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# SHARYGIN TRIANGLES AND ELLIPTIC CURVES

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Abstract. The paper is devoted to the description of family of scalene triangles for which the triangle formed by the intersection points of bisectors with opposite sides is isosceles. We call them Sharygin triangles. It turns out that they are parametrized by an open subset of an elliptic curve. Also we prove that there are infinitely many non-similar integer Sharygin triangles.

## 1. Introduction

The following problem had been stated in the Kvant journal (see [13, p. 36]) and had been included into the famous problem book on planimetry (see [14, p. 55, problem 158]).

**Problem 1.1.** *It is known that, for a given triangle, the points where the bisectors meet opposite sides form an isosceles triangle. Does it imply that the given triangle is isosceles?*

The answer is negative. Sharygin writes: *"Unfortunately, the author had not constructed any explicit example of such a triangle* (*had not provided a triple of side lengths or a triple of angles*) *with so exotic property. Maybe, the readers can construct an explicit example."*

For a given triangle, we call the triangle formed by the intersection points of the bisectors with opposite sides the *bisectral triangle* (on Figure 1a triangle $A'B'C'$ is bisectral for $ABC$).

**Definition 1.2.** We call a triangle a *Sharygin triangle* if it is scalene but its bisectral triangle is isosceles.

This work is completely devoted to the detailed study of Sharygin triangles.

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A great enthusiast of school mathematical contests Sergei Markelov told us that, amazingly, a Sharygin triangle can be constructed if we take a side of the right heptagon and two different adjacent diagonals (see the proof in Section 2).

It turns out that any Sharygin triangle has an obtuse angle (it is proved in [14]). Moreover, if $x$ denotes its cosine, then $-1 < 4x < \sqrt{17} - 5$. This implies that the angle measure is between $\approx 102.663°$ and $\approx 104.478°$. In the example arising from the right heptagon we get the obtuse angle $\frac{8\pi}{7} \approx 102.857°$. In respect that the range of suitable angles is very small, this example is totally amazing and surprising. Consequently, the following question arises naturally: are there other examples of right polygons such that three of its vertices form a Sharygin triangle? Study of this problem has led us to some beautiful formulas, but has not led yet to new examples.

In [10] the example of irrational Sharygin triangle constructed by P. Kozhevnikov

$$3\sqrt{17} + 4\sqrt{2}, \quad 3\sqrt{17} - 4\sqrt{2}, \quad 5\sqrt{2}$$

is given. Of course, an integer Sharygin triangle would be a kind of "triumph" in the problem to construct explicit examples of Sharygin triangles.

To solve this problem S. Markelov started a computation running over all triangles with side lengths not exceeding million. No such luck, after two months (in 1996 or around) the computer answered that there are no examples. Nevertheless, Sergei had not calmed down. Evidently, something suggested him the right answer.

Consider a triangle $ABC$. Let us denote by $AA'$, $BB'$, and $CC'$ its bisectors and by

$$a = BC, \qquad b = AC, \qquad c = AB$$

its side lengths (see Figure 1(A)). Sergei considered the replacement

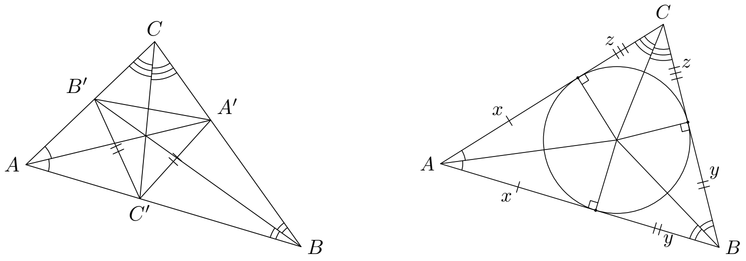
$$\begin{cases} a = y + z, \\ b = x + z, \\ c = x + y. \end{cases}$$

This replacement is well known in planimetry. For $a, b, c$ being the side lengths of a triangle, $x, y, z$ are distances from vertices to the points where the encircle meets the adjacent sides (see Figure 1(B)).

Sergei had rewritten the equation in terms of $x, y, z$:

$$4z^3 + 6xyz - 3xy(x + y) + 5z(x^2 + y^2) + 9z^2(x + y) = 0.$$

It is enough that $x, y, z > 0$ for $a, b, c$ to satisfy the triangle inequalities. We see that the condition $A'C' = B'C'$ becomes a cubic equation on $x, y, z$ without the monomials $x^3$ and $y^3$. This means that the corresponding projective curve $\mathcal{E}$ on the projective plane with coordinates $(x : y : z)$ contains the points $(1 : 0 : 0)$ and $(0 : 1 : 0)$. Sergei divided the equation by $z^3$ passing to the affine




(A) (B)

FIGURE 1

chart $\{z \neq 0\}$ with the coordinates $\widetilde{x} = \frac{x}{z}$ and $\widetilde{y} = \frac{y}{z}$, and guessed that the point $(\widetilde{x}, \widetilde{y}) = (1, -3)$ lies on the curve.

The equation of the curve $\mathcal{E}$ is quadratic in $\widetilde{x}$ and $\widetilde{y}$. On the next step Sergei reopened the addition law of the points on an elliptic curve. Namely, he guessed a rational point and started to draw vertical and horizontal lines through the points that are already constructed. If we have a point $Q$ on $\mathcal{E}$, then the vertical line intersects the curve in two points (actually, in three points, where the third point is on infinity). The equation on these two points has rational coefficients, and we know one solution that is rational. Therefore, the other one is also rational. If we make after the same construction with the horizontal line through the new point and obtain the point $Q'$, then this construction of $Q'$ from $Q$ corresponds to the addition of some point $A$ to $Q$ on the curve $\mathcal{E}$. (We need to define the origin of our elliptic curve to determine the point $A = Q' - Q$ that does not depend on the choice of $Q$.)

Iterating this algorithm (passing from $Q$ to $Q + A$), Sergei obtained on the fourth step a triangle (see [10]), i.e., a point with $\widetilde{x}, \widetilde{y} > 0$. Replacing the coordinates back to $a, b, c$, Sergei obtained $(a, b, c) = (18800081, 1481089, 19214131)$. It is not surprising that the computer program running over values before million gives nothing, while the presumably first integer triangle has so grandiose side lengths! Sergei would have found this triangle, if he started the program running the side lengths not exceeding billion on a more powerful computer some time after. Fortunately, the modern computers are not so powerful. This led us to the addition law on the elliptic curve and to some more complicated theory. As a result, we have proved that there are infinitely many non-similar integer Sharygin triangles. In this way, the school-level problem has led us to a beautiful branch of modern mathematics.

In this work we consider the question of how to construct each integer Sharygin triangle. Integer Sharygin triangles correspond to rational points of the elliptic curve $\mathcal{E}$ lying in some open subset (defined by the triangle inequalities). Therefore, we need to describe the group of rational points on the curve. Here we find the torsion subgroup, find the rank and give an element of the free part (it seems to be a generator). If this point of infinite order is a generator, then all the integer Sharygin triangles can be constructed from the points that we have found, by the addition of points on $\mathcal{E}$.

The authors are grateful to Sergei Markelov for introducing the problem and a series of useful tricks in the first rough solution. Also the authors are grateful to V. A. Voronov, N. Tsoy, D. V. Osipov, S. S. Galkin, I. A. Cheltsov, S. O. Gorchinsky and other participants of the conference "Magadan Algebraic Geometry" for useful discussions and advices. We are grateful to N. N. Osipov for the proof of Proposition 4.6.

Lastly, we should explain elementary character of the exposition. The point was to introduce wide public — students, non-algebraic mathematicians and alike — to the magic of elliptic curves, starting from the absolutely clear, school-level problem. That is why we do not restrict ourselves to refer to standard but involved results from the arithmetics and algebraic aspects of elliptic curves, instead giving absolutely elementary proofs to most of our statements. We wish that students would find their speciality in studying elliptic curves, after reading this introductory text.

## 2. The heptagonal triangle

The example below was firstly constructed in [1]. Also the problem to prove that this triangle is a Sharygin triangle was one of the problems in the mathematical contest "Tournament of Towns" (Fall '2005, A-level, grades 10–11, problem 5) and was published in [19, problem M2001].

