Bull. Korean Math. Soc. **54** (2017), No. 5, pp. 1719–1724
https://doi.org/10.4134/BKMS.b160688
pISSN: 1015-8634 / eISSN: 2234-3016

EFFICIENTLY COMPUTING TORUS CHARTS IN LANDAU–GINZBURG MODELS OF COMPLETE INTERSECTIONS IN GRASSMANNIANS OF PLANES

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Abstract. In this note, companion to the paper [10], we describe an alternative method for finding Laurent polynomials mirror-dual to complete intersections in Grassmannians of planes, in the sense discussed in [10]. This calculation follows a general method for finding torus charts on Hori–Vafa mirrors to complete intersections in toric varieties, detailed in [5] generalising the method of [8].

In [10] the authors demonstrate that Givental's integrals for the Landau–Ginzburg models for complete intersections in Grassmannians of planes proposed by Batyrev, Ciocan-Fontanine, Kim, and van Straten in [2, 3] are periods of pencils defined by Laurent polynomials. In this note we present an alternative, efficient, calculation of torus charts on these Landau–Ginzburg models and show in examples that the periods of the Laurent polynomials found by restricting the superpotential coincide with those of [10].

Theorem 1.1. *Fix values $k \in \mathbb{Z}_{\geq 1}$ and $d_i \in \mathbb{Z}_{\geq 1}, i \in [1, l]$ such that the complete intersection Y defined by a general section of $\bigoplus_{i\in[1,l]} \mathcal{O}(d_i)$ in $\mathrm{Gr}(2, 2+k)$ is a Fano variety. The algorithm presented in* [5] *defines a* mirror-dual *Laurent polynomial to this complete intersection. In particular, the period integral of this Laurent polynomial is equal to Givental's integral for Y.*

The algorithm presented in [5] gives a general method for finding torus charts on Landau–Ginzburg models for complete intersections in toric varieties, generalising the method of [8]. In this note we apply this method to the flat toric degeneration of [2]. As well as the construction of [10] the problem of finding torus charts in Landau–Ginzburg models mirror-dual to complete intersections in Grassmannians of planes is considered in [6], [9] and [11]. We first recall the general form of the procedure from [5].

Received August 17, 2016; Accepted September 21, 2016.
2010 *Mathematics Subject Classification.* 14J33, 14J45.
Key words and phrases. mirror symmetry, Grassmanians.
This work was performed at Imperial College London and the author supported by an EPSRC prize studentship.

Following the observation made just after [10, Theorem 4.1] we may replace a complete intersection in $\mathrm{Gr}(2, 2+k)$ with one in $P(2, 2+k)$. We recall the general setup for Givental's procedure, with the conventions used in [10, Section 3], in the slightly more general setting which applies to complete intersections in $P(2, 2+k)$.

Assume we have fixed a Fano toric variety X, and a collection $L_1, \ldots, L_l$ of nef line bundles on X. As in [10, Section 3] we let $\mathcal{N} \cong \mathbb{Z}^N$ denote the lattice containing the fan of X and $\mathcal{D} \cong \mathbb{Z}^{N+\rho}$ be a based lattice, with basis D_i, $i \in [1, N+\rho]$. Assuming the ray map is surjective we have the following exact sequence:

$$0 \longrightarrow \mathbb{L} \xrightarrow{D^*} \mathcal{D} \xrightarrow{r} \mathcal{N} \longrightarrow 0.$$

Defining $\mathbb{L}$ as the kernel, recall that we have an identification $\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{L}^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$, the identification induced by the identification of $\mathcal{D}$ with the lattice of torus invariant divisors. We shall denote both the basis of $\mathcal{D}^\vee$ and the image of these basis elements in $\mathbb{L}^\vee$, under the dual map $D\colon \mathcal{D}^\vee \to \mathbb{L}^\vee$ by D_i, $i \in [1, N+\rho]$. We shall denote by D_i' the dual basis in $\mathcal{D}$. We shall assume that $(X; L_1, \ldots, L_l)$ satisfies the following conditions:

(1) There is a torus fixed point of X not contained in the singular locus. This determines a maximal-dimensional smooth cone in $\mathcal{N}$, and hence there is a subset $E \subset [1, N+\rho]$ such that the vectors $D_i, i \in E$ are a basis of $\mathbb{L}^\vee$. This condition implies that the ray map r is surjective.
(2) Each L_i is a non-negative combination of the $D_i, i \in E$, under the standard identification of $\mathbb{L}^\vee$ with linear systems of divisors on X.
(3) For each L_i, $i \in [1, l]$ there is a set $E_i \subset [1, N+\rho]$ such that $L_i = \sum_{j \in E_i} D_j$, and the collection of subsets E_k together with E is pairwise disjoint.

