6(1) is valid when the field $k(\Psi)$ is replaced by the following, more general, field extension $F_\Psi|k$: for any field extension $F|k$ of at most countable transcendence degree such that k is algebraically closed in F, let $F_\Psi = F_{k,\Psi}$ be the field of fractions of the inductive limit k-algebra

$\bigotimes_{k,\ i\in\Psi} F := \varinjlim_{I\subset\Psi} \bigotimes_{k,\ i\in I} F$ of the tensor powers $\bigotimes_{k,\ i\in I} F$ over k for all finite subsets $I\subset\Psi$, consisting of finite linear combinations of tensor products of elements in F, almost all equal to 1. The group $\mathfrak{S}_\Psi$ acts on $\bigotimes_{k,\ i\in\Psi} F$ by permuting the tensor factors, and thus, it acts on the field F_Ψ. The field F_Ψ coincides with $k(\Psi)$, if $F = k(x)$ is the field of rational functions in one variable.

Proof of Theorem 2.6 (*sketch*). Proof of part (1) splits into several steps. The goal of the first three steps is to show that any smooth simple $L\langle\mathfrak{S}_\Psi\rangle$-module is isomorphic to L, where $L := k(\Psi)$.

Step 1. An explicit description of restriction of any smooth finitely generated $L\langle\mathfrak{S}_\Psi\rangle$-module M to $L\langle\mathfrak{S}_{\Psi|J}\rangle$ for a sufficiently big finite subset $J\subset\Psi$: $M\cong\bigoplus_{s=0}^N L[\binom{\Psi\smallsetminus J}{s}]^{\kappa_s}$ for some integer $N,\kappa_0,\dots,\kappa_N\geq 0$. Here $\binom{\Psi}{s}$ denotes the set of all subsets of Ψ of cardinality s. Proof proceeds by induction on (N,m), where M is dominated by a $L\langle\mathfrak{S}_\Psi\rangle$-module $L[\binom{\Psi}{N}]^m\oplus\bigoplus_{s=0}^{N-1} L[\binom{\Psi}{s}]^{m_s}$ for some $N,m,m_s\geq 0$.

Step 2. Sending M to $M' := \varinjlim_{I\subset\Psi\smallsetminus J} M^{\mathfrak{S}_{\Psi|I}}$ gives rise to an equivalence of categories of smooth $L\langle\mathfrak{S}_\Psi\rangle$-modules and smooth $L'\langle\mathfrak{S}_{\Psi|J}\rangle$-modules, where $L' = k(\Psi\smallsetminus J)$. In particular, any smooth simple $L\langle\mathfrak{S}_\Psi\rangle$-module M admits a simple $L'\langle\mathfrak{S}_{\Psi|J}\rangle$-submodule M'. This is based on identification of the smooth $\mathfrak{S}_\Psi$-sets with the sheaves on a category of finite subsets. Then M and M' correspond to the restrictions of a sheaf to finite subsets of Ψ and of $\Psi\smallsetminus J$, respectively.

Step 3. It not hard to show that there are no simple $L'\langle\mathfrak{S}_{\Psi|J}\rangle$-submodules in $L[\binom{\Psi\smallsetminus J}{s}]$ for $s>0$, so $M\cong L^m$, and thus, M' embeds into L. Specializing the variables in J of a generator $Q = Q(J)\in L^\times = L'(J)^\times$ of M', one gets an isomorphism $M'\xrightarrow{\sim} L'$, so M is isomorphic to L.

Step 4. L is injective, i.e., any essential extension E of L coincides with L. Indeed, we may assume that E is cyclic. By Step 1, there is a finite subset $J\subset\Psi$ such that the $L\langle\mathfrak{S}_{\Psi|J}\rangle$-module E is isomorphic to $\bigoplus_{s=0}^N L[\binom{\Psi\smallsetminus J}{s}]^{\kappa_s}$ for some integer $N,\kappa_0,\dots,\kappa_N\geq 0$. According to Step 2, E' is a cyclic $L'\langle\mathfrak{S}_{\Psi|J}\rangle$-submodule of $\bigoplus_{s=0}^N L[\binom{\Psi\smallsetminus J}{s}]^{\kappa_s}$ which is an essential extension of L'. The natural projection defines a morphism of $L'\langle\mathfrak{S}_{\Psi|J}\rangle$-modules $E'\to L^{\kappa_0}$ injective on $L'\subseteq E'$. Composing it with any L-linear morphism $L^{\kappa_0}\to L$, which is L'-rational and identical on the image of L', we get a morphism of $L'\langle\mathfrak{S}_{\Psi|J}\rangle$-modules $\pi : E'\to L$. It remains to show that L' is a direct summand of L. This can be seen by sending elements of J to Laurent series in $k((t))$ algebraically independent over k, so L becomes embedded into $L'((t))$ and the constant term of the Laurent series is a desired splitting.

Step 5. We have to show that for any smooth simple $L\langle\mathfrak{S}_\Psi\rangle$-module V and any nonzero $v\in V$ there is a morphism $V\to L$ non-vanishing at v. The $L\langle\mathfrak{S}_\Psi\rangle$-submodule $\langle v\rangle$ of V generated by v admits a simple quotient, which is just shown to be isomorphic to L, i.e., there is a nonzero morphism $\varphi : \langle v\rangle \to L$. As L is injective, φ extends to V.