**Example 2.1.** Consider the unit circle $|z| = 1$ on the complex plane $\mathbb{C}$. Set $\zeta = e^{\frac{2\pi i}{7}}$. Consider the triangle $(1, \zeta, \zeta^3)$. Obviously, it is scalene. Its vertices are placed at the vertices of the regular heptagon drawn by dot-and-dash line on Figure 2.

In Section 3 we reduce the property of a triangle to be a Sharygin triangle to a cubic relation on its sides. Actually, it is enough to substitute side lengths to equation (2) to verify that $(1, \zeta, \zeta^3)$ is a Sharygin triangle. It is even simpler to substitute its angles $\alpha = \frac{\pi}{7}$ and $\beta = \frac{2\pi}{7}$ into the equivalent equation (1). Let us, however, prove this fact geometrically.

Denote the points of intersection of bisectors with the opposite sides of $(1, \zeta, \zeta^3)$ by $\alpha$, $\beta$ and $\gamma$ as on Figure 2. Let $\gamma'$ denote the reflection of $\gamma$ from the line through $\sqrt{\zeta}$ and $\zeta^4$. Then $|\gamma - \beta| = |\gamma' - \beta|$. Lines $(1, \zeta^2)$ and $(\zeta, \zeta^3)$ are symmetric with respect to $(\zeta^{3/2}, \zeta^5)$. Therefore $\alpha$ lies on $(\zeta^{3/2}, \zeta^5)$. Lines $(\zeta^5, \zeta^{3/2})$ and $(\zeta^5, \sqrt{\zeta})$ are symmetric with respect to $(\zeta, \zeta^5)$. Lines $(\zeta, \zeta^3)$ and $(\zeta, 1)$ are also symmetric with respect to $(\zeta, \zeta^5)$. Therefore $|\alpha - \beta| = |\gamma' - \beta|$. Finally, $|\alpha - \beta| = |\gamma - \beta|$.

Let us prove that the triangle $(1, \zeta, \zeta^3)$ is not similar to a triangle with integer sides. Consider the ratio of two side lengths:

$$\frac{|\zeta^3 - \zeta|}{|\zeta - 1|} = |\zeta + 1| = 2\cos\frac{\pi}{7}.$$

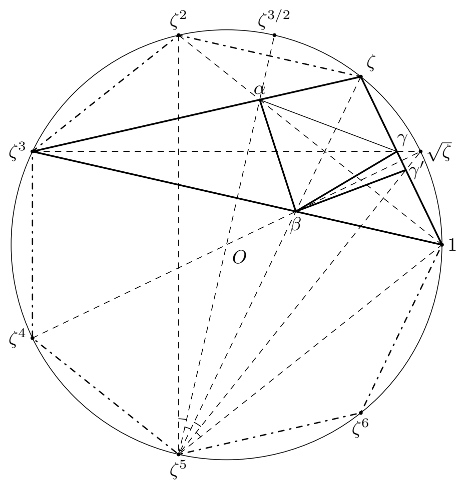



FIGURE 2

One can verify that the number $2\cos\frac{\pi}{7}$ is a root of the irreducible polynomial $z^3 - z^2 - 2z + 1$. Therefore it is irrational.

**Hypothesis 2.2.** *Suppose that vertices of Sharygin triangle coincide with vertices of some regular polygon with $n$ sides. Then $n$ is divisible by 7, and this triangle is similar to the one described above.*

Let us give some ideas about Hypothesis 2.2. From the law of sines we have

$$a = 2\sin\alpha, \quad b = 2\sin\beta, \quad c = 2\sin\gamma,$$

where $\alpha$, $\beta$ and $\gamma$ are opposite to the sides $a$, $b$, $c$ angles. We will see below that the condition on $a, b, c$ to form a Sharygin triangle is a cubic equation (2). Substituting this to (2) with $\gamma = \pi - \alpha - \beta$, we obtain

$$\sin\alpha\sin\beta(\sin\alpha+\sin\beta+\sin(\alpha+\beta)-\sin(2\alpha+\beta)-\sin(\alpha+2\beta)-\sin(2\alpha+2\beta)) = 0.$$

This can be easily checked by expansion of the brackets in both equations. Skipping $\sin\alpha\sin\beta$, we obtain the equation

$$\sin\alpha + \sin\beta + \sin(\alpha+\beta) = \sin(\alpha+2\beta) + \sin(2\alpha+\beta) + \sin(2\alpha+2\beta). \tag{1}$$

Consider complex numbers $x = \cos\alpha + i\sin\alpha$ and $y = \cos\beta + i\sin\beta$. Here $0 < \alpha, \beta < \frac{\pi}{2}$. Equation (1) implies that the imaginary part of the number $w = x + y + xy - xy^2 - xy^2 - x^2y^2$ is zero. This implies that this number coincides with its conjugate. Note that $\bar{x} = x^{-1}$ and $\bar{y} = y^{-1}$. Therefore

$$w - \bar{w} = \frac{xy-1}{x^2y^2}(1 + x + y + x^2y^3 + x^3y^2 + x^3y^3) = 0.$$

Suppose that the vertices coincide with vertices of regular $N$-gon. Therefore $\alpha = \frac{m\pi}{N}$ and $\beta = \frac{n\pi}{N}$ for some integer $m, n$. Therefore $x$ and $y$ are roots of unity of degree $2N$. So it is enough to solve the system

$$\begin{cases} 1 + x + y + x^2y^3 + x^3y^2 + x^3y^3 = 0, \\ x^N = y^N = 1 \end{cases}$$

for some $x, y \in \mathbb{C}$ such that $x, y, xy$ have positive real and imaginary parts. Numerical computation shows that there are no solutions except primitive roots of degree 7 for $N \leqslant 2000$. This suggests us to give Hypothesis 2.2. But we do not know how to prove it.

## 3. Parameterization by an open subset of elliptic curve

Take a triangle $ABC$. Let $A'B'C'$ be its bisectral triangle. Put

$$a' = B'C',\ b' = A'C',\ c' = A'B'.$$

**Proposition 3.1.** *Triples of side lengths of Sharygin triangles $ABC$ are all triples $(a, b, c)$ satisfying equation*

$$q(a,b,c) = -c^3 - c^2(a+b) + c(a^2+ab+b^2) + (a^3+a^2b+ab^2+b^3) = 0 \tag{2}$$

*and the triangle inequalities*

$$\begin{cases} 0 < a < b + c, \\ 0 < b < a + c, \\ 0 < c < a + b. \end{cases}$$

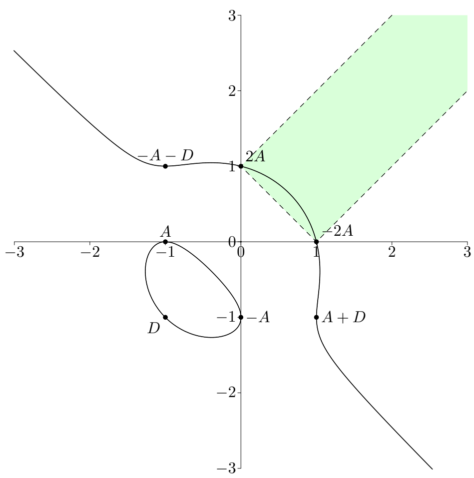



Figure 3

*Proof.* The law of cosines and the property of bisectors allows us to express $a', b', c'$. We obtain

$$a'-b' = (a-b)\frac{abc}{(a+b)(a+c)^2(b+c)^2}(a^3+a^2b+ab^2+b^3+c(a^2+ab+b^2)-c^2(a+b)-c^3).$$

The first multiplier $(a-b)$ corresponds to isosceles triangles $ABC$. The second multiplier $\frac{abc}{(a+b)(a+c)^2(b+c)^2}$ is always positive. Therefore, if $a, b, c$ are pairwise different, then the equation (2) holds. Conversely, if $a, b, c$ satisfy Eq. (2), then $a' = b'$. □

Let us denote by $\mathcal{E}$ the curve $q(a, b, c) = 0$ in the projective plane with the coordinates $(a : b : c)$. We see on Figure 3 the curve $\mathcal{E}$ in the affine plane $c = 1$ with coordinates $(a/c, b/c)$ and the domain $\mathcal{T}$ where the triangle inequalities hold:

$$\begin{cases} 0 < a < b + 1, \\ 0 < b < a + 1, \\ \quad\; 1 < a + b. \end{cases}$$

It is easy to see that the intersection is non-empty. For example, the point $(1, 0)$ lies on $\mathcal{E}$ and on the boundary of $\mathcal{T}$. The tangent of $\mathcal{E}$ at $(1, 0)$ has the equation $x + y/4 = 1$. Therefore, there are infinitely many real points of $\mathcal{E}$ in $\mathcal{T}$.

