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ON THREE-DIMENSIONAL SEMI-TERMINAL SINGULARITIES

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Abstract. We classify three-dimensional non-normal semi-terminal singularities.

1. Introduction

The notion of terminal singularities is very important in the minimal model program. For the two-dimensional case, the notion of terminal singularities is equivalent to the notion of smoothness. Three-dimensional terminal singularities are understood by explicit equations and was given by [8] and the sufficiency of the conditions was checked in [6].

On the other hand, the importance of the class of certain non-normal varieties, which are called demi-normal varieties (see Definition 2.2), has been well-understood (see [4, §5]). For example, it is natural to allow semi-log-canonical singularities, that is, demi-normal with a log-canonicity condition, in order to consider families of canonically polarized varieties (see [6]). In [2], the author introduced the notion of semi-terminal singularities (see Definition 2.3) which is a natural generalization of terminal singularities. It is important to consider the notion of semi-terminal singularities since the author proved in [2] that there exists a semi-terminal modification for *any* demi-normal pair. However, it has not been known so much about semi-terminal singularities. In this paper, we classify all of the non-normal three-dimensional semi-terminal singularities.

Theorem 1.1. *Let $0 \in X$ be a three-dimensional non-normal semi-terminal singularity. Then $0 \in X$ is analytically isomorphic to one of the following singularities*:

(1) *Double normal crossing point, that is,* $0 \in (x_1x_2 = 0) \subset \mathbb{A}^4$.
(2) *Pinch point, that is,* $0 \in (x_1^2 - x_2^2x_3 = 0) \subset \mathbb{A}^4$.

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(3) 2-*twirl point, that is,* $0 \in (x_1x_3 - x_2^2 = x_1x_4^2 - x_5^2 = x_2x_4^2 - x_5x_6 = x_3x_4^2 - x_6^2 = 0) \subset \mathbb{A}^6$.

Remark 1.2. Both double normal crossing point and pinch point are hypersurface singularities. Thus both are Gorenstein. However, as we will see in Section 6, for a 2-twirl point $0 \in X$, X is not Gorenstein but $2K_X$ is Cartier. A general element $0 \in S \in |-K_X|$ has a pinch point at $0 \in S$, the index 1 cover $\pi\colon \tilde{X} \to X$ of $0 \in X$ is double normal crossing, and π^*S is double normal crossing. See Example 2.7, Remark 2.9 and Section 6 in detail.

Now we organize the strategy of the proof of Theorem 1.1. The strategy is similar to the earlier works in [7–10]. For a demi-normal variety X, it is natural to consider its normalization $\bar{X}$, the conductor divisor $D_{\bar{X}}$ of $\bar{X}/X$ and the involution $\iota_X\colon \bar{D}_{\bar{X}} \to \bar{D}_{\bar{X}}$ obtained by the natural double cover, where $\bar{D}_{\bar{X}}$ is the normalization of $D_{\bar{X}}$. In fact, the study of demi-normal varieties X can reduced to the study of such $(\bar{X}, D_{\bar{X}})$ and $\iota_X\colon \bar{D}_{\bar{X}} \to \bar{D}_{\bar{X}}$ by [4, §9]. From Section 3 to Section 4, we consider germs $0 \in (\bar{X}, \bar{D})$ of normal pairs in place of considering non-normal singularities. For a germ $0 \in (\bar{X}, \bar{D})$ of normal semi-terminal pair, by taking the index 1 cover, we reduce to the case that $K_{\bar{X}} + \bar{D}$ is Cartier (see Theorem 4.4). Then a general hypersurface $0 \in S \subset \bar{X}$ satisfies that the germ $0 \in (S, S \cap \bar{D})$ has either canonical singularities or log-elliptic singularities (see Definition 3.1). In Section 3, we analyze log-elliptic singularities. In Section 4, we classify three-dimensional normal semi-terminal pairs with nonzero reduced boundaries. In Section 5, we see how those pairs in Section 4 glue and we prove Theorem 1.1. In Section 6, we see ring-theoretical properties of twirl singularities, which are important examples of higher-dimensional semi-terminal singularities.

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Throughout the paper, we work over the complex number field $\mathbb{C}$. In the paper, a *variety* means a reduced, separated and of finite type scheme over $\mathbb{C}$. For any variety X, the morphism $\nu_X\colon \bar{X} \to X$ denotes the normalization of X. For the minimal model program, we refer the readers to [4] and [5].

2. Preliminaries

We collect some basic definitions and results in this section.

Definition 2.1. (1) Let X be a variety, let $x \in X$ be a closed point, and let $\hat{\mathcal{O}}_{X,x}$ be the formal completion of the local ring $\mathcal{O}_{X,x}$. We say that $x \in X$ is a *double normal crossing* (*dnc*, for short) point if $\hat{\mathcal{O}}_{X,x} \simeq \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1x_2)$; a *pinch* point if $\hat{\mathcal{O}}_{X,x} \simeq \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1^2 - x_2^2x_3)$, respectively.

(2) A variety X is called a *double normal crossing* variety (*dnc* variety, for short) if any closed point $x \in X$ is either a smooth or a dnc point; a *semi-smooth* variety if any closed point $x \in X$ is one of a smooth, a dnc or a pinch point, respectively.

Definition 2.2 ([4, §5.1])**.** (1) Let X be an equi-dimensional variety. We call that X is a *demi-normal* variety if X satisfies Serre's S_2 condition and X is dnc outside codimension 2.

(2) Assume that an equi-dimensional variety X is dnc outside codimension 2. Then there exists a unique finite and birational morphism $d\colon X^d \to X$ such that X^d is a demi-normal variety and the morphism d is an isomorphism in codimension 1 over X. We call the morphism d the *demi-normalization* of X.

