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THE AP-HENSTOCK INTEGRAL OF VECTOR-VALUED FUNCTIONS

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Abstract. In this paper we introduce AP-Henstock integral of vector valued functions which is a generalization of Henstock integral of vector valued functions, investigate some of its properties, and characterize AP-Henstock integral of vector valued functions by the notion of equiintegrability.

1. Introduction and preliminaries

In 2003, Luisa Di Piazza investigated some properties concerning Henstock type integrals for vector valued functions, started in [1] and [2]. Yoon, Park, Kim, and Kim([10]) introduced the AP-Henstock extension of Dunford , Pettis, and Bochner integrals of functions mapping an interval $[a, b]$ into Banach space X and proved some properties of these integrals.

In this paper we introduce AP-Henstock integral of vector valued functions which is a generalization of Henstock integral of vector valued functions and investigate some of its properties. In particular we characterize AP-Henstock integral of vector valued functions by the notion of equiintegrability and study the absolute integrability of the AP-Henstock integrable functions.

Let E be measurable set and let c be a real number. The density of E at c is defined by

$$d_cE = \lim_{h\to 0+} \frac{\mu\left(E \cap (c-h, c+h)\right)}{2h}$$

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provided the limit exists. The point c is called a point of density of E if $d_cE = 1$.

An approximate neighborhood (or ap-nbd) of $x \in [a,b]$ is a measurable set $S_x \subset [a,b]$ containing x as a point of density. For every $x \in E \subset [a,b]$, choose an ap-nbd $S_x \subset [a,b]$ of x. Then we say that $\Delta = \{S_x : x \in E\}$ is a choice on E. A tagged interval $(x, [c,d])$ is said to be subordinate to the choice $\Delta = \{S_x\}$ if $c, d \in S_x$. Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. If $(x_i, [c_i, d_i])$ is subordinate to the choice Δ for each i, then we say that $\mathcal{P}$ is subordinate to the choice Δ. If $\mathcal{P}$ is subordinate to Δ and $[a,b] = \cup_{i=1}^n [c_i, d_i]$, we say that $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to Δ. Let $E \subset [a,b]$. If $\mathcal{P}$ is subordinate to Δ and each $x_i \in E$, $\mathcal{P}$ is called E- subordinate to Δ. For a tagged partition $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ of $[a,b]$, we will use the following notation.

$$S(f, \mathcal{P}) = \sum_{i=1}^{n} f(x_i)(d_i - c_i).$$

Throughout this paper X, Y will denote real Banach spaces and X^* its dual. The closed unit ball of X^* will be denoted by $\mathcal{B}(X^*)$.

2. AP-Henstock integral of vector-valued functions

DEFINITION 2.1. *A function $f : [a,b] \longrightarrow X$ is AP-Henstock integrable on $[a,b]$ if there exists a vector $\mathbf{L} \in X$ with the following property : for each $\epsilon > 0$ there exists a choice Δ on $[a,b]$ such that $\|S(f,\mathcal{P}) - \mathbf{L}\| < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to Δ. The vector $\mathbf{L}$ is called the AP-Henstock integral of f on $[a,b]$ and is denoted by $(AH)\int_a^b f$. The function f is AP-Henstock integrable on a measurable subset E of $[a,b]$ if $f\chi_E$ is AP-Henstock integrable on $[a,b]$. the prefix (AH) will be used to distinguish this integral from others.*

THEOREM 2.2. *Let $f : [a,b] \longrightarrow X$ is AP-Henstock integrable on $[a,b]$.*

(a) *For each $x^* \in X^*$ the function x^*f is AP-Henstock integrable on $[a,b]$ and $(AH)\int_a^b x^*f = x^*(AH)\int_a^b f$.*

(b) *If* $T : X \longrightarrow Y$ *is continuous linear operator then* $(AH)\displaystyle\int_a^b Tf = T(AH)\int_a^b f$.