This proves that there are infinitely many scalene pairwise non-similar triangles with isosceles bisectral triangles. Therefore, we obtain that there are infinitely many Sharygin triangles with real side lengths. Below we consider integer Sharygin triangles.

## 4. Integer triangles and rational points on elliptic curve

### 4.1. Smoothness and inflexion points

We have seen above that Sharygin triangles are parametrized by an open subset of a cubic curve $\mathcal{E}$ defined by the equation (2).

**Definition 4.1.** An *elliptic curve* over a field $K$ is a smooth cubic plane curve defined over $K$ with a fixed $K$-rational point (it is called the *origin*).

**Proposition 4.2.** *The curve $\mathcal{E}$ is an elliptic curve over $\mathbb{Q}$.*

*Proof.* It is sufficient to check that the curve $\mathcal{E}$ is smooth. The system $\frac{\partial q}{\partial a} = \frac{\partial q}{\partial b} = \frac{\partial q}{\partial c} = 0$ has no non-trivial solutions. This implies that the curve $\mathcal{E}$ is smooth and therefore is an elliptic curve. We will fix a rational point on $\mathbb{Q}$ below. □

The fact that an elliptic curve has 9 inflexion points is well known (see [4, Ch. IV, §2, Ex. 2.3.g, p. 305]). We want to find some inflexion point and consider it as the origin of the elliptic curve $\mathcal{E}$, because in this case the addition law is simpler then for other choices of the origin.

**Proposition 4.3.** *The only inflexion point of $\mathcal{E}$ defined over $\mathbb{Q}$ is* $(1 : -1 : 0)$.

*Proof.* The inflexion points are defined as the intersection points of $\mathcal{E}$ and its Hessian being also a smooth cubic. It can be easily verified that there are 9 intersection points and that the only point among them defined over $\mathbb{Q}$ is $(1 : -1 : 0)$. All the others become defined over the extension of $\mathbb{Q}$ by the irreducible polynomial

$$32 + 115t + 506t^2 + 1053t^3 + 1212t^4 + 1053t^5 + 506t^6 + 115t^7 + 32t^8.$$

Therefore, the proposition is proved. □

We take the point $O := (1 : -1 : 0)$ as the identity element of the elliptic curve $\mathcal{E}$. Then for a point $A$ with coordinates $(a : b : c)$ the point $-A$ has the coordinates $(b : a : c)$. Indeed, it is easy to check that these three points lie on one line and the equation is symmetric under the permutation $a \leftrightarrow b$.

### 4.2. Torsion subgroup

At first, we need to find the Weierstraß form of $\mathcal{E}$ to find its torsion subgroup. Under the change of coordinates

$$\begin{aligned} a &\mapsto x + y, \\ b &\mapsto x - y, \\ c &\mapsto 24 - 4x \end{aligned}$$

the equation $q = 0$ transforms to

$$y^2 = x^3 + 5x^2 - 32x.$$

The discriminant equals $\Delta = 2506752 = 2^{14} \cdot 3^2 \cdot 17$. The set of points of finite order can be easily described using the following result.

**Theorem 4.4** (Nagell–Lutz, [9,11])**.** *Let $\mathcal{E}$ be an elliptic curve $y^2 = x^3 + ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}$. If a point $(x, y) \neq \infty$ is a torsion point of $\mathcal{E}(\mathbb{Q})$, then*

- *$x, y \in \mathbb{Z}$,*
- *either $y = 0$, or $y$ divides $\Delta = -4a^3c + a^2b^2 + 18abc - 4b^3 - 27b^2$.*

**Proposition 4.5.** *The torsion subgroup of $\mathcal{E}(\mathbb{Q})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and consists of the points $O$ and $(1 : 1 : -1)$, which we denote hereafter by $D$.*

*Proof.* We can consider all the divisors $y$ of $\Delta$ and for each $y$ find all integer solutions $x$. It can be verified that the unique solution is $(x, y) = (0, 0)$. This point has $(a : b : c)$-coordinates equal to $(1 : 1 : -1)$. □

Let us give some elementary proof of the fact that the point $A = (1 : 0 : -1)$ has infinite order. Applying the torsion group description, we can say that $A \neq O, D$, but this proof uses the Nagell–Lutz Theorem. The proof below does not use it and is almost school-level.

**Proposition 4.6.** *The curve $\mathcal{E}$ has infinitely many rational points. In particular, the point $A = (1 : 0 : -1)$ has infinite order.*

*Proof.* Take a rational point $P_0 = (x_0, y_0)$ on the elliptic curve $\mathcal{E}$ such that $y_0 \neq 0$. Consider its *duplication*, i.e., the point $P_1 = 2P_0 = (x_1, y_1)$. It can be constructed as the intersection point of the tangent line to $\mathcal{E}$ at $P_0$. It is easy to check that

$$x_0 = \frac{(x_0^2 + 32)^2}{4x_0(x_0^2 + 5x_0 - 32)} \neq 0.$$

Let $x_0 = p_0/q_0$ and $x_1 = p_0/q_0$ be the irreducible ratios. Then

$$\frac{p_1}{q_1} = \frac{(p_0 + 32q_0^2)^2}{4p_0q_0(p_0 + 5p_0q_0 - 32q_0^2)}.$$

Suppose that $p_0$ is odd and positive (for example, $P_0 = 6A$ with $x_0 = p_0/q_0 = \frac{121}{16}$). Then $p_1$ is also odd and positive. It is easy to see that

$$\begin{aligned} d &= \mathrm{GCD}((p_0^2 + 32q_0^2)^2, 4p_0q_0(p_0^2 + 5p_0q_0 - 32q_0^2)) \\ &= \mathrm{GCD}((p_0^2 + 32q_0^2)^2, p_0^2 + 5p_0q_0 - 32q_0^2) \\ &= \mathrm{GCD}(q_0^2(64q_0 - 5p_0)^2, p_0^2 + 5p_0q_0 - 32q_0^2) \\ &= \mathrm{GCD}(25p_0^2 - 640p_0q_0 + 4096q_0^2, p_0^2 + 5p_0q_0 - 32q_0^2) \\ &= \mathrm{GCD}(153(32q_0 - 5p_0), p_0^2 + 5p_0q_0 - 32q_0^2) \\ &= \mathrm{GCD}(153p_0^2, p_0^2 + 5p_0q_0 - 32q_0) \in \{1, 3, 9, 17, 51, 153\}, \end{aligned}$$

because $p_0^2$ and $p_0^2 + 5p_0q_0 - 32q_0^2$ are coprime. We see that

$$p_1 = \frac{(p_0 + 32q_0^2)^2}{d} \geqslant \frac{(p_0^2 + 32q_0^2)^2}{153} \geqslant \frac{(p_0^2 + 32)^2}{153} > p_0.$$

Therefore, numerators of $x$ coordinates of points $P_i$, where $P_{i+1} = 2P_i$, increase (in particular, for $P_0 = 6A$). We conclude that $A$ has infinite order. □

**Theorem 4.7.** *Rational points are dense on the curve $\mathcal{E}(\mathbb{R})$ in Euclidean topology. There are infinitely many pairwise non-similar integer Sharygin triangles.*

*Proof.* Consider the rational point $A = (1 : 0 : -1)$ on $\mathcal{E}$. Since $A \neq O, D$, its order is infinite. Firstly, the fact that $\operatorname{ord} A = \infty$ was proved in other way. It was checked that points $nA$ are pairwise different for $n = 1, \dots, 12$[1]. From Mazur Theorem it follows that order of any torsion point of any elliptic curve does not exceed 12. Therefore, $A$ is not a torsion point.