(2) By part (1), $k(\Psi)$ is an injective cogenerator of the category of smooth $V_0\langle\mathfrak{S}_\Psi\rangle$-modules. To show that the subobjects $V_d \subset k(\Psi)$ form a system of injective generators, it suffices to verify that they are direct summands of $k(\Psi)$ and that $k(\Psi)$ embeds into $\prod_{d\in\mathbb{Z}} V_d$.

There is a unique discrete valuation $v : k(\Psi)^\times \to \mathbb{Z}$ trivial on $V_0^\times$ and such that $v(x) = -1$ for some (equivalently, any) $x \in \Psi$. The valuation v is $\mathfrak{S}_\Psi$-invariant and completion of $k(\Psi)$ with respect to v is isomorphic to the field of Laurent series $V_0((x^{-1})) = \varinjlim_n \prod_{d\le n} V_0 \cdot x^d = \varinjlim_n \prod_{d\le n} V_d \subset \prod_{d\in\mathbb{Z}} V_d$, so for each $d \in \mathbb{Z}$ there is a morphism of $V_0\langle\mathfrak{S}_\Psi\rangle$-modules $k(\Psi) \to V_d$ splitting the inclusion $V_d \subset k(\Psi)$. This implies that all V_d are direct summands of $k(\Psi)$, and thus, they are injective.

(3) By part (1), any smooth simple $K\langle\mathfrak{S}_\Psi\rangle$-module can be embedded into $k(\Psi)$. Let us show that any simple $K\langle\mathfrak{S}_\Psi\rangle$-submodule $V \subset k(\Psi)$ coincides with K.

Fix some $x \in \Psi$. One has $k(\Psi) = K[x] \oplus \bigoplus_R V_R$, where R runs over the $\mathfrak{S}_\Psi$-orbits of non-constant irreducible monic polynomials in $K[x]$ and V_R is the K-linear envelope of $P(x)/Q^m$ for all $Q \in R$, integer $m \ge 1$ and $P \in K[x]$ with $\deg P < m \deg R$. Clearly, this decomposition is independent of x. It is straightforward that the only $K\langle\mathfrak{S}_\Psi\rangle$-submodule $K[x]$ of length N consists of all polynomials in x of minimal degree $< N$.

It is not hard to show that there are no simple submodules in V_R for any R. Thus, any smooth $K\langle\mathfrak{S}_\Psi\rangle$-module V of finite length is embeds into a finite cartesian power of $K[x]$. Then one concludes that for any integer $N \ge 1$ the unique isomorphism class of smooth K-semilinear indecomposable representations of $\mathfrak{S}_\Psi$ of length N is presented by $\bigoplus_{j=0}^{N-1} x^j K \subset k(\Psi)$ for any $x \in \Psi$. □

Remark 2.8. Parts (1) and (2) of Theorem 2.6 provide an example of a permutation group G, an open subgroup $U \subseteq G$ and a field K endowed with a smooth G-action such that K is a cogenerator of the category of smooth K-semilinear representations of U, but K is not a cogenerator of the category of smooth K-semilinear representations of G, if we take $G = \mathfrak{S}_\Phi$, U the stabilizer of an element of Φ and K as in Theorem 2.6(2).

3. Admissible semilinear representations of automorphism groups of universal domains

For any field extension $K|k$ denote by $G_{K|k}$ its automorphism group. Now let k be a field of characteristic zero and F be an algebraically closed field extension of countable transcendence degree.

An F-semilinear representation V of $G_{F|k}$ is called *admissible* if $\dim_{F^U} V^U < \infty$ for any open subgroup $U \subseteq G_{F|k}$.

The following result generalizes to the case of arbitrary k a consequence of [7, Theorem 4.10] in the case of $k = \overline{\mathbb{Q}}$:

Theorem 3.1. *Any irreducible F-semilinear admissible representation of $G_{F|k}$ is a direct summand of the tensor algebra $\bigotimes_F^\bullet \Omega^1_{F|k}$.*

To certain extent, admissible semilinear representations of the automorphism group $G_{F|k}$ of a universal domain F over k are analogous to such notions in algebraic geometry as invariant sheaves of M. Kashiwara, [4], and functors of D. B. A. Epstein, [2] and [3] (originating from J. W. Milnor's problem on "natural" vector bundles on differentiable manifolds).

3.1. Strategy of the proof of Theorem 3.1

Proof is a modified version of [7].

To describe the admissible semilinear representations of the automorphism group $G_{F|k}$ of a universal domain F over k, one studies first their 'restrictions' to projective groups G_n, considered as subquotients of the group $G_{F|k}$, for all $n \geq 1$. More precisely, one studies the full subcategory $\mathfrak{S}\mathfrak{L}_n$ of the category of K_n-semilinear representations of G_n, whose objects are restrictions of non-degenerate finite-dimensional K_n-semilinear representations of the semigroup of dominant rational self-maps of $\mathbb{P}^n_k$, generated by G_n and the semigroup $\mathrm{End}_{\mathrm{dom}}(Y_n|k) \cong \mathrm{Mat}^{\det\neq 0}_{n\times n}\mathbb{Z} \ltimes T_n$ of dominant endomorphisms of Y_n, to the subgroup G_n. (The reason to deal with such a semigroup is that $\mathrm{End}_{\mathrm{dom}}(Y_n|k)$ is a subquotient of the group $G_{F|k}$, and thus, $V_n := V^{G_{F|K_n}} \in \mathfrak{S}\mathfrak{L}_n$ for any admissible semilinear representation V of $G_{F|k}$.)

