JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **30**, No. 1, February 2017
http://dx.doi.org/10.14403/jcms.2017.30.1.141

A REMARK ON A STABILITY IN MULTI-VALUED DYNAMICS

Hahng-Yun Chu*, Jong-suh Park**, and Seung Ki Yoo***

Abstract. In this article, we consider the Hyers-Ulam stability in multi-valued dynamics. We prove the Hyers-Ulam stability for a cubic set-valued functional equation on multi-valued dynamics by using several methods.

1. Introduction

The stability problems of functional equations originated from a question of Ulam [13] concerning the stability of group homomorphisms. D. H. Hyers [4] gave a partial answer to the question of S. M. Ulam for Banach spaces. The Hyer's theorem was generalized by Aoki [1] for additive mappings. Th. M. Rassias [12] proved the stability of a linear mapping by using a Cauchy difference. The stability for set-valued functional equations has been investigated by a number of authors[2, 3, 7, 8, 11].

It is obvious that the cubic monomial $f(x) = ax^3 (a \in \mathbb{R})$ satisfies the functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \tag{1.1}$$

Every solution of (1.1) is called a *cubic mapping*. Jun and Kim [5] proved the generalized Hyers-Ulam Rassias stability problem for equation (1.1). Jun et al. [6] studied the cubic functional equation

$$f(ax + y) + f(ax - y) = af(x + y) + af(x - y) + 2a(a^2 - 1)f(x).$$

Najati and Moradlou[10] considered general solution and investigate the generalized Hyers-Ulam-Rassias stability problem for an Euler-Lagrange type cubic functional equation $2mf(x + my) + 2f(mx - y) = (m^3 + m)[f(x + y) + f(x - y)] + 2(m^4 - 1)f(y)$ with $m \neq 0, \pm 1$.

Received January 17, 2017; Accepted January 20, 2017.
2010 Mathematics Subject Classification: 39B52.
Key words and phrases: Hyers-Ulam stability, set-valued functional equation.
Correspondence should be addressed to Seung Ki Yoo, skyoo@cnu.ac.kr.

Let Y be a Banach space. We propose several notations for subfamilies of $\mathcal{P}(Y)$. Let $CB(Y)$ be the set of all closed bounded subsets of Y and $CC(Y)$ the set of all closed convex subsets of Y. Let $CBC(Y)$ be the set of all closed bounded convex subsets of Y. For elements A, B of $CC(Y)$, we denote $A \oplus B := \overline{A + B}$. If A is convex, then we obtain that $(\alpha + \beta)A = \alpha A + \beta A$ for all $\alpha, \beta \in \mathbb{R}^+$.

In this article, we first define a *cubic set-valued functional equation of type (EL)*,

(1.2)

$$2kf(x+ky) \oplus 2f(kx-y) = (k^3+k)f(x+y) \oplus (k^3+k)f(x-y) \oplus 2(k^4-1)f(y)$$

where $k \geq 2$ is an integer. Then we prove the Hyers-Ulam stability problem for the set-valued functional equation.

2. Stability for a set-valued functional equation

In this section, we deal with the Hyers-Ulam stability for the cubic set-valued functional equation(1.2) by using direct method and the fixed point technique. For $A, A' \in CB(Y)$, the *Hausdorff distance* $d_H(A, A')$ between A and A' is defined by

$$d_H(A, A') := inf\{\alpha \geq 0 |\ A \subseteq A' + \alpha B_Y, A' \subseteq A + \alpha B_Y\},$$

where B_Y is the closed unit ball in Y. The following remark is so useful to compute set-valued equations.

Remark 2.1. Let $A, A', B, B', C \in CBC(Y)$ and $\alpha > 0$. Then we have that

(1) $d_H(A \oplus A', B \oplus B') \leq d_H(A, B) + d_H(A', B')$;
(2) $d_H(\alpha A, \alpha B) = \alpha\ d_H(A, B)$;
(3) $d_H(A, B) = d_H(A \oplus C,\ B \oplus C)$.