Proof. (a) Let $x^* \in X^*$. Since $f : [a,b] \longrightarrow X$ is AP-Henstock integrable on $[a,b]$, for each $\epsilon > 0$ there exists a choice Δ on $[a,b]$ such that

$$\|S(f,\mathcal{P}) - (AH)\int_a^b f\| < \frac{\epsilon}{\|x^*\|}$$

whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to Δ. Hence we have

$$\|S(x^*f,\mathcal{P}) - x^*(AH)\int_a^b f\| \leq \|x^*\|\|S(f,\mathcal{P}) - (AH)\int_a^b f\| < \epsilon.$$

Therefore x^*f is AP-Henstock integrable on $[a,b]$ and $(AH)\displaystyle\int_a^b x^*f = x^*(AH)\int_a^b f$.

(b) Since $T : X \longrightarrow Y$ is continuous linear operator, there exists a number $M > 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in X$. Also since f is AP-Henstock integrable on $[a,b]$, for each $\epsilon > 0$ there exists a choice Δ on $[a,b]$ such that

$$\|S(f,\mathcal{P}) - (AH)\int_a^b f\| < \frac{\epsilon}{M}$$

whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to Δ. Hence we have

$$\|S(Tf,\mathcal{P}) - T(AH)\int_a^b f\| \leq M\|S(f,\mathcal{P}) - (AH)\int_a^b f\| < \epsilon.$$

□

The following Theorem 2.3 (a) can easily be obtained from the Definition 2.1 and Theorem 2.2 (a).

THEOREM 2.3. *Let* $f : [a,b] \longrightarrow X$ *is AP-Henstock integrable on* $[a,b]$. *Then*

(a) f *is weakly measurable.*

(b) *If* $f = g$ *almost everywhere on* $[a,b]$, *then* g *is AP-Henstock integrable on* $[a,b]$ *and* $(AH)\displaystyle\int_a^b f = (AH)\int_a^b g$.

Proof. It is sufficient to prove (b) that if $f = \mathbf{0}$(the zero of X) almost everywhere on $[a, b]$, then f is AP-Henstock integrable on $[a, b]$ and $(AH)\int_a^b f = \mathbf{0}$. Since $\|f\| = 0$ almost everywhere on $[a, b]$, $\|f\|$ is AP-Henstock integrable on $[a, b]$ and $(AH)\int_a^b \|f\| = 0$. For any $\epsilon > 0$, there is a choice Δ on $[a, b]$ such that $\|f\|(\mathcal{P}) < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to Δ. Let $\mathcal{P}$ be a tagged partition of $[a, b]$ that is subordinate to Δ, then

$$\|f(\mathcal{P}) - \mathbf{0}\| = \|f(\mathcal{P})\| \leq \|f\|(\mathcal{P}) < \epsilon.$$

Hence, f is AP-Henstock integrable on $[a, b]$ and $(AH)\int_a^b f = \mathbf{0}$. □

Definition 2.4. A function $f : [a, b] \longrightarrow X$ is said to be scalarly AP-Henstock integrable on $[a, b]$ if for each x^* in X^* the function $x^* f$ is AP-Henstock integrable on $[a, b]$. If f is AP-Henstock integrable on $[a, b]$ and for every interval I in $[a, b]$ there exists a vector $\mathbf{L_I} \in X$ such that $x^*(\mathbf{L_I}) = \int_I x^* f$ for all x^* in X^*, then f is AP-Henstock-Pettis integrable on $[a, b]$ and set $\mathbf{L_I} = (AHP)\int_I f$. We denote the set of all AP-Henstock integrable functions $f : [a, b] \longrightarrow X$ by $AH([a, b], X)$.

Remark 2.5. By (a) of Theorem 2.2, it follows that each AP-Henstock integrable function is also AP-Henstock-Pettis integrable. But the reverse implication is not true ([9]).

Definition 2.6. A falmily $\mathcal{A} \subset AH([a, b], X)$ is AP-Henstock equi-integrable on $[a, b]$ if for each $\epsilon > 0$, there exists a choice Δ on $[a, b]$ such that

$$\sup_{f \in \mathcal{A}} \|S(f, \mathcal{P}) - (AH)\int_a^b f\| < \epsilon$$

whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to Δ.

Using the notion of AP-Henstock equiintegrability, we may characterize the vector valued AP-Henstock integrable functions.