Let $\mathfrak{S}\mathfrak{L}^u_n$ be the category of the finite-dimensional K_n-semilinear representations of the group G_n, whose restrictions to the maximal torus T_n in G_n is isomorphic to $K_n \otimes_k W$ for a unipotent representation W of the torus T_n (where T_n is considered as a discrete group). This means that W is a finite-dimensional k-vector space, admitting a T_n-invariant filtration with the trivial T_n-action on the graded quotients of that filtration. Clearly, $\mathfrak{S}\mathfrak{L}^u_n$ is an abelian and neutral tannakian category, while $H^0(T_n^{\mathrm{tors}}, -) : \mathfrak{S}\mathfrak{L}^u_n \longrightarrow \mathcal{V}ec_k$ is a fibre functor.

One says that a sheaf $\mathcal{V}$ on $\mathbb{P}^n_k$ is a G_n-*sheaf* if it is endowed with a G_n-*structure*, i.e., with a compatible collection of isomorphisms (satisfying the chain rule) $\alpha_g : \mathcal{V} \xrightarrow{\sim} g^*\mathcal{V}$ for each $g \in G_n$, i.e., such that $\alpha_{hg} = g^*\alpha_h \circ \alpha_g$ for all $g, h \in G_n$.

One proves that for any integer $n \geq 2$ one has an inclusion $\mathfrak{S}\mathfrak{L}_n \subseteq \mathfrak{S}\mathfrak{L}^u_n$, and construct a fully faithful tensor functor

$$\mathfrak{S}\mathfrak{L}^u_n \xrightarrow{\mathcal{S}_n} \{\text{coherent } G_n\text{-sheaves on } \mathbb{P}^n_k\}, \tag{1}$$

whose composition with the generic fibre functor is the identical full embedding of $\mathfrak{S}\mathfrak{L}^u_n$ into {finite-dimensional semilinear K_n-representations of G_n} as a Serre subcategory.

The next step is to identify simple objects of $\mathfrak{SL}_n^u$ with generic fibres of G_n-equivariant coherent sheaves on $\mathbb{P}_k^n$. In the proofs one uses a description of 'abstract' homomorphisms to reductive groups with Zariski dense images, due to A. Borel and J. Tits ([1]).

It is well-known that any simple G_n-equivariant coherent sheaf on $\mathbb{P}_k^n$ is a direct summand of the sheaf $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}((\Omega^n_{\mathbb{P}_k^n|k})^{\otimes^i_{\mathcal{O}_{\mathbb{P}_k^n}}}, \bigotimes^{\bullet}_{\mathcal{O}_{\mathbb{P}_k^n}}\Omega^1_{\mathbb{P}_k^n|k})$ for an appropriate $i \geq 0$.

In [7, Theorem 2.4] one shows that certain simple objects of $\mathfrak{SL}_n^u$ do not occur as subquotients of admissible semilinear $G_{F|k}$-representation:

Theorem 3.2. *For any F-semilinear admissible $G_{F|k}$-representation V any irreducible subquotient of the K_n-semilinear $\mathrm{PGL}_{n+1}k$-representation $V_n := V^{G_{F|K_n}}$ is a direct summand of $\bigotimes^{\bullet}_{K_n}\Omega^1_{K_n|k}$.*

One shows that the category $\mathcal{A}$ of admissible F-semilinear representations of $G_{F|k}$ is abelian, that the functor $\mathcal{A} \longrightarrow \mathfrak{SL}_n^u$, $V \mapsto V_n$, is exact, and the groups Ext^1 between the simple object of the category $\mathfrak{SL}_u^n$ are calculated. For example, there exists a non-trivial extension of $\Omega^1_{K_n|k}$ by K_n if and only if k is transcendental over $\mathbb{Q}$, and in that case they are parametrized by k-hyperplanes $\mathcal{H} \subset \Omega^1_k$: $0 \longrightarrow K_n \xrightarrow{v\cdot} \Omega^1_{K_n}/\mathcal{H} \otimes_k K_n \longrightarrow \Omega^1_{K_n|k} \longrightarrow 0$, where $v \in \Omega^1_k/\mathcal{H} \cong k$ is a nonzero element. (Note, that the coherent G_n-sheaves $\mathcal{S}_n(\Omega^1_{K_n}/\mathcal{H} \otimes_k K_n)$ are not equivariant.) The calculation of Ext^1 uses a description of certain 'abstract' homomorphisms to the groups with commutative unipotent radical, due to L. Lifschitz and A. Rapinchuk ([5]). This implies, in particular, that if a $V \in \mathcal{A}$ admits no subobjects isomorphic to F, then any simple subquotient of $V_n \in \mathfrak{SL}_n^u$ is a direct summand of $\bigotimes^{\geq 1}_{K_n}\Omega^1_{K_n|k}$.

After establishing the principal structural results concerning the category $\mathcal{A}$, the category $\mathcal{A}$ is identified with the category of 'coherent' sheaves in the smooth topology, when $k = \overline{\mathbb{Q}}$.