Let X be a real vector space. We define the *cubic set-valued functional equation of type (EL).*

Definition 2.2. Let $f : X \to CBC(Y)$ be a mapping and $x, y \in X$. The *cubic set-valued functional equation of type (EL)* is defined by

$$f(2x + y) \oplus f(2x - y) = 2f(x + y) \oplus 2f(x - y) \oplus 12f(x).$$

Every solution of the cubic set-valued functional equation is said to be *a cubic set-valued mapping of type (EL).*

In the following theorem, we prove the Hyers-Ulam stability of the cubic set-valued functional equation of type (EL).

THEOREM 2.3. *Let $k \geq 2$ be an integer and let $\phi : X^2 \to [0, \infty)$ be a function with satisfying the property that for every $x, y \in X$,*

$$\sum_{i=0}^{\infty} \frac{1}{k^{3i}} \phi(k^i x, 0) < \infty, \ \lim_{n\to\infty} \frac{1}{k^{3n}} \phi(k^n x, k^n y) = 0. \tag{2.1}$$

Suppose that $f : X \longrightarrow (CBC(Y), d_H)$ is a set-valued mapping with $f(0) = \{0\}$ and

$$\begin{aligned} & d_H\big(2kf(x+ky) \oplus 2f(kx-y), (k^3+k)f(x+y) \oplus (k^3+k)f(x-y) \\ & \qquad \oplus 2(k^4-1)f(y)\big) \leq \phi(x, y) \end{aligned} \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique cubic set-valued mapping of type (EL) $T : X \to (CBC(Y), d_H)$ such that

$$d_H\big(f(x), T(x)\big) \leq \frac{1}{2k^3} \sum_{i=0}^{\infty} \frac{1}{k^{3i}} \phi(k^i x, 0) \tag{2.3}$$

for all $x \in X$.

Proof. Put $y = 0$ in (2.2). Thus we have

$$d_H\big(2kf(x) \oplus 2f(kx), 2(k^3+k)f(x) \oplus 2(k^4-1)f(0)\big) \leq \phi(x, 0) \tag{2.4}$$

for all $x \in X$. By remark2.1, we get

$$d_H\big(2f(kx), 2k^3 f(x)\big) \leq \phi(x, 0) \tag{2.5}$$

for all $x \in X$. Divide by $2k^3$ in (2.5). We get

$$d_H\big(\frac{1}{k^3} f(kx), f(x)\big) \leq \frac{1}{2k^3} \phi(x, 0) \tag{2.6}$$

for all $x \in X$. Replace x by kx and multiply by $\frac{1}{k^3}$ in (2.6), so we obtain

$$d_H\big(\frac{1}{k^6} f(k^2 x), \frac{1}{k^3} f(kx)\big) \leq \frac{1}{2k^6} \phi(kx, 0) \tag{2.7}$$

for all $x \in X$. From (2.6) and (2.7), we have

$$d_H\big(f(x), \frac{1}{k^6} f(k^2 x)\big) \leq \frac{1}{2k^3} \phi(x, 0) + \frac{1}{2k^6} \phi(kx, 0) \tag{2.8}$$

for all $x \in X$. Using the induction on n, we get
(2.9)

$$d_H\big(f(x), \frac{1}{k^{3n}} f(k^n x)\big) \leq \frac{1}{2k^3} \sum_{i=0}^{n-1} \frac{1}{k^{3i}} \phi(k^i x, 0) \leq \frac{1}{2k^3} \sum_{i=0}^{\infty} \frac{1}{k^{3i}} \phi(k^i x, 0)$$

for all $x \in X$. Divide by k^{3m} in (2.9) and let x by $k^m x$. Thus we obtain

$$\begin{aligned} d_H\big(\frac{1}{k^{3m}}f(k^m x), \frac{1}{k^{3(n+m)}}f(k^{n+m}x)\big) &= \frac{1}{k^{3m}}d_H\big(f(k^m x), \frac{1}{k^{3n}}f(k^{n+m}x)\big) \\ &\leq \frac{1}{k^{3m}}\sum_{i=0}^{\infty}\frac{1}{2k^{3i}}\phi(k^{m+i}x, 0) \qquad (2.10) \end{aligned}$$