(3) Let X be a demi-normal variety and $\nu_X\colon \bar{X} \to X$ be the normalization of X. The *conductor ideal* of X is defined to be $\mathrm{cond}_X := \mathcal{H}om_{\mathcal{O}_X}((\nu_X)_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X) \subset \mathcal{O}_X$. This ideal can be seen as an ideal sheaf $\mathrm{cond}_{\bar{X}}$ on $\bar{X}$. Set

$$D_X := \mathrm{Spec}_X(\mathcal{O}_X/\,\mathrm{cond}_X) \text{ and } D_{\bar{X}} := \mathrm{Spec}_{\bar{X}}(\mathcal{O}_{\bar{X}}/\,\mathrm{cond}_{\bar{X}}).$$

We call the subscheme D_X (resp., $D_{\bar{X}}$) as the *conductor divisor* of X (resp., of $\bar{X}/X$). It has been known that both $D_{\bar{X}}$ and D_X are reduced and of pure codimension 1. Moreover, for the normalization morphism $\nu_{D_{\bar{X}}}\colon \bar{D}_{\bar{X}} \to D_{\bar{X}}$, we get the Galois involution $\iota_X\colon \bar{D}_{\bar{X}} \to \bar{D}_{\bar{X}}$ defined from ν_X unless ν_X is an isomorphism.

Definition 2.3. (1) The pair (X, Δ) is called a *demi-normal pair* if X is a demi-normal variety, Δ is a formal $\mathbb{Q}$-linear sum $\Delta = \sum_{i=1}^k a_i\Delta_i$ of reduced and irreducible closed subvarieties Δ_i of codimension 1 with $\Delta_i \not\subset \mathrm{Supp}\, D_X$ and $a_i \in [0,1] \cap \mathbb{Q}$ for all $1 \le i \le k$. Moreover, if X is normal, then the pair (X, Δ) is called a *normal pair*.

(2) Let (X, Δ) be a demi-normal pair, let $\nu_X\colon \bar{X} \to X$ be the normalization of X, and set $\Delta_{\bar{X}} := (\nu_X)_*^{-1}\Delta$.

(i) [6, Definition 4.17] The pair (X, Δ) is said to be *purely semi-log-terminal* if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and the pair $(\bar{X}, \Delta_{\bar{X}} + D_{\bar{X}})$ is purely log-terminal.

(ii) [6, Definition 4.17] The pair (X, Δ) is said to be *semi-canonical* if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and the pair $(\bar{X}, \Delta_{\bar{X}} + D_{\bar{X}})$ has canonical singularities.

(iii) [2, Definition 2.3] The pair (X, Δ) is said to be *semi-terminal* if the pair (X, Δ) is semi-canonical and for any exceptional prime divisor E over $\bar{X}$ we have the inequality $a(E, \bar{X}, \Delta_{\bar{X}} + D_{\bar{X}}) > 0$ unless $\mathrm{center}_{\bar{X}}\, E \subset \mathrm{Supp}\lfloor \Delta_{\bar{X}} + D_{\bar{X}} \rfloor$ and $\mathrm{codim}_{\bar{X}}(\mathrm{center}_{\bar{X}}\, E) = 2$.

A demi-normal variety X is said to be semi-log-terminal (resp., semi-canonical, semi-terminal) if the demi-normal pair $(X, 0)$ is purely semi-log-terminal (resp., semi-canonical, semi-terminal).

Remark 2.4 ([2, Remark 2.4]). Let us consider a normal pair $(Y, \Delta + S)$ such that $S = \lfloor S \rfloor$.

(1) If $(Y, \Delta + S)$ has canonical singularities, then $\mathrm{Diff}_S \Delta = 0$ and the variety S with the reduced structure has canonical singularities. In particular, S is a normal variety.
(2) If $(Y, \Delta + S)$ is semi-terminal, then the variety S with the reduced structure has terminal singularities.

In particular, for any demi-normal pair (X, Δ), the following holds. (1) If (X, Δ) is semi-canonical, then $\mathrm{Supp}\lfloor \Delta_{\bar{X}} + D_{\bar{X}} \rfloor$ with the reduced structure has canonical singularities. (2) If (X, Δ) is semi-terminal, then $\mathrm{Supp}\lfloor \Delta_{\bar{X}} + D_{\bar{X}} \rfloor$ with the reduced structure has terminal singularities.

Example 2.5. (1) [5, Corollary 2.31] Assume that (X, Δ) is a normal pair such that X is a smooth variety and $\mathrm{Supp}\, \Delta \subset X$ is a (possibly non-connected) smooth divisor. Then (X, Δ) is semi-terminal.
(2) If X is a semi-smooth variety, then the variety X is semi-terminal by (1).
(3) [5, Theorem 4.5] Let (S, C) be a two-dimensional normal pair with C reduced and $0 \in C$ be a point. Then (S, C) has canonical singularities around 0 if and only if both S and C are smooth at 0.
(4) [6, Proposition 4.12] Let X be a demi-normal surface and $0 \in X$ be a closed point. The variety X is semi-canonical around $0 \in X$ if and only if $0 \in X$ is one of a smooth, a du Val, a dnc or a pinch point. Thus, X is semi-terminal around 0 if and only if X is semi-smooth around $0 \in X$.

Lemma 2.6. *Let X, X' be semi-log-terminal varieties.*

(1) *All of the varieties X, $\bar{X}$ and $D_{\bar{X}}$ are Cohen-Macaulay. The variety $D_{\bar{X}}$ is normal.*
(2) *The variety D_X is equal to the quotient $D_{\bar{X}}/\iota_X$ (thus D_X is normal) and the variety X is obtained by the universal push-out* (see [4, Theorem 9.30]) *of the following diagram*:

$$\begin{array}{ccc} D_{\bar{X}} & \hookrightarrow & \bar{X} \\ \downarrow & & \\ D_{\bar{X}}/\iota_X & & \end{array}$$

(3) *For two singularities $p \in X$ and $p' \in X'$ are analytically isomorphic to each other if and only if there exist analytical neighborhoods of $\bar{X}$ and $\bar{X}'$ around $\nu_X^{-1}(p)$ and $\nu_{X'}^{-1}(p')$ such that the triplets $(\bar{X}, D_{\bar{X}}, \iota_X)$ and $(\bar{X}', D_{\bar{X}'}, \iota_{X'})$ are analytically isomorphic around those neighborhoods.*

Proof. (1) Both the varieties $D_{\bar{X}}$ and $\bar{X}$ are normal and Cohen-Macaulay by [5, Corollary 5.25 and Proposition 5.51]. We show that X is Cohen-Macaulay.