Namely, for any object V of $\mathcal{A}$ and any smooth k-variety Y embeddings of generic points of Y into F gives rise to a locally free coherent sheaf $\mathcal{V}_Y$ on Y. Any dominant morphism $X \xrightarrow{\pi} Y$ of smooth k-varieties induces an embedding of coherent sheaves $\pi^*\mathcal{V}_Y \hookrightarrow \mathcal{V}_X$, which is an isomorphism if π is étale.

This is where an equivalence $\mathcal{S} : \mathcal{A} \xrightarrow{\sim} \{$'coherent' sheaves in the smooth topology$\}$, $V \longmapsto (Y \mapsto \mathcal{V}_Y(Y))$, comes from. Slightly more generally, 'coherent' sheaves are contained in the category $\mathcal{F}l$ of flat 'quasicoherent' sheaves in the smooth topology. For any flat 'quasicoherent' sheaf $\mathcal{V}$ in the smooth topology the space $\Gamma(Y, \mathcal{V}_Y)$ is a birational invariant of a proper Y. This gives rise to a left exact functor Γ from $\mathcal{F}l$ to the category of smooth k-representations of $G_{F|k}$, given by $V \mapsto \varinjlim \Gamma(Y, \mathcal{V}_Y)$, where Y runs over the smooth proper models of the subfields in F of finite type over k.

The functor $\Gamma \circ \mathcal{S}$ is faithful, since $\Gamma(Y', \mathcal{V}_{Y'})$ generates the (general fibre of the) sheaf $\mathcal{V}_{Y'}$ for appropriate finite covers Y' of Y, if $\mathcal{V}$ is 'coherent'. However, it is not full and the objects in its image are 'very' reducible. If $\Gamma(Y,\mathcal{V}_Y)$ has Galois descent property, then $\Gamma(V)$ is admissible. However, the Galois descent property does not hold in general.

3.1.1. *Inclusion* $\mathfrak{SL}_n \subseteq \mathfrak{SL}_n^u$. Let $k_\infty := \cup_{j\ge 0}k_j$, where $k_0=\mathbb{Q}$ and k_{j+1} is generated over $\mathbb{Q}$ by all roots in k of all elements of the field k_j.

The first step in the proof of the inclusion $\mathfrak{SL}_n \subseteq \mathfrak{SL}_n^u$, is to show that restriction of (an extension to the corresponding semigroup of) $V\in\mathfrak{SL}_n$ to $\mathbb{Z}^n_{\neq 0}\ltimes T_n$, where $\mathbb{Z}^n_{\neq 0}$ is the 'maximal split torus' in $\mathrm{Mat}_{n\times n}^{\det\neq 0}\mathbb{Z}$, comes from a k-linear representation by extending coefficients to K_n. Some analytic arguments (using the existence of all ℓ-primary roots of unity in k) reduce the problem to a local result, stating that any finite-dimensional $k((t))$-semilinear representation of the semigroup $\mathbb{N}$ (acting on the field of formal Laurent series $k((t))$ by raising the indeterminate to the corresponding powers $p: t\mapsto t^p$) is obtained from a k-linear representation by extending coefficients to $k((t))$.

This implies that $V\longmapsto V^{T_n^{\mathrm{tors}}}$ is a 'fibre functor' to the category of unipotent k-representations of T_n, i.e., $V=V^{T_n^{\mathrm{tors}}}\otimes_k K_n$, which shows the inclusion $\mathfrak{SL}_n\subseteq\mathfrak{SL}_n^u$.

3.1.2. *The functor* $\mathcal{S}_n$. In order to construct the functor $\mathcal{S}_n$ from (1), one has to check that $\mathcal{S}_n(V)|_{Y_n} := H^0(T_n^{\mathrm{tors}},V)\otimes_k\mathcal{O}_{Y_n}\subset V$ happily glue together, when Y_n varies.

The principal step is to show that for any hyperplane $H\subset\mathbb{P}^n_k\smallsetminus Y_n$ (i.e., stabilized by the torus T_n) and any k_∞-lattice U_0 in the unipotent radical U of the stabilizer P of the hyperplane H, the functor $H^0(U_0,-): V\longmapsto V^{U_0}$ from $\mathfrak{SL}_n^u$, this time to the category of unipotent k-representations of U, is a 'fibre functor' as well. Then the $\mathcal{O}_{\mathbb{P}^n_k}(\mathbb{P}^n_k\smallsetminus H)$-lattice $\mathcal{V}_H$ in V, spanned by V^{U_0}, is P-invariant and independent of U_0.

Localizing this lattice and varying H, we get a coherent G_n-subsheaf $\mathcal{V}$ of the constant sheaf V on $\mathbb{P}^n_k$ such that $\mathcal{V}|_{Y_n}=V^{T_n^{\mathrm{tors}}}\otimes_k\mathcal{O}_{Y_n}=V^{U_0}\otimes_k\mathcal{O}_{Y_n}$ and $\Gamma(\mathbb{P}^n_k\smallsetminus H,\mathcal{V})=\mathcal{V}_H$.

One checks that if $k=k_\infty$, then the action of G_n on the total space E of the vector bundle corresponding to the sheaf $\mathcal{V}$ comes from a morphism of k-varieties $G_n\times_k E\longrightarrow E$, so that the functor $\mathcal{S}_n$ factors through the category of G_n-equivariant coherent sheaves on $\mathbb{P}^n_k$, which is equivalent to the category of rational representations (over k) of finite degree of the stabilizer of a point of the space $\mathbb{P}^n_k$.