for all $x \in X$. The right-hand side of the inequality (2.10) tends to zero as $m \to \infty$. Hence the sequence $\{\frac{1}{k^{3n}}f(k^n x)\}$ is a Cauchy sequence in $CBC(Y)$. From the completeness of $CBC(Y)$, we define a mapping $T : X \to (CBC(Y), d_H)$ as

$$T(x) := \lim_{n\to\infty}\frac{1}{k^{3n}}f(k^n x)$$

for all $x \in X$. By setting $n \to \infty$ in (2.9), we have the inequality (2.3). Replacing x by $k^n x$ and y by $k^n y$ and dividing by k^{3n} in (2.2), we get

$$\begin{aligned} \frac{1}{k^{3n}}d_H\big(2kf(k^n(x+ky)) &\oplus 2f(k^n(kx-y)), \\ (k^3+k)[f(k^n(x+y)) &\oplus f(k^n(kx-y))] \oplus 2(k^4-1)f(k^n y)\big) \\ &\leq \frac{1}{k^{3n}}\phi(k^n x, k^n y) \qquad (2.11) \end{aligned}$$

for all $x, y \in X$. Taking the limit as $n \to \infty$, we obtain that T satisfies equation (1.2) for all $x, y \in X$.

To prove uniqueness of the mapping T, let $T' : X \to (CBC(Y), d_H)$ be another cubic set-valued mapping of type (EL) satisfying (1.2). Then we have $T'(k^n x) = k^{3n}T'(x)$ for all $x \in X$ and $n \in \mathbb{N}$.

$$\begin{aligned} d_H\big(T(x), T'(x)\big) &= \frac{1}{k^{3n}}d_H\big(T(k^n x), T'(k^n x)\big) \\ &\leq \frac{1}{k^{3n}}\Big(d_H\big(T(k^n x), f(k^n x)\big) + d_H\big(f(k^n x), T'(k^n x)\big)\Big) \qquad (2.12) \\ &\leq \lim_{n\to\infty}\frac{1}{k^{3n}}\sum_{i=0}^{\infty}\phi(k^{i+n}x, 0) \\ &= 0 \end{aligned}$$

for all $x \in X$. Thus we get $T(x) = T'(x)$ for all $x \in X$ which completes this proof. $\square$

Remark 2.4. Let $\phi : X^2 \to [0,\infty)$ be a function with satisfying the property

$$\sum_{i=0}^{\infty} k^{3i}\phi(\frac{1}{k^i}x,0) < \infty, \ \lim_{n\to\infty} k^{3n}\phi(\frac{1}{k^n}x, \frac{1}{k^n}y) = 0 \ \text{ for all } x,\, y \in X.$$

Suppose that $f : X \longrightarrow (CBC(Y), d_H)$ is a set-valued mapping with $f(0) = \{0\}$ and

$$d_H\big(2kf(x+ky) \oplus 2f(kx-y), (k^3+k)f(x+y) \oplus (k^3+k)f(x-y) \\ \oplus 2(k^4-1)f(y)\big) \le \phi(x,y)$$

for all $x, y \in X$. Then there exists a unique cubic set-valued mapping of type (EL) $T : X \to (CBC(Y), d_H)$ such that for each $x \in X$,

$$d_H\big(f(x), T(x)\big) \le \frac{1}{2k^3}\sum_{i=1}^{\infty} k^{3i}\phi(\frac{1}{k^i}x, 0).$$

Corollary 2.5. *Let $\epsilon \ge 0$, $0 < p < 3$ be real numbers. Let $f : X \to (CBC(Y), d_H)$ be a set-valued mapping with satisfying the property $d_H\big(2kf(x+ky) \oplus 2f(kx-y), (k^3+k)f(x+y) \oplus (k^3+k)f(x-y) \oplus 2(k^4-1)f(y)\big) \le \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then there exists a unique cubic set-valued mapping of type (EL)*

$$T : X \to (CBC(Y), d_H)$$

that satisfies (1.2) *and* $d_H\big(f(x), T(x)\big) \le \frac{\epsilon}{2(k^3-k^p)}\|x\|^p$ *for all* $x \in X$.