Consider the equation (2) in three-dimensional affine space. It is homogeneous, i.e., if a point $(a, b, c) \neq 0$ is a solution, then any point of the line $(\lambda a, \lambda b, \lambda c)$ for any $\lambda$ is a solution. In other words, the set of solutions is a cone. Consider the unit sphere $S = \{a^2 + b^2 + c^2 = 1\}$. The cone intersects it in a curve $\widetilde{\mathcal{E}}$ that consists of three ovals (see Figure 4). Under the map of $S$ into the projective plane $\mathbb{P}^2 = \{(a : b : c)\}$ points of projective plane correspond to pairs of opposite points on $S$. Two of the ovals are opposite on the sphere, and another one is opposite to itself.

[1]Thanks to N. Tsoy for this calculation.

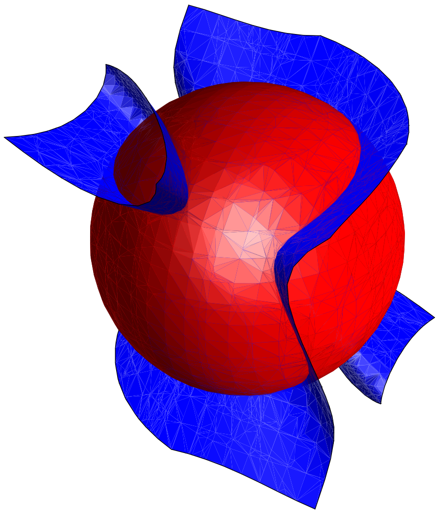


FIGURE 4

For any point $nA = (a_n : b_n : c_n)$ of $\mathcal{E}$ denote by $A_n$ one of two points of intersection of the line $(\lambda a_n, \lambda b_n, \lambda c_n)$ and the sphere $S$. Since all these lines are different and any pair of them intersects only at the origin, all the points $A_n$ are pairwise different. Therefore, $\{A_n\}$ is an infinite subset of the compact set $S \subset \mathbb{R}^3$. From Bolzano–Weierstrass theorem it follows that there is a limit point $\widetilde{L}$ of the set $\{A_n\}$. The curve $\widetilde{\mathcal{E}}$ is closed, and $A_n \in \widetilde{\mathcal{E}}$ for any $n$, therefore $\widetilde{L} \in \widetilde{\mathcal{E}}$. The point $\widetilde{L} \in \widetilde{\mathcal{E}}$ corresponds to a point $L \in \mathcal{E}$ that is a limit point of the set $nA$.

It is easy to see that the addition $(X, Y) \mapsto X + Y$ and inversion $X \mapsto -X$ of points of $\mathcal{E}$ are continuous operations. Denote by $M$ the point $(18800081 : 1481089 : 19214131)$ corresponding to the known Sharygin triangle. Let us introduce the function $f(X, Y) = M + X - Y$ of points of the elliptic curve $\mathcal{E}$. Obviously, it is continuous, and $f(L, L) = M$. Denote by $O_\varepsilon(P)$ the $\varepsilon$-neighborhood of a point $P$. From the definition, for any $\varepsilon > 0$ exists $\delta > 0$ such that if $X, Y \in O_\delta(L)$, then $f(X, Y) \in O_\varepsilon(M)$. We can take such $\varepsilon > 0$ that $O_\varepsilon \subset \mathcal{T}$, i.e., any point of $\mathcal{E} \cap O_\varepsilon(M)$ corresponds to a Sharygin triangle. For the corresponding $\delta > 0$ there are $n_\varepsilon A, m_\varepsilon A \in O_\delta(L)$.

The point $f(n_\varepsilon A, m_\varepsilon A)$

- corresponds to a Sharygin triangle,
- is rational.

Therefore, we obtain an integer Sharygin triangle in arbitrary small neighborhood of $M$. So we get infinitely many integer Sharygin triangles.

Now let us prove that rational points are dense on $\mathcal{E}(\mathbb{R})$. We see that topologically $\mathcal{E}(\mathbb{R})$ is a union of two circles. It is obvious, because in the Weierstraß form the curve is given by equation $y^2 = x^3 + 5x^2 - 32x$ and the polynomial $x^3 + 5x^2 - 32x$ has three real roots. One can see that as a topological group $\mathcal{E}(\mathbb{R})$ is a product $S^1 \oplus \mathbb{Z}/2\mathbb{Z}$. (Actually, this isomorphism can be constructed applying theory of elliptic functions and identifies $\mathbb{E}(\mathbb{C})$ with the quotient group $\mathbb{C}/\Lambda$ for some lattice $\Lambda$. Then real points can be described as points preserved by complex conjugation. Under this identification multiplication of points is described as the addition of complex numbers modulo $\Lambda$. So we obtain usual addition of point of circle $S^1$ or on $S^1 \oplus \mathbb{Z}/2\mathbb{Z}$ for any elliptic curve. Details of the constructions can be found in [15, Ch. VI].) Any of the two circles has a rational point (for example, $O$ and $D$). Also there is a rational point $A$ of infinite order. Therefore the set of points $\{nA\} \sqcup \{nA + D\}$ consists of rational points and is dense in $\mathcal{E}(\mathbb{R})$ in Euclidean topology (because it is obvious for $S^1 \oplus \mathbb{Z}/2\mathbb{Z}$). □

*Remark* 4.8. Actually, we do not use any specific properties of our elliptic curve. We prove that if a curve has a $\mathbb{Q}$-point of infinite order, then in any Euclidean neighborhood of any $\mathbb{Q}$-point there is another $\mathbb{Q}$-point. This Theorem is a particular case of a known Theorem (see [16, Ch. V, Satz 11]), which says that if there is a point of infinite order, then there are infinitely many rational points in every neighbourhood of any one of them. In our case it is enough to find a point of infinite order and one Sharygin triangle to prove that there are infinitely many Sharygin triangles.

### 4.3. Rank

The *rank* of an elliptic curve $E$ over the field $\mathbb{Q}$ is defined as the rank of it as an abelian group. Unlike the torsion subgroup, there is no known algorithm to calculate the rank of any elliptic curve. But some results are known and allow to find ranks of some curves. One of them is related to the Hasse–Weil function defined below. To give the definition, we need to introduce some notations.

For the elliptic curve $E$ defined as $y^2 = x^3 + ax + b$ for $a, b \in \mathbb{Q}$ we can change the variables $x$ and $y$ in such a way that the curve $E$ would be defined as $y^2 = x^3 + a'x + b'$ for $a', b' \in \mathbb{Z}$. Indeed, if $d$ is a common denominator of $a$ and $b$, then we can replace $x \mapsto d^2 x$ and $y \mapsto d^3 y$.

Suppose that the elliptic curve $E$ is defined as $y^2 = x^3 + ax + b$ for $a, b \in \mathbb{Z}$. Then we can consider its reduction modulo $p$ for each prime number $p$. If $p$ does not divide $\Delta$, then the group homomorphism arises:

$$E(\mathbb{Q}) \to E(\mathbb{F}_p).$$

For the proof and details see [12].

**Definition 4.9.** For each prime number $p$ not dividing $\Delta$ denote by $N_p$ the number of points on the curve $E(\mathbb{F}_p)$, i.e., the number of pairs $(x, y)$, where

$0 \leqslant x, y \leqslant p-1$, such that

$$y^2 \equiv x^3 + ax + b \mod (p).$$

**Definition 4.10.** Define the *Hasse-Weil L-function* of $E$, a function in complex variable $s$, by

$$L(E,s) = \prod_{p \nmid \Delta} \left(1 - \frac{1+p-N_p}{p^s} + \frac{p}{p^{2s}}\right)^{-1} \times \prod_{p \mid \Delta} \ell_p(E,s)^{-1},$$

where $\ell_p(E,s)$ is a certain polynomial in $p^{-s}$ such that $\ell_p(E,1) \neq 0$ (see [17, p. 196]).