By taking the index 1 cover (see [4, Definition 2.49]), we can assume that X is semi-canonical and K_X is Cartier by [5, Proposition 5.7]. Take a semi-resolution $f\colon Y \to X$ of X in the sense of [4, Theorem 10.54]. Since X is semi-canonical and K_X is Cartier, there exists an effective f-exceptional Cartier divisor B on Y such that $\omega_Y(-B) = f^*\omega_X$ holds. By [1, Theorem 1.10], $R^i f_*\mathcal{O}_Y(B) = 0$ and $R^i f_*\omega_Y = 0$ for all $i > 0$. The composition of the following natural morphisms

$$f_*\mathcal{O}_Y \to \mathbb{R}f_*\mathcal{O}_Y \to \mathbb{R}f_*\mathcal{O}_Y(B) \simeq_{\mathrm{qis}} f_*\mathcal{O}_Y(B) = f_*\mathcal{O}_Y$$

in the derived category of coherent sheaves on Y is a quasi-isomorphism. By [4, Corollary 2.75], the variety X is Cohen-Macaulay.

(2) We know that the set of log-canonical centers of the pair $(\bar{X}, D_{\bar{X}})$ is equal to the set of connected components of the variety $D_{\bar{X}}$. Thus (2) is a very special case of [4, §9.1].

(3) Follows from (2) immediately. □

We see important examples of semi-terminal singularities.

Example 2.7. Fix $m \in \mathbb{Z}_{>0}$. Set $\bar{X}_m := \mathbb{A}^{m+1}_{x_1,\dots,x_{m+1}}$ and $\bar{D}_m := (x_{m+1} = 0) \subset \bar{X}_m$. We set the involution $\iota\colon \bar{D}_m \to \bar{D}_m$ defined by $x_i \mapsto -x_i$ for $1 \le i \le m$. Let X_m be the demi-normal variety obtained by the triplet $(\bar{X}_m, \bar{D}_m, \iota)$ (see [4, Corollaries 5.33, 9.31(3) and Theorem 5.38]). In fact, by a direct calculation in Lemma 2.6(2), $X_m = \operatorname{Spec} R_m$ with

$$R_m = \mathbb{C}[\{x_i x_j\}_{1\le i\le j\le m}, x_{m+1}, \{x_i x_{m+1}\}_{1\le i\le m}].$$

Let $\nu\colon \bar{X}_m \to X_m$ be the normalization morphism and let $0 \in X_m$ be the image of $0 \in \bar{X}_m$. Consider a section $\phi := 1/x_{m+1}(dx_1 \wedge \cdots \wedge dx_{m+1})$ of $\omega_{\bar{X}_m}(\bar{D}_m)$. Then $\operatorname{Res}_{\bar{X}_m\to\bar{D}_m}(\phi) = (-1)^m dx_1 \wedge \cdots \wedge dx_m \in \omega_{\bar{D}_m}$ is ι-anti-invariant if m is odd and ι-invariant if m is even, where $\operatorname{Res}_{\bar{X}_m\to\bar{D}_m}$ is the residue map. By [4, Proposition 5.8], $2K_{X_m}$ is Cartier. Moreover, K_{X_m} is Cartier if m is odd. In fact, X_m is Gorenstein if and only if m is odd (see Section 6). From now on, we consider the index 1 cover $\pi\colon \tilde{X}_m \to X_m$ of X_m with respects to ϕ^2 for the case m is even. By [4, Proposition 5.8], the global section $\Gamma(X_m, \omega_{X_m})$ is equal to

$$\sum_{i=1}^{m+1} R_m \cdot \frac{x_i}{x_{m+1}} dx_1 \wedge \cdots \wedge dx_{m+1}.$$

Thus $\tilde{X}_m = \operatorname{Spec} \tilde{R}_m$ with

$$\begin{aligned}\tilde{R}_m &= R_m[\{y_i\}_{1\le i\le m+1}]/(\{y_i y_j - x_i x_j\}_{1\le i\le j\le m+1})\\ &\simeq \mathbb{C}[x_{m+1}, y_1, \dots, y_{m+1}]/(y_{m+1}^2 - x_{m+1}^2).\end{aligned}$$

Hence $\tilde{X}_m$ is a dnc variety.

Definition 2.8. An n-dimensional demi-normal singularity $0 \in X$ is called an *m-twirl point* if $0 \in X$ is analytically isomorphic to the singularity $0 \in X_m \times \mathbb{A}^{n-m-1}$, where $0 \in X_m$ is the singularity defined in Example 2.7.

Remark 2.9. (1) The notion of 1-twirl points is equal to the notion of pinch points since there exists a natural isomorphism $\mathbb{C}[x_1^2, x_2, x_1x_2] \simeq \mathbb{C}[y_1, y_2, y_3]/(y_2^2 - y_3^2 y_1)$.

(2) We consider a three-dimensional 2-twirl point $0 \in X_2$. Let $\pi\colon \tilde{X}_2 \to X_2$ be the index 1 cover as in Example 2.7. Take a general element $0 \in S \in |-K_{X_2}|$ and set $\tilde{S} := \pi^* S \subset \tilde{X}_2$. By a suitable coordinate change, we may assume that the embedding $S \subset X_2$ corresponds to the following surjection

$$\mathbb{C}[x_1^2, x_1x_2, x_2^2, x_3, x_1x_3, x_2x_3] \twoheadrightarrow \mathbb{C}[x_2^2, x_3, x_2x_3]$$

such that x_1^2, x_1x_2 and x_1x_3 map to zero. Thus the double cover $(0 \in \tilde{S}) \to (0 \in S)$ is from a double normal crossing point to a pinch point.