Theorem 2.7. *A function $f : [a, b] \longrightarrow X$ is AP-Henstock integrable on $[a, b]$ if and only if the family $\{x^* f : x^* \in \mathcal{B}(X^*)\}$ is AP-Henstock equiintegrable on $[a, b]$.*

Proof. Let $f : [a,b] \longrightarrow X$ is AP-Henstock integrable on $[a,b]$. For each $\epsilon > 0$, there exists a choice Δ on $[a,b]$ such that $\|S(f,\mathcal{P}) - (AH)\int_a^b f\| < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to Δ. Because

$$\|S(f,\mathcal{P}) - (AH)\int_a^b f\| = \sup_{x^*\in\mathcal{B}(X^*)} |S(x^*f,\mathcal{P}) - x^*(AH)\int_a^b f|,$$

$\{x^*f : x^* \in \mathcal{B}(X^*)\}$ is AP-Henstock equiintegrable on $[a,b]$. Conversely, it is sufficient to show that there exists $\mathbf{L} \in X$ such that $x^*(\mathbf{L}) = (AH)\int_a^b x^*f$ for all x^* in X^*. Define $T_f : X^* \longrightarrow \mathbb{R}$ by $T_f(x^*) = (AH)\displaystyle\int_a^b x^*f$. For each $\alpha \in \mathbb{R}$, let $Q(\alpha) = \{x^* \in X^* : T_f(x^*) \le \alpha\}$, then we claim that $Q(\alpha)$ is w^*-closed. Since $Q(\alpha)$ is convex, according to the Banach-Dieudonne Theorem, it suffices to show that $Q(\alpha)\cap\mathcal{B}(X^*)$ is w^*-closed.

Let x_0^* be a w^*-cluster point of $Q(\alpha)\cap\mathcal{B}(X^*)$ and let $(x_r^*)_{r\in I} \subset Q(\alpha)\cap\mathcal{B}(X^*)$ be a net converging to x_0^* in the w^*-topology. Since $\{x^*f : x^* \in \mathcal{B}(X^*)\}$ is AP-Henstock equiintegrable on $[a,b]$, for each $\epsilon > 0$, there exists a choice Δ on $[a,b]$ such that

$$\sup_{\|x^*\|\le 1}\left|(AH)\int_a^b x^*f - S(x^*f,\mathcal{P})\right| < \epsilon \tag{1}$$

whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to Δ. Let $\mathcal{P} = \{(I_1,t_1),\cdots,(I_p,t_p)\}$ be a tagged partition of $[a,b]$ that is subordinate to Δ. Using the convergence of $(x_r^*)_{r\in I}$ we choose an index $r_0 \in I$ such that

$$\sum_{i=1}^{p}\left|x_{r_0}^*f(t_i) - x_0^*f(t_i)\right| < \epsilon. \tag{2}$$

Since $x_0^* \in \mathcal{B}(X^*)$, by (1) and (2) we have

$$\begin{aligned} T_f(x_0^*) \le& \left|T_f(x_0^*) - \sum_{i=1}^{p} x_0^*f(t_i)|I_i|\right| + \sum_{i=1}^{p}\left|x_0^*f(t_i) - x_{r_0}^*f(t_i)\right||I_i| \\ &+ \left|\sum_{i=1}^{p} x_{r_0}^*f(t_i)|I_i| - (AH)\int_a^b x_{r_0}^*f\right| + T_f(x_{r_0}^*) \\ <& \ \alpha + 3\epsilon. \end{aligned}$$

Since ϵ is arbitrary, $x_0^* \in Q(\alpha) \cap \mathcal{B}(X^*)$. Thus T_f is w^*-continuous. Since X is the w^*-dual of X^*, there exists $\mathbf{L} \in X$ such that $x^*(\mathbf{L}) = T_f(x^*)$. □

By Gordon [5, Theorem 17] and Pettis Measurability Theorem, we obtains the following Theorem 2.8.