From the estimation

$$p + 1 - 2\sqrt{p} \leqslant N_p \leqslant p + 1 + 2\sqrt{p}$$

proved by Hasse (see [5, 6]) it follows that $L(E,s)$ converges absolutely and uniformly on compact subsets of the half-plane $\{\mathrm{Re}(s) > 3/2\}$. It was proved by Breuil, Conrad, Diamond, Taylor and Wiles (see [2, 18, 20]) that $L(E,s)$ has an analytic continuation to $\mathbb{C}$. It turns out that the behavior of $L(E,s)$ at $s = 1$ is related to the rank of $E$ in the following way. Define the *analytical rank* $\mathrm{rk}_{an}(E)$ as the order of vanishing of $L(E,s)$ at $s = 1$.

**Theorem 4.11** (see [3, 7, 8])**.**
- *If* $\mathrm{rk}_{an}(E) = 0$*, then* $\mathrm{rk}(E) = 0$*,*
- *If* $\mathrm{rk}_{an}(E) = 1$*, then* $\mathrm{rk}(E) = 1$*.*

**Theorem 4.12.** *The rank of the curve* $\mathcal{E}(\mathbb{Q})$ *equals* 1.

It can be checked by the `pari/gp` computer algebra system that the Hasse-Weil $L$-function has order 1 at $s = 1$. It has the form

$$L(E,s) = s \cdot 0.67728489801666901020123734615355993155... + O(s^2).$$

By 4.12 it implies that the curve has rank 1. But this method uses some computer algorithm that is not well known. We give below a method to check this in a way that does not use complicated computer algorithms and applies more complicated geometrical methods instead.

The corresponding method is called 2-*descent* and is related to computation of weak Mordell–Weil groups of elliptic curves. We will not give proofs and details of the underlying constructions (they all can be found in [15]), but briefly recall some basic notation needed for the algorithm and proceed the algorithm for our elliptic curve.

Take an elliptic curve $E$ defined over a number field $K$. We do not specify the field here, because we should take some extension of $\mathbb{Q}$ below. Denote

- by $E(K)$ the set of $K$-points on $E$,
- by $mE(K)$ the subgroup of points $[m]A$ for all $A \in E(K)$,
- by $E[m] \simeq \left(\frac{\mathbb{Z}}{m\mathbb{Z}}\right)^2$ the subgroup of points of order $m$ over the algebraic closure $\bar{K}$ of $K$.

The main idea is to find an embedding of the group $E(K)/mE(K)$ to some group and then find the preimages of elements in $E(K)$. If we know the image of $E(K)/mE(K)$ and the torsion subgroup of $E(K)$, then we can calculate the rank of $E$.

Denote by $S$ the set of all valuations of $K$ including all non-archimedian, dividing $m$ and all bad reductions of $E$. Then we can define the group

$$K(S,m) = \{b \in K^\times/(K^\times)^m : \mathrm{ord}_v(b) \vdots m, v \notin S\}.$$

From now on, we restrict ourselves with the case $m = 2$. Below, a morphism

$$E(K)/mE(K) \to K(S,m) \times K(S,m)$$

is defined as follows:

$$P = (x,y) \mapsto \begin{cases} (x-e_1, x-e_2), & x \neq e_1, e_2 \\ \left(\frac{e_1-e_3}{e_1-e_2}, e_1-e_2\right), & x = e_1, \\ \left(e_2-e_1, \frac{e_2-e_3}{e_2-e_1}\right), & x = e_2, \\ (1,1), & x = \infty. \end{cases}$$

Here $P = (x,y)$ is a point on the elliptic curve in the Weierstraß form

$$y^2 = (x-e_1)(x-e_2)(x-e_3).$$

All the constructions leading to this morphism can be found in [15] with an example of elliptic curve and the computation. In our case the equation has the form

$$y^2 = x^3 + 5x^2 - 32x = x\left(x + \frac{5+3\sqrt{17}}{2}\right)\left(x + \frac{5-3\sqrt{17}}{2}\right).$$

Some technical details of the construction require that three points $e_1, e_2, e_2$ must be defined over $K$. So, in our case we need to extend $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{17})$. This makes some computations more difficult than in the example given in [15]. The computations below illustrate this, but in our case they are not very large and show where the difficulties appear. Also this illustrates how the initial problem of classification of Sharygin triangles leads to computations in finite fields $\mathbb{F}_q$, fields of $p$-adic numbers $\mathbb{Q}_p$ and their extensions.

A point $(b_1, b_2) \in K(S,2) \times K(S,2)$ is the image of a point

$$P = (x,y) \in E(K)/2E(K)$$

not coinciding with $O$, $(e_1, 0)$, $(e_2, 0)$ if and only if the system of equations

$$b_1 z_1^2 - b_2 z_2^2 = e_2 - e_1,$$
$$b_1 z_1^2 - b_1 b_2 z_3^2 = e_3 - e_1$$

has a solution $(z_1, z_2, z_3) \in K^\times \times K^\times \times K$. (We omit the calculation leading to this form.) If such a solution exists, we can take

$$P = (x,y) = (b_1 z_1^2 + e_1, b_1 b_2 z_1 z_2 z_3).$$

We put

$$e_1 = 0,\; e_2 = -\frac{5+3\sqrt{17}}{2},\; e_3 = -\frac{5-3\sqrt{17}}{2}.$$

Over $\mathbb{Q}$ the set $S$ equals $\{2, 3, 17, \infty\}$. The group of unities of $\mathbb{Q}(\sqrt{17})$ is $\langle -1\rangle_2 \oplus \langle 4+\sqrt{17}\rangle_\infty$. Therefore the group of unites modulo squares has the set of representatives $\{\pm 1, \pm(4+\sqrt{17})\}$. Over $\mathbb{Q}(\sqrt{17})$ the number 2 splits as $\frac{5+\sqrt{17}}{2} \cdot \frac{5-\sqrt{17}}{2}$, 3 remains prime, 17 is the square of $\sqrt{17}$. Denote $\tilde{\imath} = 4+\sqrt{17}$ and $2_\pm = \frac{5\pm\sqrt{17}}{2}$. Denote $\overline{a+\sqrt{17}b} = a - \sqrt{17}b$ the image of the unique nontrivial automorphism of $\mathbb{Q}(\sqrt{17})$ over $\mathbb{Q}$. Therefore, we can choose the set of representatives of $K(S, 2)$ each of them is a product of some of numbers in the set

$$\left\{-1, \tilde{\imath}, 2_+, 2_-, 3, \sqrt{17}\right\}.$$

So the set of all pair $(b_1, b_2)$ in $K(S, 2)$ consists of $(2^6)^2 = 4096$ pairs. We will see below that the system

$$\begin{aligned} b_1 z_1^2 - b_2 z_2^2 &= e_2, \\ b_1 z_1^2 - b_1 b_2 z_3^2 &= e_3 \end{aligned} \tag{3}$$

has a solution $(z_1, z_2, z_3) \in K^\times \times K^\times \times K$ with $x, y \in \mathbb{Q}$ only for the values $b_1$ and $b_2$ listed in the following table:

| | | | | |
|---|---|---|---|---|
| $b_1$ | 1 | $-2$ | $-1$ | 2 |
| $b_2$ | 1 | $\tilde{\imath} \cdot 2_-$ | $\tilde{\imath} \cdot 3$ | $2_- \cdot 3$ |

The rest of the proof is analogous to the proof in example given in [15], but it is some bit more complicated due to the difficulties arising from the field extension and prime decomposition in its ring of integral elements. Proceeding systematically, we list our results in the following table:

<table>
<tr><th>$b_2 \backslash b_1$</th><th>1</th><th>$-1$, $\pm 2$</th><th>$\tilde{\imath}$, $2_\pm$</th><th>3</th><th>$\sqrt{17}$</th></tr>
<tr><td>1</td><td>$O$①</td><td rowspan="5">×⑦</td><td rowspan="6">$\mathbb{Q}$⑤</td><td rowspan="6">$\mathbb{Q}_3(i)$④</td><td rowspan="6">$\mathbb{Q}_{17}(\sqrt{17})$③</td></tr>
<tr><td>$\tilde{\imath}$</td><td>$\mathbb{Q}_2$⑪</td></tr>
<tr><td>3</td><td>$\mathbb{Q}_3(i)$⑩</td></tr>
<tr><td>$2_+$</td><td>$\mathbb{Q}(\sqrt{17})$⑨</td></tr>
<tr><td>$\sqrt{17}^*$</td><td>$\mathbb{R}$⑧</td></tr>
<tr><td>$2_-^*$</td><td colspan="2">$\mathbb{Q}_2$⑥</td></tr>
<tr><td>$-1$</td><td colspan="5">$\mathbb{R}$②</td></tr>
</table>

On the first step we find $z_1, z_2, z_3$ for $b_1$ and $b_2$ listed in the table above. On other steps we consequently exclude possible factors of $b_1$ and $b_2$. Also on the fifth step we prove that multiplicities of $2_-$ and $2_+$ in $b_1$ are equal, on seventh step we deduce the cases $b_1 \in \{\pm 1, \pm 2\}$ to the case $b_1 = 1$ using the homomorphism property of $f$. It remains to proceed this sequence step by step.