Observe that the second condition is independent of the third since the sets E_i are necessarily disjoint from E. In other words we can define a *nef-partition* for $L_1, \ldots, L_l$ analogously to [10, Section 3]. We shall also fix an element $s_j \in E_j$ for each j and let $E_j^\circ = E_j \backslash \{s_j\}$. The map $D\colon \mathcal{D}^\vee \to \mathbb{L}^\vee$ is now a map of based lattices, and from now on we shall use D to refer to this *matrix* and for the corresponding map of (split) tori, obtained by tensoring with $\mathbb{C}^*$.

Givental's Landau–Ginzburg model is defined by setting the superpotential

$$w = u_1 + \cdots + u_{N+\rho} \in \mathbb{C}[\mathcal{D}],$$

where the u_i are variables corresponding to the basis elements, and restricting to the subvariety defined by $D = 1$, $F_m = 1$ for $m \in [1, l]$, where the F_m are defined by the following:

$$F_m = \sum_{j \in E_m} u_j.$$

Observe that the conditions $D(u_1, \ldots, u_{N+\rho}) = 1$ are monomial, and applying only these equations yields the complex torus $T_{\mathcal{M}} = \operatorname{Spec} \mathbb{C}[\mathcal{N}]$, equipped with the regular function

$$w = \sum_{i\in[1,N+\rho]} z^{r(D_i')},$$

where z^a denotes the monomial in $\mathbb{C}[\mathcal{N}]$ corresponding to $a \in \mathcal{N}$. This is a reformulation of the condition [10, (3.3)], setting $q = 1$. Since we have fixed an echelon form for D, $T_{\mathcal{M}}$ has coordinates $\{u_i \mid i \notin E\}$. To pass to the subvariety defined by $\{F_m = 1\}_{m\in[1,l]}$ we introduce new variables y_i for $i \in \bigcup_{m\in[1,l]} E_m^\circ$ defined by the following:

$$u_i = \begin{cases} \frac{1}{1+\sum_{k\in E_m^\circ} y_k} & \text{if } i = s_m, \\ \frac{y_i}{1+\sum_{k\in E_m^\circ} y_k} & \text{if } i \in E_m^\circ. \end{cases}$$

So there is a birational torus chart on Givental's Landau–Ginzburg model given by the torus with coordinates $y_j, j \in \bigcup_{m\in[1,l]} E_m^\circ$ and u_j for $j \notin E \cup \bigcup_{m\in[1,l]} E_m$. It is easy to see from our assumptions that w remains Laurent after applying this change of coordinates.

We now apply this procedure explicitly to complete intersections in $P(2, 2+k)$, this is singular, but we recall replacing $\mathrm{Gr}(2, 2+k)$ with $P(2, 2+k)$ is justified in [10, Section 4]. Indeed, inspecting its definition, $P(2, 2 + k)$ satisfies the condition (1) above. Given line bundles defining a Fano complete intersection we shall represent each line bundle as a sum of divisors D_i satisfying conditions (2) and (3). Recall that the superpotential $w = \sum_{i\in[1,N+\rho]} z^{r(D_i')}$ for $P(2, 2+k)$, in the coordinates $a_{i,j}$ introduced in [10, Section 4], is given by the Laurent polynomial

$$f_{\mathrm{Gr}(2,2+k)} = a_{1,1} + \sum_{j=2}^{k} \frac{a_{1,j}}{a_{1,j-1}} + \sum_{j=1}^{k} \frac{a_{2,j}}{a_{1,j}} + \sum_{j=2}^{k} \frac{a_{2,j}}{a_{2,j-1}} + \frac{1}{a_{2,k}}.$$

Also consider the $k + 2$ equations

$$f_1 = a_{1,1}, \qquad f_j = \frac{a_{1,j}}{a_{1,j-1}} + \frac{a_{2,j}}{a_{2,j-1}},\ j \in [2, k], \qquad f_{k+1} = \frac{1}{a_{2,k}}.$$