3. Log-elliptic singularities

We consider log-elliptic singularities. The concept of log-elliptic singularities is a logarithmic analogue of the concept of elliptic singularities. In this section, many arguments are similar to the arguments in [5, §4.4] based on the works [7] and [9].

Definition 3.1. A germ $0 \in (S, C)$ of a two-dimensional normal pair is called a *log-elliptic singularity* if C is nonzero, $0 \in C$, $K_S + C$ is Cartier and for any projective birational morphism $f\colon T \to S$ such that T is smooth and $C_T := f_*^{-1}C$ is smooth, $f_*\omega_T(C_T) = \mathfrak{m}_{0,S} \cdot \omega_S(C)$ holds, where $\mathfrak{m}_{0,S}$ is the maximal ideal sheaf corresponds to $0 \in S$.

Remark 3.2. For a two-dimensional normal pair (S, C) with C reduced and $K_S + C$ Cartier and for a projective birational morphism $f\colon T \to S$ such that T is smooth and $C_T := f_*^{-1}C$ is smooth, $f_*\omega_T(C_T) = \omega_S(C)$ holds if and only if the pair (S, C) has canonical singularities. The proof is essentially same as the proof of [4, Claim 2.3.1]. Thus in Definition 3.1, it is enough to check the condition $f_*\omega_T(C_T) = \mathfrak{m}_{0,S} \cdot \omega_S(C)$ for only one birational morphism f.

For the reason to consider log-elliptic singularities, see Lemma 4.1.

Notation 3.3. Let $0 \in (S, C)$ be a germ of a two-dimensional normal pair such that $K_S + C$ is Cartier and (S, C) has not canonical singularities. Let $g\colon S' \to S$ be the canonical modification (see [2, Definition 2.6]) of the normal pair (S, C), $h\colon T \to S'$ be the minimal resolution, $f\colon T \to S$ be the composition and C_T be the strict transform of C on T. We note that $f\colon T \to S$ is a semi-terminal modification (see [2, Definition 2.6]) of the normal pair (S, C). By [4, Claim 2.26.4], there exists a unique f-exceptional effective Cartier divisor Z on T such that $K_T + C_T + Z \sim 0$ and the support of Z is equal to the exceptional locus of the morphism f.

The following two propositions are essentially same as [5, Propositions 4.45 and 4.47].

Proposition 3.4. *Fix Notation 3.3. Let L be an f-nef line bundle on T. Then the following hold*:

(1) *The homomorphism $H^0(T, L) \to H^0(Z, L|_Z)$ is surjective.*
(2) *The homomorphism $H^1(T, L) \to H^1(Z, L|_Z)$ is an isomorphism.*
(3) *$L \simeq \mathcal{O}_T$ if and only if $L \equiv_f 0$ and $L|_Z \simeq \mathcal{O}_Z$.*
(4) *$f_*\omega_T(C_T + Z) = \omega_S(C)$ holds.*
(5) *$\omega_S(C)/f_*\omega_T(C_T) \simeq H^0(Z, \omega_Z(C_T|_Z))$ holds.*
(6) *$\omega_S(C)/f_*\omega_T(C_T)$ and $H^1(Z, \mathcal{O}_Z(-C_T|_Z))$ are dual to each other.*

Proposition 3.5. *Under Notation 3.3, for any nonzero effective divisor $Z' \lneq Z$, we have $h^1(Z', \mathcal{O}_{Z'}(-C_T|_{Z'})) < h^1(Z, \mathcal{O}_Z(-C_T|_Z))$.*

Lemma 3.6 (cf. [5, Proposition 4.51])**.** *Fix Notation 3.3. Assume that $0 \in (S, C)$ is a log-elliptic singularity. Then $h^1(Z, \mathcal{O}_Z) = 0$ holds. Moreover, for any reduced and irreducible component $E \leq Z$, E is isomorphic to $\mathbb{P}^1$ and $((C_T + Z - E).E) = 2$ holds.*

Proof. From the exact sequence

$$0 \to \mathcal{O}_{Z'}(-C_T|_{Z'}) \to \mathcal{O}_{Z'} \to \mathcal{O}_{Z' \cap C_T} \to 0,$$

we have $h^1(Z', \mathcal{O}_{Z'}(-C_T|_{Z'})) \geq h^1(Z', \mathcal{O}_{Z'})$ for any nonzero effective divisor $Z' \leq Z$. By Proposition 3.4(6), $h^1(Z, \mathcal{O}_Z(-C_T|_Z)) = 1$ holds. Thus $h^1(Z, \mathcal{O}_Z) = 0$ or 1. Assume that $h^1(Z, \mathcal{O}_Z) = 1$. Then $h^0(Z, \omega_Z) = 1$. Thus $(\omega_Z.Z) \geq 0$. Moreover, since the support of Z is equal to the exceptional locus of f, $(C_T.Z) > 0$ holds. However, since $K_T + C_T + Z \sim 0$, we have $0 = (\omega_Z.Z) + (C_T.Z)$. This leads to a contradiction. Thus $h^1(Z, \mathcal{O}_Z) = 0$. By Proposition 3.5, we have $h^1(E, \mathcal{O}_E) = 0$ for any reduced and irreducible component $E \leq Z$. Moreover, we have $0 = ((K_T + C_T + Z).E) = -2 + ((C_T + Z - E).E)$. □

Proposition 3.7 (cf. [5, Lemma 4.53])**.** *Fix Notation 3.3. Assume that $0 \in (S, C)$ is a log-elliptic singularity. Let L be a nef line bundle on Z. Then $H^1(Z, L) = 0$ and there exists a section $s \in H^0(Z, L)$ such that the associated subscheme $V := (s = 0) \subset Z$ does not intersect C_T and the singular locus of* $\mathrm{red}(Z)$, *and $s|_{\mathrm{red}(Z)}$ is smooth. Moreover, for such s and V, if we set $A := \mathcal{O}_V$, then the natural homomorphism $H^0(Z, L) \to A \otimes L$ is surjective.*