① Let us find the solution for the cases listed above and the corresponding points on the elliptic curve.

(1) Take $(b_1, b_2) = (1, 1)$. Note that $\overline{e_2} = e_3$. Therefore, if we assume $z_1 \in \mathbb{Q}$ and find $z_1, z_2$ such that $z_1^2 - z_2^2 = e_2$, then it is enough to take $z_3 = \overline{z_2}$. We can take $z_1 = 2$, $z_2 = \frac{3+\sqrt{17}}{2}$. This choice corresponds to the point $P = (4, 4) = 2A$.

(2) Take $(b_1, b_2) = (-2, \tilde{\imath} \cdot 2_-)$. We can take $z_1 = 2$, $z_2 = \frac{5-\sqrt{17}}{2}$, $z_3 = -\frac{1+\sqrt{17}}{4}$ as a solution of

$$\begin{cases} -2z_1^2 - \frac{3+\sqrt{17}}{2} z_2^2 = -\frac{5+3\sqrt{17}}{2}, \\ -2z_1^2 + (3+\sqrt{17}) z_3^2 = -\frac{5-3\sqrt{17}}{2}. \end{cases}$$

This gives the point $P = (-8, 8) = 2A + D$.

(3) Take $(b_1, b_2) = (-1, \tilde{\imath} \cdot 3)$. We can take $z_1 = 1$, $z_2 = \frac{-3+\sqrt{17}}{2}$, $z_3 = \frac{5-\sqrt{17}}{4}$ as a solution of

$$\begin{cases} -z_1^2 - 3(4+\sqrt{17}) z_2^2 = -\frac{5+3\sqrt{17}}{2}, \\ -z_1^2 + 3(4+\sqrt{17}) z_3^2 = -\frac{5-3\sqrt{17}}{2}. \end{cases}$$

This gives the point $P = (-1, 6) = 3A$.

(4) Take $(b_1, b_2) = (2, 2_- \cdot 3)$. We can take $z_1 = 2$, $z_2 = \frac{3\sqrt{17}}{2}$, $z_3 = -\frac{1+\sqrt{17}}{4}$ as a solution of

$$\begin{cases} 2z_1^2 - 3\frac{5+\sqrt{17}}{2} z_2^2 = -\frac{5+3\sqrt{17}}{2}, \\ 2z_1^2 + 3(5+\sqrt{17}) z_3^2 = -\frac{5-3\sqrt{17}}{2}. \end{cases}$$

This gives the point $P = (8, -24) = A + D$.

② Either $b_1 > 0$, or $b_1 < 0$:

(1) if $b_1 > 0$ and $b_2 < 0$, then the equation $b_1 z_1^2 - b_2 z_2^2 = e_2 < 0$ has no solution over $\mathbb{R}$;

(2) if $b_1 < 0$ and $b_2 < 0$, then the equation $b_1 z_1^2 - b_1 b_2 z_3^2 = e_3 > 0$ has no solution over $\mathbb{R}$.

This excludes the factor $-1$ from $b_2$, because $\tilde{\imath}, 2_\pm, 3, \sqrt{17} > 0$.

③ Consider the ramified field extension $\mathbb{Q}_{17} \subset \mathbb{Q}_{17}(\sqrt{17})$. The valuation $\operatorname{ord}_{17}$ can be continued to the extension and has there half-integer values. Here we exclude the factor $\sqrt{17}$ from $b_1$ (all others has zero $\operatorname{ord}_{17}$). Assume that $\operatorname{ord}_{17} b_1 = 1/2$, i.e., the factor $\sqrt{17}$ does contribute to $b_1$. Either $\operatorname{ord}_{17} b_2 = 0$, or $\operatorname{ord}_{17} b_2 = 1/2$:

(1) Suppose $b_1 \vdots \sqrt{17}$ and $b_2 \not\vdots \sqrt{17}$. The expression $b_1 z_1^2 - b_2 z_2^2 = e_2$ has 17-order 0, because $\operatorname{ord}_{17}\left(-\frac{5+3\sqrt{17}}{2}\right) = 0$. Since $\operatorname{ord}_{17} z_i \in \frac{1}{2}\mathbb{Z}$, we have $\operatorname{ord}_{17} z_i^2 \in \mathbb{Z}$, $i = 1, 2$. Therefore $b_1 z_1^2$ and $b_2 z_2^2$ has different 17-orders. Their sum has order 0. Therefore $z_1$ and $z_2$ are integral

in $\mathbb{Q}_{17}(\sqrt{17})$. Consider the second equation:

$$b_1 z_1^2 - b_1 b_2 z_3^2 = \tilde{\imath}^{-3} 2_+^5 = e_3.$$

Right hand side has 17-order 0. In the left hand side the first summand has order 1/2 and the second summand has half-integer 17-order. Either the last is positive or negative, anyway the equation has no solutions.

(2) Suppose $b_1, b_2 \not\vdots \sqrt{17}$. From

$$b_1 z_1^2 - b_1 b_2 z_3^2 = e_3 = \tilde{\imath}^{-3} \cdot 2_+^5$$

in the same way it follows that $z_1$ and $\sqrt{17} z_3$ are integral in $\mathbb{Q}_{17}(\sqrt{17})$. In the equation

$$b_1 z_1^2 - b_2 z_2^2 = -\tilde{\imath}^3 \cdot 2_-^5$$

the right hand side has order 0, in the left hand side the first summand has positive order and the second summand has half-integer order. Therefore there are no solutions.

Therefore $\sqrt{17}$ does not contribute to $b_1$.

④ Consider the unramified field extension $\mathbb{Q}_3 \subset \mathbb{Q}_3(i)$. Since $x^2 = 17$ has a root in the residue field $\mathbb{F}_3(i) = \mathbb{F}_9$, by Hensel's lemma it has a root in $\mathbb{Q}_3(i)$. The valuation $\operatorname{ord}_3$ has integer values on the extension. The following reasoning literally repeats the previous case, except the half-integer and integer valuations are replaced with odd and even integer valuations. Therefore 3 does not contributes to $b_1$.

⑤ Consider the valuations $\nu_-$ and $\nu_+$ at $2_-$ and $2_+$. They both are even for $z_1^2$. For $b_1$ they have the values in $\{0, 1\}$, and $b_1 z_1^2 \in \mathbb{Q}$, i.e., they coincide. In particular, their parities coincide. Therefore, they coincide for $b_1$.

It is well known that the integral domain of $\mathbb{Q}(\sqrt{17})$ is Euclidean and therefore an UFD. Any element there is a product of some irreducible elements and a unity. The element $b_1$ is a product of some subset in $\{-1, 2, \tilde{\imath}\}$. Therefore, the element $b_1 z_1^2$ is not a square under the projection into the unity subgroup, because $\tilde{\imath}$ is not a square there.

Hence, $\tilde{\imath}$ does not contribute to $b_1$ and $\operatorname{ord}_{2_-} b_1 = \operatorname{ord}_{2_+} b_1$. Therefore, $b_1 \in \mathbb{Q}$.