Given a complete intersection of hypersurfaces of degree d_i, $i \in [1, l]$ we fix a partition

$$[1, k + 1] = E_0 \sqcup E_1 \sqcup \cdots \sqcup E_l$$

with $|E_j| = d_j$ for $j > 0$, and we form Givental's Landau–Ginzburg mirror to the complete intersection in $\mathrm{Gr}(2, 2 + k)$ by restricting to the subvariety

$$\left\{ F_j = \sum_{r\in E_j} f_r = 1 \mid j = 1, \ldots, l \right\}.$$

In order to apply the method explained above for finding a birational torus chart we apply a change of variables

$$x_{1,1} = a_{1,1}, \qquad x_{1,j} = \frac{a_{1,j}}{a_{1,j-1}},\ \ j \in [2, k],$$

$$x_{2,j} = \frac{a_{2,j}}{a_{2,j-1}}, \quad j \in [2,k], \qquad x_{2,k+1} = \frac{1}{a_{2,k}}.$$

With these changes of variables, of the $3k$ terms of $f_{\mathrm{Gr}(2,2+k)}$, we see that $2k$ are now simply variables $x_{i,j}$ and the remaining k terms are monomials

$$M_i = \frac{a_{2,i}}{a_{1,i}} = \big(\prod_{j\le i} x_{1,j} \prod_{j \ge i} x_{2,j}\big)^{-1}.$$

The polynomials f_j have become the following:

$$f_1 = x_{1,1}, \qquad f_j = x_{1,j} + x_{2,j},\ j \in [2,k], \qquad f_{k+1} = x_{2,k+1}.$$

We may now apply the general procedure given above, namely we define the E_m and E used there according to the terms appearing in the polynomials F_m and $\sum_{j=1}^{k} M_j$. Observe that all three requirements for finding a birational torus chart in the first section are met, so we may apply the change of co-ordinates above. This provides a completely explicit way of forming, potentially several, Laurent polynomial mirrors for a given complete intersection in $\mathrm{Gr}(2, k+2)$. To complete the proof of Theorem 1.1 we need to compare Givental's integral with the period of the candidate Laurent polynomial. This however is a well known residue calculation see, for example, the proof of [10, Theorem 10.4] for details. We now consider some examples of this method.

Example 1.2 (see also [10, Example 12.12])**.** Consider a cubic intersected with the quadric $\mathrm{Gr}(2,4)$. The weight matrix D for $P(2,4)$ is the following:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The Picard group is the sublattice generated by the column $(1,1)^t$. The Laurent polynomial mirror may be obtained either by applying the condition $D = 1$ above, or by changing $f_{\mathrm{Gr}(2,4)}$ to the $x_{i,j}$ variables, in either case there is a Laurent polynomial presentation given by the following:

$$f_{\mathrm{Gr}(2,4)} = x_{1,1} + x_{1,2} + \frac{1}{x_{1,1}x_{1,2}x_{2,3}} + \frac{1}{x_{1,1}x_{2,2}x_{2,3}} + x_{2,2} + x_{2,3}.$$

In general the first k and final k columns of D correspond to basis elements in $\mathcal{N}$ and to variables $x_{1,j}$, $j \in [1,l]$ and $x_{2,j}$, $j \in [2, l+1]$ respectively. In this example ($k = 2$) the column vector $(3,3)^t$ may be obtained by adding the first, second, fifth, and sixth columns, giving the relation

$$x_{1,1} + x_{1,2} + x_{2,2} + x_{2,3} = 1.$$

Let $E_1 = \{1,2,5,6\}$, $E = \{3,4\}$, and $s_1 = 1$. Denoting the new variables $y_{i,j}$, consistent with the variables $x_{i,j}$ we have the following:

$$x_{1,1} = \frac{1}{1 + y_{1,2} + y_{2,2} + y_{2,3}}, \qquad x_{1,2} = \frac{y_{1,2}}{1 + y_{1,2} + y_{2,2} + y_{2,3}},$$
$$x_{2,2} = \frac{y_{2,2}}{1 + y_{1,2} + y_{2,2} + y_{2,3}}, \qquad x_{2,3} = \frac{y_{2,3}}{1 + y_{1,2} + y_{2,2} + y_{2,3}}.$$