Proof. By Lemma 3.6, we have $h^1(Z, \mathcal{O}_Z) = 0$. Thus the assertion follows from [5, Lemma 4.50]. □

Proposition 3.8 (cf. [5, Proposition 4.54])**.** *Under the notation in Proposition 3.7, assume that the integer $k := (L.Z)$ satisfies that $k \in \mathbb{Z}_{>0}$. Then there exists an isomorphism*

$$\bigoplus_{n \geq 0} H^0(Z, L^{\otimes n}) \simeq \mathbb{C}[s, t, x_1, \ldots, x_{k-1}]/(\{x_i x_j + q_{ij}(s, t)\}_{1 \leq i \leq j \leq k-1})$$

of graded $\mathbb{C}$-algebras, where s, t, $x_1, \ldots, x_{k-1}$ are of degree one and $q_{ij}(s, t) \in \mathbb{C}[s, t]$ are homogeneous polynomials of degree two.

Proof. We set

$$R_Z(n) := H^0(Z, L^{\otimes n}), \qquad R_Z := \bigoplus_{n\geq 0} R_Z(n),$$

$$R_V(n) := H^0(V, L|_V^{\otimes n}), \text{ and } \qquad R_V := \bigoplus_{n\geq 0} R_V(n).$$

For any $n \geq 0$, there exists a natural exact sequence

$$0 \to R_Z(n) \xrightarrow{\cdot s} R_Z(n+1) \to R_V(n+1) \to 0.$$

Since $\dim_{\mathbb{C}} R_Z(0) = 1$, we have $\dim_{\mathbb{C}} R_Z(n) = kn + 1$ for any $n \geq 0$. Let $T \in R_V(1) = A \otimes L$ be an element generating $A \otimes L$ and $t \in R_Z(1)$ be an extension of T. Since $R_V(n) = A \cdot T^n$ for any $n \geq 0$, we have

$$(R_Z/sR_Z)(n) = \begin{cases} \mathbb{C} & (n = 0), \\ A \cdot T^n & (n \geq 1). \end{cases}$$

Thus there exists elements $x_1, \ldots, x_{k-1} \in R_Z(1)$ such that

$$R_Z/(s,t)R_Z = \mathbb{C}[\bar{x}_1, \ldots, \bar{x}_{k-1}]/(\{\bar{x}_i\bar{x}_j\}_{1\leq i\leq j\leq k-1}),$$

where $\bar{x}_i \in (R_Z/(s,t)R_Z)(1)$ is the image of x_i. Therefore the assertion follows from [5, Lemma 4.55]. □

Theorem 3.9 (cf. [5, Theorem 4.57])**.** *Fix Notation 3.3. Assume that $0 \in (S, C)$ is a log-elliptic singularity. Let $g\colon S' \to S$ be the blowing up along the maximal ideal sheaf $\mathfrak{m}_{0,S}$ corresponds to $0 \in S$, that is, $S' = \mathrm{Proj}_S \bigoplus_{n\geq 0} \mathfrak{m}_{0,S}^n$. Let $\mathcal{O}_{S'}(1)$ be the g-ample line bundle on S' corresponds to the projectivization. Then the morphism is equal to the canonical modification of the normal pair (S, C). Thus there exists a morphism $h\colon T \to S'$ such that $g \circ h = f$ holds. Moreover, $K_{S'} + C_{S'} \sim \mathcal{O}_{S'}(1) \sim -h_*Z$ holds, where $C_{S'} \subset S'$ be the strict transform of C.*

Proof. Set $L := \mathcal{O}_T(-Z) \simeq \omega_T(C_T)$. Then $L|_Z$ is nef and $(L.Z) \in \mathbb{Z}_{>0}$. Since f^* gives a natural isomorphism $H^0(S, \mathcal{O}_S) \simeq H^0(T, \mathcal{O}_T)$, we get an ideal $I_n \subset H^0(S, \mathcal{O}_S)$ defined by

$$H^0(T, L^{\otimes n}) = H^0(T, \mathcal{O}_T(-nZ)) =: I_n \subset H^0(S, \mathcal{O}_S)$$

for any $n \geq 0$. Since $H^0(Z, \mathcal{O}_Z) \simeq \mathbb{C}$, we have $I_1 = \mathfrak{m}_{0,S}$. Take general global sections $s_1, s_2 \in H^0(T, L)$. Since there exists an exact sequence

$$0 \to L^{\otimes n-1} \xrightarrow{{}^t(s_2,-s_1)} L^{\otimes n} \oplus L^{\otimes n} \xrightarrow{(s_1,s_2)} L^{\otimes n+1} \to 0$$

and $H^1(T, L^n) = 0$ for any $n \geq 0$ (see [5, Corollary 2.68]), we have $I_{n+1} = I_n \cdot I_1$ for any $n \geq 0$. Since there exists a natural exact sequence

$$\begin{aligned} 0 \to H^0(T, \mathcal{O}_T(-(n+1)Z)) &\to H^0(T, \mathcal{O}_T(-nZ)) \\ &\to H^0(Z, \mathcal{O}_Z(-nZ)) \to 0 \end{aligned}$$

for any $n \geq 0$, we have an isomorphism

$$\bigoplus_{n\geq 0} I_n/I_{n+1} \simeq \bigoplus_{n\geq 0} H^0(Z, L^{\otimes n}|_Z)$$

of graded $\mathbb{C}$-algebras. By Proposition 3.8, the algebra $\bigoplus_{n\geq 0} I_n/I_{n+1}$ is generated by I_1/I_2. Hence $I_n = I_{n+1} + I_1^n = I_n \cdot I_1 + I_1^n$ holds for any $n \geq 0$. By Nakayama's lemma, $I_1^n = I_n$ for any $n \geq 0$. Hence $I_n = \mathfrak{m}_{0,S}^n$ for any $n \geq 0$. Thus there exists isomorphisms