⑥ Here we prove that the prime $2_-$ has the same degree at $b_1$ and $b_2$. Consider the field $\mathbb{Q}_2$. The root $\sqrt{17}$ can be extracted there, because a solution of $x^2 = 17$ exists over the residue field $\mathbb{F}_2$ and a solution over $\mathbb{Q}_2$ exists by Hensel's lemma. (We need to note here that the general version of Hensel's Lemma is not applicable here, and we need to use more general one. Actually, there are *four* solution of $x^2 = 17$

modulo $2^k$ for $k \geqslant 3$. But there are only *two* 2-adic limits.) Therefore, $\mathbb{Q}(\sqrt{17}) \subset \mathbb{Q}_2$. Actually, $\sqrt{17}$ can be written as

$$\sqrt{17} = ...10011011101001$$

or

$$\sqrt{17} = ...01100100010111.$$

We fix the embedding corresponding to the first choice of $\sqrt{17}$ and consider the restriction of the valuation $\operatorname{ord}_2$ to $\mathbb{Q}(\sqrt{17})$. Note that this valuation on $\mathbb{Q}(\sqrt{17})$ is not Galois-invariant and depends on the embedding. Here $\sqrt{17} = 5 + O(8)$. Therefore,

$$\frac{5+\sqrt{17}}{2} = \frac{14+O(8)}{2} = 3 + O(4),$$

$$\frac{5-\sqrt{17}}{2} = \frac{-2+O(8)}{2} = 2 + O(4).$$

In other word, $\operatorname{ord}_2 2_- = 1$ and $\operatorname{ord}_2 2_+ = 0$. Also

$$e_2 = -\frac{5+3\sqrt{17}}{2} = -\frac{5+3(105+O(128))}{2} = 32 + O(64),$$

$$e_2 = -\frac{5-3\sqrt{17}}{2} = -\frac{5-3(105+O(128))}{2} = 27 + O(64).$$

Suppose the degrees of $2_-$ in $b_1$ and $b_2$ differ. Then on of the following cases holds:

(1) Let $b_1 \vdots 2_-$ and $b_2 \not\vdots 2_-$. Since $\operatorname{ord}_2 b_1 z_1^2$ is odd and $\operatorname{ord}_2 b_2 z^2$ is even, the equation

$$b_1 z_1^2 - b_2 z_2^2 = e_2 = 2^5 + O(2^6)$$

implies that $z_1$ and $z_2$ are integral in $\mathbb{Q}_2$, and $\operatorname{ord}_2 z_1 = 2$. In the equation

$$b_1 z_1^2 - b_1 b_2 z_3^2 = e_3$$

we have $\operatorname{ord}_2 b_1 z_1^2 = 5$, and $\operatorname{ord}_2 b_1 b_2 z_3^2$ is odd, and $\operatorname{ord}_2 e_3 = 0$. Therefore, there are no solutions.

(2) Let $b_1 \not\vdots 2_-$ and $b_2 \vdots 2_-$. Since $\operatorname{ord}_2 b_1 z_1^2$ is even and $\operatorname{ord}_2 b_2 z^2$ is odd, the equation

$$b_1 z_1^2 - b_2 z_2^2 = e_2 = 2^5 + O(2^6)$$

implies that $z_1$ and $z_2$ are integral in $\mathbb{Q}_2$, and $\operatorname{ord}_2 z_1 \geqslant 3$. In the equation

$$b_1 z_1^2 - b_1 b_2 z_3^2 = e_3$$

we have $\operatorname{ord}_2 b_1 z_1^2 \geqslant 6$, and $\operatorname{ord}_2 b_1 b_2 z_3^2$ is odd, and $\operatorname{ord}_2 e_3 = 0$. Therefore, there are no solutions.

⑦ Recall that correspondence $E(\mathbb{Q})/2E(\mathbb{Q}) \to K(S,2)\times K(S,2)$ taking $P$ to $(b_1(P), b_2(P))$ is a homomorphism. We prove below that for $b_1 = 1$ system (3) has a solution only for $b_2 = 1$. Therefore, for each $b_1$ there is at most one $b_2$ such that system (3) has a solution. In ① we have constructed these solutions. Hereafter we consider only $b_1 = 1$ and the system

$$\begin{cases} z_1^2 - b_2 z_2^2 = e_2, \\ z_1^2 - b_2 z_3^2 = e_3 = \overline{e_2}. \end{cases} \tag{4}$$

⑧ We have $x = z_1^2 \in \mathbb{Q}$. Therefore $\overline{b_2 z_2^2} = b_2 z_3^2$ and $z_3/\overline{z_2} = \pm\sqrt{\overline{b_2}/b_2} \in \mathbb{Q}(\sqrt{17})$. If $\overline{b_2}/b_2$ is not a square in $\mathbb{Q}(\sqrt{17})$, then there are no solutions of (4). The expression $\overline{b_2}/b_2$ is multiplicative in $b_2$ and is positive for $b_2 = 1, 3, 2_+$ and is negative for $b_2 = \tilde{\imath}, \sqrt{17}$. Since $\mathbb{Q}(\sqrt{17}) \subset \mathbb{R}$, the degrees of $\sqrt{17}$ and $\tilde{\imath}$ in $b_2$ coincide.

⑨ Consider again the expression $B = \overline{b_2}/b_2$ that must be an element of $\mathbb{Q}\sqrt{17}$, if there are a solution of (4). Multiplication of $b_2$ with 3 does not change $B$, multiplication with $\tilde{\imath}$ divides $B$ by $\tilde{\imath}^2$. Therefore, we need only to consider $b_2 = 2_+$ and $b_2 = 2_+\sqrt{17}$ to exclude the factor $2_+$ from $b_2$. Take $b_2 = 2_+$. One can see that the minimal polynomial of $B = \sqrt{\overline{b_2}/b_2} = \sqrt{\frac{5-\sqrt{17}}{5+\sqrt{17}}}$ is $2x^4 - 21x^2 + 2$. Therefore $[\mathbb{Q}(B) : \mathbb{Q}] = 4$, and $B$ can not lie in any quadratic extension of $\mathbb{Q}$, in particular, $\mathbb{Q}(\sqrt{17})$.

For $b_2 = 2_+\sqrt{17}$ we get the minimal polynomial $2x^2 + 21x^2 + 2$. In the same way $B \notin \mathbb{Q}(\sqrt{17})$.

⑩ It remains to consider $b_2 = 3^{\varepsilon_1}\tilde{\imath}^{\varepsilon_2}$, where $\varepsilon_1, \varepsilon_2 = 0, 1$. If a pair $(z_1, z_2)$ is a solution of $z_1^2 - b_2 z_2^2 = e_2$, then the triple $(z_1, z_2, z_3 = \tilde{\imath}^{-\varepsilon_2}\overline{z_2})$ is a solution of (4). Therefore $b_2 z_2 z_3 = 3^{\varepsilon_1}(\tilde{\imath}\sqrt{17})^{\varepsilon_2} z_2 (-\tilde{\imath}\sqrt{17})^{-\varepsilon_2}\overline{z_2} = (-1)^{\varepsilon_2} 3^{\varepsilon_1} z_2 \overline{z_2} \in \mathbb{Q}$. In the same time $y = b_2 z_1 z_2 z_3 \in \mathbb{Q}$. This implies that $z_1 \in \mathbb{Q}$.

Since $x^2 = 17$ has a non-zero root in $\mathbb{F}_3(i) = \mathbb{F}_9$, we have an embedding $\mathbb{Q}(\sqrt{17}) \subset \mathbb{Q}_3(i)$. The valuation $\operatorname{ord}_3$ can be extended from $\mathbb{Q}_3$ to $\mathbb{Q}_3(i)$ and has there integer values. We have $z_1^2 - 3(\tilde{\imath}\sqrt{17})^{\varepsilon_2} z_2^2 = e_2$. Here $\operatorname{ord}_3 e_2 = \operatorname{ord}_3 \tilde{\imath} = \operatorname{ord}_3 \sqrt{17} = 0$. Therefore in the residue field $\mathbb{F}_9$ we obtain $z_1^2 \equiv e_2$.