3.1.3. Fix a transcendence base $x_1,x_2,x_3,\dots$ of $F|k$ and set $K_j{:=}k(x_1,\dots,x_j)$ for each integer $j\ge 0$.

Fix some integer $m\ge 0$ and a rational irreducible representation $B\cong S_k^\lambda(k^m)$ of $\mathrm{GL}_m k := \mathrm{Aff}_m/(\mathrm{Aff}_m)_u$. Let $W^\circ := \{K_m \overset{/k}{\hookrightarrow} F\}/(\mathrm{Aff}_m)_u$ be considered as a left $G_{F|k}$-set and a right $\mathrm{GL}_m k$-set. As any element of W° is determined by its restriction to the k-vector space $(K_m/k)^{(\mathrm{Aff}_m)_u}$ (and also by its restriction to the basis $\{\overline{x_1}, \dots, \overline{x_m}\}$ of $(K_m/k)^{(\mathrm{Aff}_m)_u})$, one can consider W° as a subset of $W := \mathrm{Hom}_k((K_m/k)^{(\mathrm{Aff}_m)_u}, F/k) \cong (F/k)^m$ consisting of all elements of W containing m algebraically independent elements in the image (of all m-tuples with entries algebraically independent over k).

Let $(y_1, \dots, y_m) \mapsto [y_1, \dots, y_m]$ be the map $W \longrightarrow \{0\} \cup W^\circ$ identical on W° and sending $W \setminus W^\circ$ to 0. Then $[\mu y_1, \dots, \mu y_m] \otimes b = \mu^{|\lambda|}[y_1, \dots, y_m] \otimes b$ in U for any $\mu \in k$. If $y_1, \dots, y_m$ belong to the k-linear envelope of $x_1, \dots, x_M$ for some integer $M \geq 1$, then $[y_1, \dots, y_m] \otimes b \in U_M^{(\mathrm{Aff}_M)_u}$ is a weight $|\lambda|$ eigenvector of the centre of $\mathrm{GL}_M k$.

Let $\mathfrak{S}\mathfrak{L}_n^u$ be the category of finite-dimensional semilinear representations of $\mathrm{PGL}_{n+1}k$ over K_n whose restrictions to the maximal torus T_n in $\mathrm{PGL}_{n+1}k$ are isomorphic to $K_n \otimes_k W$ for unipotent representations W of T_n (where T_n is considered as a discrete group).

Let $\mathfrak{S}\mathfrak{L}_n^{eq} \subset \mathfrak{S}\mathfrak{L}_n^u$ be the full subcategory consisting of generic fibres of coherent $\mathrm{PGL}_{n+1}k$-*equivariant* sheaves on $\mathbb{P}_k^n$. It is equivalent to the category of rational representations of the stabilizer of a point of $\mathbb{P}_k^n$.

According to [7, §1], the category $\mathfrak{S}\mathfrak{L}_n^u$ is abelian and its simple objects are simple objects of $\mathfrak{S}\mathfrak{L}_n^{eq}$, i.e., direct summands of $\mathrm{Hom}_{K_n}((\Omega^n_{K_n|k})^{\otimes M}, \bigotimes^\bullet_{K_n} \Omega^1_{K_n|k})$ for appropriate integer $M \geq 0$.

Let Q_n be a k-vector space and $X_0, \dots, X_n$ be a base of the space $Q_n^\vee$ of linear functionals with X_i/X_0 and x_i identified for $1 \leq i \leq n$. Let Φ_{-1} be the (one-dimensional) K_n-vector subspace of homogeneous elements of degree -1 in the field of the rational functions on Q_n.

Lemma 3.3 ([7], Lemma 3.10)**.** *Let $n \geq 2$. Suppose that $\mathrm{Ext}^1_{\mathfrak{S}\mathfrak{L}_n^u}(K_n, V_\circ) \neq 0$ for some irreducible object $V_\circ$ of $\mathfrak{S}\mathfrak{L}_n^u$. Then either $V_\circ \cong \Omega^1_{K_n|k}$, or $V_\circ \cong \mathrm{Der}(K_n|k)$. One has $\mathrm{Ext}^1_{\mathfrak{S}\mathfrak{L}_n^u}(K_n, \Omega^1_{K_n|k}) = k$* (*generated by the class*

$$(2) \quad 0 \to \Omega^1_{K_n|k} \xrightarrow{\iota} Q_n^\vee \otimes_k \Phi_{-1} \xrightarrow{\times} K_n \to 0, \quad \iota : f\frac{l_0}{l_1}d\frac{l_1}{l_0} \mapsto l_1 \otimes \frac{f}{l_1} - l_0 \otimes \frac{f}{l_0}$$

of the generic fibre of the Euler extension) *and $\mathrm{Ext}^1_{\mathfrak{S}\mathfrak{L}_n^u}(K_n, \mathrm{Der}(K_n|k)) = \mathrm{Der}(k)$.*

Lemma 3.4. *For any $V \in \mathfrak{S}\mathfrak{L}_M^u$ the representation $V^{(\mathrm{Aff}_M)_u}$ of $\mathrm{GL}_M k$ is rational* (*and thus, semisimple*)*.*