$$\bigoplus_{n\geq 0} \mathfrak{m}_{0,S}^n \simeq \bigoplus_{n\geq 0} f_*\mathcal{O}_T(-nZ) \simeq \bigoplus_{n\geq 0} f_*\mathcal{O}_T(n(K_T + C_T))$$

of graded $\mathcal{O}_S$-algebras. Thus $S' \simeq \operatorname{Proj}_S \bigoplus_{n\geq 0} f_*\mathcal{O}_T(n(K_T + C_T))$ is the canonical modification of (S, C) by [2, Proposition 3.2]. Since $\mathcal{O}_T(-nZ)$ is generated by global sections for any $n \gg 0$, the induced morphism $h\colon T \to S'$ satisfies that $h^*\mathcal{O}_{S'}(1) \sim \mathcal{O}_T(-Z)$. Thus we have $\mathcal{O}_{S'}(1) \sim -h_*Z \sim K_{S'} + C_{S'}$ since $K_T + C_T + Z \sim 0$. □

4. Normal semi-terminal pairs

For a semi-terminal variety X, the pair $(\bar{X}, D_{\bar{X}})$ is a normal semi-terminal pair. In this section, we consider three-dimensional such objects with nonzero $D_{\bar{X}}$.

Lemma 4.1 (cf. [5, Lemma 5.30])**.** *Let $0 \in (X, D)$ be a germ of a three-dimensional canonical singularity with $0 \in D \neq 0$ and $K_X + D$ Cartier. Let $0 \in S \subset X$ be a general hypersurface passing through $0 \in X$ and let $C := D \cap S$. Then the two-dimensional singularity $0 \in (S, C)$ is either a canonical singularity or a log-elliptic singularity.*

Proof. Let $f\colon Y \to X$ be a log resolution of the normal pair (X, D) which dominates the blowing up of X along the maximal ideal sheaf $\mathfrak{m}_{0,X}$ corresponds to $0 \in X$. Then there exists an f-exceptional effective divisor E on Y such that $f^*\mathfrak{m}_{0,X} = \mathcal{O}_Y(-E)$ holds. Moreover, since $0 \in S \subset X$ is general, we have $f^*S = S' + E$, S' is smooth and $C' := D' \cap S' \subset S'$ is equal to $(f|_{S'})^{-1}_*C$, where $S' := f^{-1}_*S$ and $D' := f^{-1}_*D$. Since X and D are Cohen-Macaulay, S is normal, C is reduced and $K_S + C = (K_X + D + S)|_S$ is Cartier. There exists an f-exceptional effective divisor F on Y such that $\omega_Y(D') = f^*\omega_X(D)(F)$ holds since (X, D) has canonical singularities. Since

$$\begin{aligned}\omega_{S'}(C') &= \omega_Y(D' + S')|_{S'} \\ &= f^*(\omega_X(D + S))(F - E)|_{S'} = (f|_{S'})^*(\omega_S(C))(F - E|_{S'}),\end{aligned}$$

we have

$$\begin{aligned}(f|_{S'})_*\omega_{S'}(C') &= \omega_S(C) \otimes (f|_{S'})_*\mathcal{O}_{S'}(F - E|_{S'}) \\ &\supset \omega_S(C) \otimes (f|_{S'})_*\mathcal{O}_{S'}(-E|_{S'}) = \mathfrak{m}_{0,S} \cdot \omega_S(C),\end{aligned}$$

where $\mathfrak{m}_{0,S}$ is the maximal ideal sheaf of $\mathcal{O}_S$ corresponds to $0 \in S$. Thus the assertion follows. □

Theorem 4.2 (cf. [5, Theorems 5.34 and 5.35])**.** *Let $0 \in (X, D)$ be a germ of a three-dimensional normal semi-terminal singularity with $0 \in D \neq 0$ and $K_X + D$ Cartier. Let $0 \in S \subset X$ be a general hypersurface passing through $0 \in X$ and let $C := D \cap S$. Then the two-dimensional singularity $0 \in (S, C)$ has a canonical singularity.*

Proof. Assume not. Then the singularity $0 \in (S, C)$ is a log-elliptic singularity by Lemma 4.1. Let $f\colon Y \to X$ be the blowing up along the maximal ideal sheaf $\mathfrak{m}_{0,X}$ corresponding to $0 \in X$ and let $\bar{f}\colon \bar{Y} \to X$ be the composition $f \circ \nu_Y$. Let $E \subset Y$ be the f-exceptional Cartier divisor on Y defined by $f^*\mathfrak{m}_{0,X} = \mathcal{O}_Y(-E)$ and let $S' \subset Y$ be the strict transform of S. By Theorem 3.9, the morphism $f|_{S'}\colon S' \to S$ is the canonical modification of the normal pair (S, C). In particular, S' is normal. Since $f^*S = S' + E$ and ν_Y is an isomorphism around S', $\nu_Y^*S' (\simeq S')$ is $\bar{f}$-ample and ν_Y^*S' intersects any component of $\bar{f}$-exceptional divisors. Since (X, D) has canonical singularities, there exists an $\bar{f}$-exceptional effective Cartier divisor F on $\bar{Y}$ such that $K_{\bar{Y}} + D_{\bar{Y}} = \bar{f}^*(K_X + D) + F$ holds, where $D_{\bar{Y}} := \bar{f}_*^{-1}D$. Since $0 \in S \subset X$ is general, $D_{\bar{Y}}|_{\nu_Y^*S'} = C_{S'}$ holds, where $C_{S'} \subset \nu_Y^*S'$ is the strict transform of $C \subset S$. By Theorem 3.9, we have

$$\begin{aligned} -\nu_Y^*E|_{\nu_Y^*S'} &\equiv K_{\nu_Y^*S'} + C_{S'} = K_{\bar{Y}} + D_{\bar{Y}} + \nu_Y^*S'|_{\nu_Y^*S'} \\ &= (\bar{f}^*(K_X + D + S) + F - \nu_Y^*E)|_{\nu_Y^*S'} \equiv (F - \nu_Y^*E)|_{\nu_Y^*S'}. \end{aligned}$$