Let ψ be the (rational) map defined by this co-ordinate change, pulling back the superpotential we find

$$\begin{aligned}\psi^* f_{\mathrm{Gr}(2,4)} &= \frac{(1+y_{1,2}+y_{2,2}+y_{2,3})^3}{y_{1,2}y_{2,3}} + \frac{(1+y_{1,2}+y_{2,2}+y_{2,3})^3}{y_{2,2}y_{2,3}} \\ &= \frac{(y_{2,2}+y_{1,2})}{y_{1,2}y_{2,2}y_{2,3}}(1+y_{1,2}+y_{2,2}+y_{2,3})^3.\end{aligned}$$

We shall now show this is equivalent to the result of [10, Example 12.12] up to symplectomorphisms of cluster type ([4], [7]) and in fact factorizes into two *algebraic mutations* (as defined in [1]). To see this first consider the birational map ϕ_1, defined by the following:

$$\phi_1^* y_{1,2} = y_{1,2}, \qquad \phi_1^* y_{2,2} = y_{2,2}, \qquad \phi_1^* y_{2,3} = (y_{1,2}+y_{2,2})y_{2,3}.$$

Computing $\phi_1^*\psi^* f_{\mathrm{Gr}(2,4)}$ we obtain

$$g_{\mathrm{Gr}(2,4)} = \phi_1^*\psi^* f_{\mathrm{Gr}(2,4)} = \frac{1}{y_{1,2}y_{2,2}y_{2,3}}(1+y_{1,2}+y_{2,2}+(y_{1,2}+y_{2,2})y_{2,3})^3.$$

Now, changing the co-ordinates on this torus,

$$y_{1,2} = a_{2,1}, \qquad y_{2,2} = \frac{a_{2,1}^2}{a_{1,1}}, \qquad y_{2,3} = a_{1,2}$$

we have that

$$g_{\mathrm{Gr}(2,4)} = \frac{a_{1,1}}{a_{1,2}}\left(\frac{1}{a_{2,1}} + \left(1+\frac{a_{2,1}}{a_{1,1}}\right)(1+a_{1,2})\right)^3.$$

Apply another birational change of coordinates ϕ_2, sending

$$\phi_2 \colon a_{i,j} \mapsto a_{i,j}\left(1+\frac{a_{2,1}}{a_{1,1}}\right)^{-1}$$

for each (i,j). We obtain

$$h_{\mathrm{Gr}(2,4)} = \phi_2^* g_{\mathrm{Gr}(2,4)} = \frac{a_{1,1}}{a_{1,2}}\left(a_{1,2} + \frac{1+a_{1,1}a_{2,1}+a_{1,1}+a_{2,1}}{a_{1,1}a_{2,1}}\right)^3.$$

Which is the resulting polynomial of [10, Example 12.12].

Example 1.3 (see also [10, Example 12.6])**.** A fourfold of index 2 given by 4 hyperplane sections of $\mathrm{Gr}(2,6)$. The matrix D for $P(2,6)$ is

$$\begin{pmatrix} 1&0&0&0&0&0&0&1&1&1&1&1\\ 1&1&0&0&0&0&1&0&0&1&1&1\\ 1&1&1&0&0&1&0&0&0&0&1&1\\ 1&1&1&1&1&0&0&0&0&0&0&1 \end{pmatrix}.$$

The four bundles L_i are all equal to $(1,1,1,1) \in \mathbb{L}^\vee$. We fix the nef-partition by taking the collections of basis elements D_i corresponding to columns $\{1\}$, $\{12\}$,

$\{2,9\}$, $\{3,10\}$ of the matrix D. Applying the notation employed in Example 1.2 we compute

$$\psi^* f_{\mathrm{Gr}(2,6)} = x_{1,4} + x_{2,4} + \frac{(1+y_{2,2})(1+y_{2,3})}{y_{2,2}y_{2,3}x_{2,4}}(1+y_{2,2}+y_{2,2}y_{2,3}) + \frac{(1+y_{2,2})(1+y_{2,3})}{x_{1,4}},$$

noting that columns 4, 11 are in neither E nor any of the sets E_m, so the variables $x_{1,4}$ and $x_{2,4}$ persist. As in the previous example, this polynomial also agrees with the result of [10, Example 12.6] up to symplectomorphisms of cluster type, which in particular preserve the period sequence of the Laurent polynomial.

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