Denote $\sqrt{17} = i$ in $\mathbb{F}_9$. Then $e_2 = -\frac{5-3\sqrt{17}}{2} = -\frac{5-3i}{2} = 2 + O(3)$ in $\mathbb{Q}_3(i)$. As above, $z_1$ and $z_2$ are integral in $\mathbb{Q}_3(i)$. Therefore, $z_1^2 \equiv 2$ in $\mathbb{F}_9$. But it was proved above that $z_1 \in \mathbb{Q}$. This means that its square is not 2 modulo 3.

⑪ It only remains to check the case $b_2 = \tilde{\imath}\sqrt{17}$. Existence of solution $(z_1, z_2) \subset K^\times \times K^\times$ is equivalent to existence of a point defined over $\mathbb{Q}(\sqrt{17})$ on the plane conic

$$z_1^2 - \tilde{\imath}\sqrt{17}z_2^2 = e_2 = -\tilde{\imath}^3 2_-^5.$$

At first, we can projectivize this conic multiplying the right hand side with $\tilde{\imath}^{-2}2_-^{-4}z_4^2$. Therefore, we can consider the equation

$$\tilde{\imath}z_5^2 + 2_- z_4^2 = \sqrt{17},$$

where $z_5 = \tilde{\imath}z_1$. As usual, we obtain that $z_4$ and $z_5$ are integral in $\mathbb{Q}_2$ if we take the embedding $\mathbb{Q}(\sqrt{17})$ mentioned above. We have $\sqrt{17} = 1 + O(8)$ and $\tilde{\imath} = 1 + O(8)$. Therefore, $\operatorname{ord}_2 z_4 > 0$, and $\operatorname{ord}_2 2_- z_4^2 \geqslant 3$.

Consequently, $z_5^2 = \frac{1+O(8)}{5+O(8)} = 5+O(8)$, where $z_5$ is an integral 2-adic numbed. Taking it modulo 8, we get $z_1^2 \equiv 5 \mod 8$. But 5 is not a square in $\mathbb{Z}/8\mathbb{Z}$. Therefore the equation has no solutions.

**Conclusion 4.13.** $\mathcal{E}(\mathbb{Q}) = \mathbb{Z} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$.

## Appendix A. Examples

The curve $\mathcal{E}$ can be written in some different coordinates:

(1) $y^2 + y = x^3 + x^2 - 2x$ (minimal form),
(2) $y^2 = x^3 + 5x^2 - 32$ (the form we use in 2-descent algorithm),
(3) $c^3 + c^2(a+b) = c(a^2 + ab + b^2) + a^3 + a^2b + ab^2 + b^3$ (initial form).

In the following table some points of $\mathcal{E}(\mathbb{Q})$ are listed:

| pt | 1 | 2 | 3 |
|---|---|---|---|
| $O$ | $(0:1:0)$ | | $(1:-1:0)$ |
| $D$ | $(0:0:1)$ | | $(-1:-1:1)$ |
| $A$ | $(-1:-1:1)$ | $(-4:-12:1)$ | $(1:0:-1)$ |
| $A+D$ | $(2:-4:1)$ | $(8:-24:1)$ | $(1:-1:1)$ |
| $2A$ | $(1:0:1)$ | $(4:4:1)$ | $(0:1:1)$ |
| $2A+D$ | $(-2:2:1)$ | $(-8:8:1)$ | $(1:3:-5)$ |
| $3A$ | $(-2:7:8)$ | $(-1:6:1)$ | $(1:5:-4)$ |
| $3A+D$ | $(8:20:1)$ | $(32:192:1)$ | $(-9:7:5)$ |
| $4A$ | $(9:-33:1)$ | $(36:-228:1)$ | $(25:-32:17)$ |
| $4A+D$ | $(-6:-16:27)$ | $(-24:-152:27)$ | $(49:11:-39)$ |
| $5A$ | $(-245:-14:125)$ | $(-980:-1092:125)$ | $(128:37:-205)$ |
| $5A+D$ | $(350:-370:343)$ | $(1400:-1560:343)$ | $(121:-9:119)$ |
| $6A$ | $(968:913:512)$ | $(484:1397:64)$ | $(-1369:1425:1424)$ |
| $6A+D$ | $(-1408:2736:1331)$ | $(-5632:16256:1331)$ | $(3:4067:-4147)$ |
| $7A$ | $(-37:1955:50653)$ | $(-148:15492:50653)$ | $(16245:17536:-16909)$ |
| $7A+D$ | $(2738:141932:1)$ | $(10952:1146408:1)$ | $(-48223:47311:1825)$ |
| $8A$ | $(392673:-808088:185193)$ | $(1570692:-4894012:185193)$ | $(600608:-622895:600153)$ |
| $8A+D$ | $(-539334:-570570:571787)$ | $(-2157336:-6721896:571787)$ | $(1681691:1217:-1650455)$ |
| $9A$ | $(-41889394:39480443:20570824)$ | $(-20944697:18535746:2571353)$ | $(7659925:20017089:-34783204)$ |
| $9A+D$ | $(58709432:-3376228:59776471)$ | $(234837728:207827904:59776471)$ | $(1481089:18800081:19214131)$ |

Recall that for given point $Z = (x : y : z)$ the point $-Z$ is $(x : z - y : z)$ in the first case and is $(x : -y : z)$ in the second case, for the point $Z = (a : b : c)$ in the third case the point $-Z$ is $(b : a : c)$.

Let us find some of integer triangles. Minimal among them are the following.

- given in the introduction triangle corresponding to $9A + D$:

$$(83^2 \cdot 2729, 1217^2, 17 \cdot 23 \cdot 157 \cdot 313).$$

- the triangle corresponding to $16A$:

$$(2^5 \cdot 5^2 \cdot 17 \cdot 23 \cdot 137 \cdot 7901 \cdot 943429^2,\\ 29^2 \cdot 37 \cdot 1291 \cdot 3041^2 \cdot 11497^2,\\ 3 \cdot 19 \cdot 83 \cdot 2593 \cdot 14741 \cdot 227257 \cdot 7704617).$$

- the triangle corresponding to $23A + D$:

$$(5 \cdot 17 \cdot 29 \cdot 97 \cdot 17182729 \cdot 32537017 \cdot 254398174040897 \cdot 350987274396527,\\ 7 \cdot 1093889^2 \cdot 4941193 \cdot 894993889^2 \cdot 331123185233,\\ 83^2 \cdot 571^2 \cdot 13873 \cdot 337789537 \cdot 16268766383521^2).$$

- the triangle corresponding to $30A$:

$$(3 \cdot 19 \cdot 83 \cdot 347 \cdot 853^2 \cdot 14741 \cdot 197609 \cdot 1326053 \cdot 9921337^2 \cdot 2774248223\\ \times 16439698126501721^2,\\ 37 \cdot 53 \cdot 113 \cdot 1291 \cdot 6301^2 \cdot 11057 \cdot 70717^2 \cdot 419401^2\\ \times 56702749^2 \cdot 75758233^2 \cdot 58963203163,\\ 2^4 \cdot 5 \cdot 7 \cdot 13 \cdot 281 \cdot 1361 \cdot 4519 \cdot 943429 \cdot 1277496791 \cdot 58636722172129\\ \times 434222192069971469300337687991080717947321).$$

- the triangle corresponding to $37A + D$:

$$(5 \cdot 2299159 \cdot 138049208211121 \cdot 2760426916410799 \cdot 728165182513369014929\\ \times 245724452275360800414766 9717\\ \times 364631251477476883895926270727199434262 7321,\\ 3^6 \cdot 41 \cdot 43^2 \cdot 59^2 \cdot 71 \cdot 1753^2 \cdot 4271 \cdot 6449^2 \cdot 306193^2 \cdot 258408497^2\\ \times 294583400141651^2 \cdot 59171155940313829798393591825074372 87191,\\ 7^2 \cdot 79 \cdot 3529 \cdot 2812999081^2 \cdot 5544800297^2 \cdot 16078869119\\ \times 138608471911744191743 77^2 \cdot 306179686612030303942777).$$

- the following triangles correspond to $44A$, $51A{+}D$, $58A$, $65A{+}D$, $72A$, $79A + D$, $86A$, $93A + D$, $100A$, $107A + D$, $114A$, $121A + D$, $132A, \ldots$.

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