Hence $F|_{\nu_Y^*S'} \equiv 0$. Any component of F maps onto $0 \in X$. Thus $F|_{\nu_Y^*S'} \subset \nu_Y^*S'$ is exceptional with respects to the morphism $\nu_Y^*S' \to S$. By the negativity lemma [5, Lemma 3.39], $F|_{\nu_Y^*S'} = 0$. Since any component of F intersects ν_Y^*S', we have $F = 0$. Therefore, there exists an exceptional prime divisor G over X such that $\operatorname{center}_X G = \{0\}$ and $a(G, X, D) = 0$. This leads to a contradiction since (X, D) is semi-terminal. Thus the assertion follows. □

By Example 2.5(3) and Theorem 4.2, we have the following:

Corollary 4.3. *Let $0 \in (X, D)$ be a germ of a three-dimensional normal semi-terminal singularity with $D \neq 0$, $0 \in D$ and $K_X + D$ Cartier. Then both X and D are smooth at 0.*

The following theorem is proven similar to [10, Theorem (3.1)].

Theorem 4.4. *Let $0 \in (X, D)$ be a germ of a three-dimensional normal semi-terminal singularity such that $0 \in \operatorname{Supp} D$ and D is a nonzero reduced divisor. Then both X and D are smooth at 0.*

Proof. We set

$$r := \min\{r \in \mathbb{Z}_{>0} \mid r(K_X + D) \text{ is Cartier}\}.$$

Take the index 1 cover $\pi\colon 0 \in (\tilde{X}, \tilde{D}) \to 0 \in (X, D)$ of $0 \in (X, D)$ (see [4, Proposition 2.50(2)]). Then we have $K_{\tilde{X}} + \tilde{D} = \pi^*(K_X + D)$, $\pi^{-1}(0) = \{0\}$, $K_{\tilde{X}} + \tilde{D}$ is Cartier, the normal pair $(\tilde{X}, \tilde{D})$ is semi-terminal, the group μ_r of r-th roots of unity acts on $(\tilde{X}, \tilde{D})$ and the normal pair (X, D) is the quotient of the group action. We note that the group action is free outside $0 \in \tilde{X}$ by Example 2.5(3). By Corollary 4.3, both $\tilde{X}$ and $\tilde{D}$ are smooth at 0. Therefore the assertion follows if $r = 1$. Assume that $r > 1$. By taking an analytical neighborhood of $0 \in \tilde{X}$, we can assume that $0 \in (\tilde{X}, \tilde{D})$ is equal to $0 \in (\mathbb{A}^3_{x_1,x_2,x_3}, \mathbb{A}^2_{x_1,x_2} = (x_3 = 0))$ and the action $\mu_r \curvearrowright (\mathbb{A}^3_{x_1,x_2,x_3}, \mathbb{A}^2_{x_1,x_2})$ is given by $x_i \mapsto \varepsilon^{a_1} x$ for some $0 \le a_i \le r-1$ $(1 \le i \le 3)$, where $\varepsilon \in \mu_r$ is a generator. Since the group action is free outside $0 \in \tilde{X}$, all of a_1, a_2 and a_3 are nonzero. By replacing a generator $\varepsilon \in \mu_r$ if necessary, we can assume that $\gcd(a_1, a_2, a_3) = 1$ and $a_1 + a_2 \le r$.

Let $f\colon Y \to \mathbb{A}^3_{x_1,x_2,x_3}$ be the weighted blowup with weights (a_1, a_2, a_3). By [4, Theorem 3.21], a local chart is

$$f\colon \mathbb{A}^3_{y_1,y_2,y_3}/\frac{1}{a_1}(1, -a_2, -a_3) \to \mathbb{A}^3_{x_1,x_2,x_3}$$

with $f^*x_1 = y_1^{a_1}$, $f^*x_2 = y_1^{a_2} y_2$ and $f^*x_3 = y_1^{a_3} y_3$. Set $F := (y_1 = 0)$. Since

$$f^*(x_3^{-1} dx_1 \wedge dx_2 \wedge dx_3) = a_1 y_3^{-1} y_1^{a_1+a_2-1} dy_1 \wedge dy_2 \wedge dy_3,$$

we have $a(F, \tilde{X}, \tilde{D}) = a_1 + a_2 - 1$. Let E be the exceptional prime divisor over X which is dominated by F. We note that $\mathrm{center}_X E = \{0\}$. By [4, Theorem 3.21], we have $a(E, X, D) = (a_1 + a_2)/r - 1 \le 0$. Since (X, D) is semi-terminal, this leads to a contradiction. Thus r must be equal to one. $\square$

5. Proof of Theorem 1.1

As a corollary of Theorem 4.4, we can prove Theorem 1.1. Let $0 \in X$ be a germ of a three-dimensional non-normal semi-terminal singularity. By Theorem 4.4, both $\bar{X}$ and $\bar{D}$ are smooth.