Sketch of the proof. By induction argument, one can reduce the problem to the case of an extension between simple objects. In view of Lemma 3.3, the problem reduces further to the case of extension (2) and the extension $0 \to K_M \xrightarrow{\cdot\eta} \Omega^1_{K_M}/K_M \cdot \Lambda \to \Omega^1_{K_M|k} \to 0$ for a k-hyperplane $\Lambda \subset \Omega^1_k$ and a nonzero $\eta \in \Omega^1_k/\Lambda$. It is clear that $(Q_M^\vee \otimes_k \Phi_{-1})^{(\mathrm{Aff}_M)_u} = \{l \otimes \frac{1}{l_0} - l_0 \otimes \frac{l}{l_0^2} \mid l \in Q_M^\vee\} \oplus k \cdot l_0 \otimes \frac{1}{l_0}$ and $(\Omega^1_{K_M}/K_M \cdot \Lambda)^{(\mathrm{Aff}_M)_u} = k \cdot \eta$. $\square$

Lemma 3.5. *Let $A \subset B$ be a pair of finite-dimensional k-vector spaces and λ be a finite partition* (*a Young diagram*)*. Then any nonzero morphism $\varphi_{A,B}$*

from $k[\mathrm{Aut}_k(B)]\otimes_{k[\mathrm{Aut}_k(A,B)]}S^\lambda A$ *to a rational representation of* $\mathrm{Aut}_k(B)$ *is a composition of the natural morphism* $\psi : k[\mathrm{Aut}_k(B)]\otimes_{k[\mathrm{Aut}_k(A,B)]}S^\lambda A\to S^\lambda B$ *with an injection.*

Proof. Any rational representation of $\mathrm{Aut}_k(B)$ is a direct sum of representations of type $S^\mu B\otimes_k(\det B)^{\otimes(-s)}$ for partitions μ and integer $s\geq 0$. One has

$$\mathrm{Hom}_{k[\mathrm{Aut}_k(A,B)]}(A^{\otimes_k m},B^{\otimes_k n})=\mathrm{Hom}_{k[\mathrm{Aut}_k(A)]}(A^{\otimes_k m},A^{\otimes_k n})\subseteq\mathrm{Hom}_{k[k^\times]}(A^{\otimes_k m},A^{\otimes_k n})$$

and this space vanishes, unless $m=n$, so

$$\mathrm{Hom}_{k[\mathrm{Aut}_k(A,B)]}(S^\lambda A,S^\mu B)=\mathrm{Hom}_{k[\mathrm{Aut}_k(A)]}(S^\lambda A,S^\mu A)=\begin{cases}k, & \text{if } \mu=\lambda \text{ and } S^\lambda A\neq 0,\\ 0, & \text{if } \mu\neq\lambda \text{ or } S^\lambda A=0,\end{cases}$$

which means (by adjunction) that $\varphi_{A,B}$ factors through ψ. □

Let $W^{M\circ}\subset W^M$ be the subset consisting of M-tuples $(y_1,\dots,y_M)$ such that $\sum_{i\in I}y_i\in W^\circ$ for any non-empty subset $I\subseteq\{1,\dots,M\}$.

Let $k[W^{M\circ}]\longrightarrow k[W^\circ]\otimes_{k[k^\times]}k(M)$ be the k-linear map sending $(y_1,\dots,y_M)$ to

$$\langle y_1,\dots,y_M\rangle:=\sum_{I\subseteq\{1,\dots,M\}}(-1)^{\#I}[\sum_{i\in I}y_i]\in k[W^\circ]\otimes_{k[k^\times]}k(M).$$

Here $k(M)$ denotes a one-dimensional k-vector space with $k^\times$-action by M-th powers. As $(y,\dots,y)$ is sent to

$$\sum_{j\geq 0}(-1)^j\binom{M}{j}j^M[y]=(t\frac{d}{dt})^M(1-t)^M|_{t=1}\cdot[y]=(-1)^M M!\cdot[y],$$

it is surjective. Clearly, $\langle y_1,\dots,y_M\rangle=\langle y_{\theta(1)},\dots,y_{\theta(M)}\rangle$ for any permutation $\theta\in\mathfrak{S}_M$. Let $\tilde{U}:=F[W^{|\lambda|\circ}]\longrightarrow U$ be the F-linear surjection sending $(y_1,\dots,y_{|\lambda|})$ to $\langle y_1,\dots,y_{|\lambda|}\rangle\otimes b$.

Lemma 3.6 ([7], Lemma 4.3). *Let the k-linear map* $\alpha : k[W^\circ]\longrightarrow\bigotimes_k^M W$ *be given by* $[w]\mapsto w^{\otimes M}$. *Then* α *factors through* $k[W^\circ]\otimes_{k[k^\times]}k(M)$ *and* $\langle y_1,\dots,y_M\rangle\mapsto(-1)^M\sum_{\theta\in\mathfrak{S}_M}y_{\theta(1)}\otimes\cdots\otimes y_{\theta(M)}$ *if* $(y_1,\dots,y_M)\in W^{M\circ}$.