THEOREM 2.8. *Let X be separable. If $f : [a,b] \longrightarrow X$ is Pettis integrable on $[a,b]$, then f is AP-McShane integrable on $[a,b]$.*

3. Absolute AP-Henstock integrability

DEFINITION 3.1. *A function $f : [a,b] \longrightarrow X$ is absolute AP-Henstock integrable on $[a,b]$ if f and $\|f\|$ are AP-Henstock integrable on $[a,b]$.*

THEOREM 3.2. *If $f : [a,b] \longrightarrow X$ is absolute AP-Henstock integrable on $[a,b]$, f is Pettis integrable.*

Proof. Since f is AP-Henstock integrable on $[a,b]$, for each $x^* \in \mathcal{B}(X^*)$, x^*f is measurable. Moreover $\|f\|$ being AP-Henstock integrable, by [6, Theorem 16.15 (b)], it is also Lebesgue integrable.

For every measurable set $E \subset [a,b]$ and for each $x^* \in \mathcal{B}(X^*)$, it follows that

$$\int_E |x^*f| \leq \int_E \|f\| < \infty.$$

Thus, f is Dunford integrable. Let $v(E)$ be its Dunford integral. If an interval $J \subset [a,b]$, the AP-Henstock integrability of f implies that $v(J) \in X$. Fix $\epsilon > 0$, The Lebesgue integrability of $\|f\|$ implies the existence of a positive number $\eta > 0$ such that if the Lebesgue measure $|E| < \eta$, then $\int_E \|f\| < \epsilon$. Thus if $|E| < \eta$, we have

$$\|v(E)\| = \sup_{x^* \in \mathcal{B}(X^*)} \left| \int_E x^*f \right| \leq \sup_{x^* \in \mathcal{B}(X^*)} \int_E |x^*f| \leq \int_E \|f\| < \epsilon.$$

Therefore, the assertion follows from [2, Proposition 2B]. □

By Gordon [5, Theorem 17] and Theorem 3.2, we obtain the following Corollary 3.3.

COROLLARY 3.3. *Let $f : [a,b] \longrightarrow X$ be a function. If X be separable and f is absolute AP-Henstock integrable on $[a,b]$, then f is McShane integrable.*

THEOREM 3.4. *If $f : [a, b] \longrightarrow X$ is absolute AP-Henstock integrable on $[a, b]$, then for any measurable subset $E \subset [a, b]$ we have*

$$\left\|(P)\int_E f\right\| \le (AH)\int_E \|f\|\,.$$

Proof. Since f is absolute AP-Henstock integrable on $[a, b]$, By Theorem 3.2 it is Pettis integrable. Morever for any measurable subset $E \subset [a, b]$ we have

$$\left\|(P)\int_E f\right\| = \sup_{x^*\in\mathcal{B}(X^*)}\left|\int_E x^* f\right| \le \sup_{x^*\in\mathcal{B}(X^*)}\int_E |x^* f| \le (AH)\int_E \|f\|.$$

□

References

[1] S. Cao, *The Henstock integral for Banach-valued functions,* SEA Bull. Math. **16** (1992), 35-40.

[2] D. H. Fremlin and J. Mendoza, *On the integration of vector-valued functions,* Illnois J. Math. **38** (1994), 127-147.

[3] J. L. Gamez and J. Mendoza, *On Denjoy-Dunford and Denjoy-Pettis integral,* Studia Math. **130** (1998), 115-133.

[4] R. A. Gordon, *The Denjoy extension of the Bochner, Pettis, and Dunford integrals,* Studia Math. **92** (1989), 73-91.

[5] R. A. Gordon, *The McShane integral of Banach-valued functions,* Illnois J. Math. **34** (1990), 556-567.

[6] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock,* Amer. Math. Soc. 1994.

[7] J. M. Park, *The Denjoy extension of the Riemann and McShane integrals,* Czechslovak Math. J. **50** (2000), 615-625.

[8] J. M. Park, D. H. Lee and J. H. Yoon, *The integrals of Banach space-valued functions,* J. Chungcheong Math. Soc. **20** (2008), 79-89.

[9] L. D. Piazza, *Kurzweil-Henstock type Integration on Banach Spaces,* Real Analysis Exchange **29** (2003-2004), no. 2, 543-556.

[10] J. H. Yoon, J. M. Park, Y. K. Kim and B. M. Kim, *The AP-Henstock extension of the Dunford and Pettis integrals,* J. Chungcheong Math. Soc. **23** (2010), 879-884.

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