We consider the case that the inverse image $\nu_X^{-1}(0)$ does not consist of only one point. By Lemma 2.6(2), $\nu_X^{-1}(0) = \{q_1, q_2\}$. By taking analytical neighborhoods of q_1, $q_2 \in \tilde{X}$, we can assume that q_1, $q_2 \in (\tilde{X}, D_{\tilde{X}})$ is equal to the disjoint union of

$$(q_1 =)0 \in (\mathbb{A}^3_{x_1,x_2,x_3}, \mathbb{A}^2_{x_1,x_2} = (x_3 = 0)),$$
$$(q_2 =)0 \in (\mathbb{A}^3_{y_1,y_2,y_3}, \mathbb{A}^2_{y_1,y_2} = (y_3 = 0)),$$

and the involution $\iota_X\colon D_{\tilde{X}} \to D_{\tilde{X}}$ is given by $x_i \mapsto y_i$ and $y_i \mapsto x_i$ for $1 \le i \le 2$. Then the coordinate ring $\mathcal{O}_X$ is equal to

$$\{(f, g) \in \mathbb{C}[x_1, x_2, x_3] \times \mathbb{C}[y_1, y_2, y_3] | f(x_1, x_2, 0) = g(x_1, x_2, 0)\}.$$

Consider the ring surjection $\mathbb{C}[x_1, x_2, x_3, y_3] \to \mathcal{O}_X$ defined by $x_1 \mapsto (x_1, y_1)$, $x_2 \mapsto (x_2, y_2)$, $x_3 \mapsto (x_3, 0)$ and $y_3 \mapsto (0, y_3)$. The kernel of the surjection is generated by $x_3 y_3$. Thus X is a dnc variety.

We consider the case that the inverse image $\nu_X^{-1}(0)$ consists of only one point, say $0 \in \bar{X}$. Then $0 \in D_{\tilde{X}}$ is a fixed point of the involution $\iota_X \colon D_{\tilde{X}} \to D_{\tilde{X}}$. By taking an analytical neighborhood of $0 \in \tilde{X}$, we can assume either

$$0 \in \bar{X} = \mathbb{A}^3_{x_1,x_2,x_3}, D_{\tilde{X}} = \mathbb{A}^2_{x_1,x_2} = (x_3 = 0), \iota_X \colon \begin{cases} x_1 \mapsto x_1, \\ x_2 \mapsto -x_2, \end{cases}$$

or

$$0 \in \bar{X} = \mathbb{A}^3_{x_1,x_2,x_3}, D_{\tilde{X}} = \mathbb{A}^2_{x_1,x_2} = (x_3 = 0), \iota_X \colon \begin{cases} x_1 \mapsto -x_1, \\ x_2 \mapsto -x_2. \end{cases}$$

As we have seen in Example 2.7, $0 \in X$ is a 1-twirl point (that is, a pinch point) for the former case; a 2-twirl point for the latter case.

As a consequence, we have completed the proof of Theorem 1.1.

6. Appendix: ring-theoretical properties of twirl singularities

In this section, we determine whether a given m-twirl point is Gorenstein or not by using [3, Theorem (2)]. Fix a positive integer m and a lattice $N := \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{m+1}$. Set $H \subset N$ such that

$$H := \sum_{1 \leq i \leq j \leq m} \mathbb{Z}_{\geq 0}(e_i + e_j) + \mathbb{Z}_{\geq 0}e_{m+1} + \sum_{1 \leq i \leq m} \mathbb{Z}_{\geq 0}(e_i + e_{m+1}).$$

Then $H \subset N$ is a finitely generated additive semigroup with identity. Moreover, the semigroup ring $\mathbb{C}[H]$ is equal to R_m in Example 2.7. We set $f_i := 2e_i$ $(1 \leq i \leq m)$ and $f_{m+1} := e_{m+1}$. Then $f_1, \ldots, f_{m+1}$ satisfies the conditions (1) and (2) in [3, p. 1]. Set

$$F_i := H \cap \sum_{1 \leq p \leq m+1, p \neq i} \mathbb{Q}_{\geq 0} f_p$$

for $1 \leq i \leq m+1$, that is,

$$\begin{cases} F_p = \displaystyle\sum_{1 \leq i \leq j \leq m, i \neq p, j \neq p} \mathbb{Z}_{\geq 0}(e_i + e_j) + \mathbb{Z}_{\geq 0}e_{m+1} \\ \qquad + \displaystyle\sum_{1 \leq i \leq m, i \neq p} \mathbb{Z}_{\geq 0}(e_i + e_{m+1}) \quad (p \neq m+1), \\ F_{m+1} = \displaystyle\sum_{1 \leq i \leq j \leq m} \mathbb{Z}_{\geq 0}(e_i + e_j). \end{cases}$$

Set

$$H_i := \{w \in N \mid \text{there exists } g \in F_i \text{ such that } w + g \in H\}$$

for $1 \leq i \leq m+1$, that is,

$$\begin{cases} H_p = \displaystyle\sum_{1\leq i\leq j\leq m, i\neq p, j\neq p} \mathbb{Z}(e_i+e_j) + \mathbb{Z}e_{m+1} \\ \qquad + \displaystyle\sum_{1\leq i\leq m, i\neq p} \mathbb{Z}(e_i+e_{m+1}) + \sum_{1\leq i\leq m} \mathbb{Z}_{\geq 0}(e_i+e_p) \\ \qquad + \mathbb{Z}_{\geq 0}(e_p+e_{m+1}) \quad (p\neq m+1), \\ H_{m+1} = \displaystyle\sum_{1\leq i\leq j\leq m} \mathbb{Z}(e_i+e_j) + \mathbb{Z}_{\geq 0}e_{m+1} + \sum_{1\leq i\leq m} \mathbb{Z}_{\geq 0}(e_i+e_{m+1}). \end{cases}$$

Hence the set $N \setminus \bigcup_{1\leq i\leq m+1} H_i$ is equal to

$$\left\{ \sum_{1\leq i\leq m} a_i e_i \,\middle|\, a_i \in \mathbb{Z}_{<0}, \sum_{1\leq i\leq m} a_i \text{ is odd} \right\} \cup \sum_{1\leq i\leq m+1} \mathbb{Z}_{<0} e_i.$$

By [3, Theorem (2)], $\mathbb{C}[H]$ is Gorenstein if and only if there exists $c \in N$ such that $c - H = N \setminus \bigcup_{1\leq i\leq m+1} H_i$. Thus $\mathbb{C}[H]$ is Gorenstein if and only if m is odd. Therefore we have the following:

Proposition 6.1. *m-twirl point is Gorenstein if and only if m is odd.*

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