Lemma 3.7 ([7], Lemma 4.4). *If* $M=|\lambda|$, $\mu\in k$, $y_0,y_1,y_0+y_1\in W^\circ$ *and all coordinates of* $t_2,\dots,t_M\in W$ *are algebraically independent over* $k(y_0,y_1)$, *then*

$$\langle y_0+y_1,t_2,\dots,t_M\rangle\otimes b\equiv\langle y_0,t_2,\dots,t_M\rangle\otimes b+\langle y_1,t_2,\dots,t_M\rangle\otimes b \bmod \ker\varphi,$$

and $\langle\mu y_1,t_2,\dots,t_M\rangle\otimes b\equiv\mu\langle y_1,t_2,\dots,t_M\rangle\otimes b \bmod \ker\varphi$.

Lemma 3.8 ([7], Lemma 4.5)**.** *Let* $(y_1, \ldots, y_M) \in \{0\} \times W^{(M-1)\circ} \cup W^{M\circ}$ *and let the coordinates of* $t_{ij} \in W^\circ$ *be algebraically independent over* $k(y_1, \ldots, y_M)$*, where* $1 \leq i \leq M$ *and* $2 \leq j \leq M$*. Set* $[0] := 0$ *and* $\langle 0, y_2, \ldots, y_M \rangle := 0$*. Then*

$$(3)\quad \langle y_1, \ldots, y_M \rangle \otimes b$$
$$\equiv \sum_{J \subseteq \{2,\ldots,M\}} (-1)^{\#J} \langle y_1, \sum_{s \in \{1\} \cup J} t_{s2}, \ldots, \sum_{s \in \{1\} \cup J} t_{sM} \rangle \otimes b$$
$$- \sum_{\emptyset \neq I \subseteq \{2,\ldots,M\}} (-1)^{\#I} \langle y_1, y_2 + \sum_{i \in I} t_{2i}, \ldots, y_M + \sum_{i \in I} t_{Mi} \rangle \otimes b \bmod \ker \varphi.$$

Lemma 3.9 ([7], Lemma 4.6)**.** *If* $M = |\lambda|$, $\mu \in k$ *and*

$$(z_j, y_2, \ldots, y_M), (\sum_{i=1}^{N} z_i, y_2, \ldots, y_M), (\mu z_1, y_2, \ldots, y_M) \in W^{M\circ}$$

for all $1 \leq j \leq N$*, then*

$$(4)\qquad \langle \sum_{j=1}^{N} z_j, y_2, \ldots, y_M \rangle \otimes b \equiv \sum_{j=1}^{N} \langle z_j, y_2, \ldots, y_M \rangle \otimes b \bmod \ker \varphi,$$

and $\langle \mu z_1, y_2, \ldots, y_M \rangle \otimes b \equiv \mu \langle z_1, y_2, \ldots, y_M \rangle \otimes b \bmod \ker \varphi$*.*

Lemma 3.10 ([7], Lemma 4.7)**.** *The* k*-linear map* $k[W^{M\circ}] \longrightarrow \bigotimes_k^M W$*, given by* $[(y_1, \ldots, y_M)] \mapsto y_1 \otimes \cdots \otimes y_M$*, is surjective and its kernel is spanned over* k *by* $[(y_0, \ldots, y_{j-1} + y_j, \ldots, y_M)] - [(y_0, \ldots, \widehat{y_{j-1}}, \ldots, y_M)] - [(y_0, \ldots, \widehat{y_j}, \ldots, y_M)]$ *and* $\mu[(y_1, \ldots, y_M)] - [(y_1, \ldots, \mu y_j, \ldots, y_M)]$ *for all* $y_0, \ldots, y_M \in W^\circ$ *and all* $\mu \in k^\times$*.*

Let $\mathfrak{m} = \mathfrak{m}_{F|k}$ be the kernel of the multiplication map $F \otimes_k F \stackrel{\times}{\longrightarrow} F$. We consider $\mathfrak{m}$ as an ideal and as an F-vector space with F-multiplication via $F \otimes 1$. The map $F \otimes_k (F/k) \longrightarrow \mathfrak{m}$, given by $\sum_j z_j \otimes \overline{y_j} \mapsto \sum_j z_j \otimes y_j - (\sum_j z_j y_j) \otimes 1$ is clearly an isomorphism of F-vector spaces.

Corollary 3.11. *Let* $k|k_0$ *be an extension of fields of characteristic 0,* $F|k$ *be an algebraically closed field extension and* $\mathfrak{m}$ *be the kernel of the multiplication homomorphism* $F \otimes_{k_0} F \stackrel{\times}{\longrightarrow} F$*. Then any homomorphism* ξ *from* $F \otimes_{k_0} \bigotimes_k^M (F/k_0) \cong \bigotimes_F^M \mathfrak{m}$ *to any* F*-semilinear admissible representation* V *factors through* $\bigotimes_F^M (\mathfrak{m}/\mathfrak{m}^s)$ *for some* $s \geq 1$*.*

Proof. For each $M \geq 1$, set $H_M := \{\sigma \in G \mid \sigma x_i - x_i \in k_0,\ 1 \leq i \leq M\}/G_{F|k(x_1,\ldots,x_M)}$. For any $x \in F$ set $\delta x := x \otimes 1 - 1 \otimes x \in \mathfrak{m}$, so $(\delta x)^s = \sum_{j=0}^{s} (-1)^j \binom{s}{j} x^{s-j} \otimes x^j \in \mathfrak{m}^s$. For any collection of integers $\underline{s} = (s_1, \ldots, s_M)$, $s_i \geq 1$, the element $\alpha_{\underline{s}} := (\delta x_1)^{s_1} \otimes \cdots \otimes (\delta x_M)^{s_M} \in (\mathfrak{m}^{s_1} \otimes_F \cdots \otimes_F \mathfrak{m}^{s_M})_M^{H_M}$ has weight $(s_1, \ldots, s_M)$ with respect to the torus $\{\sigma \in G \mid \sigma x_i / x_i \in k^\times,\ 1 \leq i \leq M\}/G_{F|k(x_1,\ldots,x_M)} \cong (k^\times)^M$. In particular, the nonzero images in V (more precisely, in V_M) of the elements $\alpha_{\underline{s}}$ are linearly independent over k. As V_M

is finite-dimensional, the k-vector space $V_M^{H_M}$ is finite-dimensional as well, and therefore, the image of the element $\alpha_{\underline{s}}$ in V_M is zero for all but finite number of $\underline{s}$.

It is a little calculation done in [7, Lemma 4.8], that the element $\alpha_{\underline{s}} \in \bigotimes_F^M \mathfrak{m}$ generates the F-semilinear subrepresentation $\mathfrak{m}^{s_1} \otimes_F \cdots \otimes_F \mathfrak{m}^{s_M}$ of G. This implies that the homomorphism ξ factors through the quotient of $\bigotimes_F^M \mathfrak{m}$ by $\bigoplus_{i=1}^M \bigotimes_F^{M-i} \mathfrak{m} \otimes_F \cdots \otimes_F \mathfrak{m}^s \otimes_F \cdots \otimes_F \bigotimes_F^{i-1} \mathfrak{m}$ for some $s \geq 1$, i.e., ξ factors through $\bigotimes_F^M (\mathfrak{m}/\mathfrak{m}^s)$ for some $s \geq 1$. □

Proof of Theorem 3.1. Let V be an irreducible F-semilinear admissible representation of $G_{F|k}$. Fix any $m \geq 0$ with a nonzero $V_m := V^{G_{F|K_m}}$. As the K_m-semilinear representation V_m of $\mathrm{PGL}_{m+1}k$ is finite-dimensional (and thus, it is of finite length), it admits a simple subobject A. By Theorem 3.2, A is isomorphic to $S_{K_m}^{\lambda}\Omega_{K_m|k}^1$ for a partition λ. Clearly, $A = B \otimes_k K_m$, where $B := A^{(\mathrm{Aff}_m)_u} \cong (S_{K_m}^{\lambda}\Omega_{K_m|k}^1)^{(\mathrm{Aff}_m)_u} \cong S_k^{\lambda}(k^m)$ is a rational irreducible representation of $\mathrm{GL}_m k := \mathrm{Aff}_m/(\mathrm{Aff}_m)_u$.

Then there is a nonzero (and therefore, surjective) morphism of semilinear representations $U := F[W^\circ] \otimes_{k[\mathrm{GL}_m k]} B \xrightarrow{\varphi} V$.

By Lemmas 3.9 and 3.10, V is a quotient of $F \otimes_k \bigotimes_k^M (F/k)$ for some $M \geq 0$. Then the conclusion follows from Corollary 3.11 and the identities $\mathfrak{m}^j/\mathfrak{m}^{j+1} = \mathrm{Sym}_F^j(\mathfrak{m}/\mathfrak{m}^2)$ and $\mathfrak{m}/\mathfrak{m}^2 = \Omega_{F|k}^1$. □

Acknowledgements. I am grateful to the organizers of Magadan conference for the invitation. The author was partially supported by the Russian Academic Excellence Project '5-100' and by an RSF grant. Section 2 is prepared with the support from RSF grant, project 14-21-00053 dated 11.08.14.

References

[1] A. Borel and J. Tits, *Homomorphismes "abstraits" de groupes algébriques simples*, Ann. of Math. (2) **97** (1973), 499–571.

[2] D. B. A. Epstein, *Natural vector bundles*, Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three) pp. 171–195 Springer, Berlin, 1969.

[3] D. B. A. Epstein and M. Kneser, *Functors between categories of vector spaces*, in Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three) pp. 154–170 Springer, Berlin, 1969.

[4] M. Kashiwara, *Invariant Sheaves*, Publ. Res. Inst. Math. Sci. **36** (2000), no. 4, 491–509.

[5] L. Lifschitz and A. Rapinchuk, *On abstract homomorphisms of Chevalley groups with nonreductive image. I*, J. Algebra **242** (2001), no. 1, 374–399.

[6] M. Rovinsky, *Semilinear representations of PGL*, Selecta Math. New Ser. **11** (2005), no. 3-4, 491–522.

[7] ———, *Admissible semilinear representations*, J. Reine Angew. Math. **604** (2007), 159–186.

[8] ———, *An analogue of Hilbert's Theorem 90 for infinite symmetric groups*, arXiv: 15038.02267v5.

[9] ———, *Stable birational invariants with Galois descent and differential forms*, Moscow Math. J. **15** (2015), no. 4, 777–803.
[10] A. Speiser, *Zahlentheoretische Sätze aus der Gruppentheorie*, Math. Z. **5** (1919), no. 1-2, 1–6.

Marat Rovinsky
National Research University Higher School of Economics
AG Laboratory
HSE, 6 Usacheva str., Moscow 119048, Russia
and
Institute for Information Transmission Problems
of Russian Academy of Sciences
Russia
E-mail address: marat@mccme.ru