Proof. The result directly follows theorem 2.3 by setting

$$\phi(x,y) := \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. □

Remark 2.6. In the corollary 2.5, we have results by setting $p > 3$. That is, we obtain a unique cubic set-valued mapping of type (EL) T given by

$$T : X \to (CBC(Y), d_H)$$

that satisfies (1.2) and $d_H\big(f(x), T(x)\big) \le \frac{\epsilon}{2(k^p-k^3)}\|x\|^p$ for all $x \in X$.

Next we investigate the Hyers-Ulam stability of the cubic set-valued functional equation of type (EL) using the alternative fixed point.

LEMMA 2.7. [9] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \ \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

THEOREM 2.8. *Let $f : X \to (CBC(Y), d_H)$ be a mapping with $f(0) = \{0\}$ such that*

$$d_H\big(2kf(x+ky) \oplus 2f(kx-y), (k^3+k)f(x+y) \oplus (k^3+k)f(x-y) \oplus 2(k^4-1)f(y)\big) \leq \phi(x,y) \tag{2.13}$$

for all $x, y \in X$. Suppose that a function $\phi : X^2 \to [0, \infty]$ satisfies

$$\phi(kx, ky) \leq k^3 L \phi(x, y) \tag{2.14}$$

for all $x, y \in X$. Then there exists a unique cubic set-valued mapping of type (EL) $T : X \to (CBC(Y), d_H)$ such that

$$d_H\big(f(x), T(x)\big) \leq \frac{1}{2k^3(1-L)} \phi(x, 0) \tag{2.15}$$

for all $x \in X$.

Proof. Setting $y = 0$ in (2.13) we have

$$d_H\big(\ 2kf(x) \oplus 2f(kx), \quad (k^3+k)f(x) \oplus (k^3+k)f(x) \oplus 2(k^4-1)f(0) \ \big) \leq \phi(x, 0)$$

for all $x \in X$. By the remark 2.1 , we get

$$d_H\big(\frac{1}{k^3} f(kx), f(x)\big) \leq \frac{1}{2k^3} \phi(x, 0) \tag{2.16}$$

for all $x \in X$. Let S be the set of all mapping $g : X \to CBC(Y)$ with $g(0) = \{0\}$. We define a generalized metric on S given by

$$d(g_1(x), g_2(x)) := inf\{M \in [0, \infty) | d_H(g_1(x), g_2(x) \leq M\phi(x, 0), x \in X\},$$

and also define a mapping $J : S \to S$ by

$$(Jg)(x) := \frac{1}{k^3} g(kx)$$

for every $g \in S$ and $x \in X$. Let M be an arbitrary nonnegative constant with $d(g_1(x), g_2(x)) \leq M$. Then we have $d_H(g_1(x), g_2(x)) \leq M\phi(x, 0)$ for all $x \in X$. Thus we have

$$\begin{aligned} d_H\big((Jg_1)(x), (Jg_2)(x)\big) &= \frac{1}{k^3} d_H\big(g_1(kx), g_2(kx)\big) \\ &\leq \frac{1}{k^3} M\phi(kx, 0) \\ &\leq ML\phi(x, 0) \end{aligned}$$

for all $x \in X$. By the definition of the generaizedl metric, we get that for each $g_1, g_2 \in S$,

$$d(Jg_1, Jg_2) \leq Ld(g_1, g_2).$$

So J is a strictly contractive mapping with the Lipschitz contant L. Using (2.16), we easily obtain that $d(Jf, f) \leq \frac{1}{2k^3}$. By lemma 2.7, there exists a unique fixed point T of J given by

$$T : X \to (CBC(Y), d_H) \text{ such that } J^n f \to 0 \text{ as } n \to \infty.$$

Thus we have $T(x) = \lim_{n\to\infty}(J^n f)(x) = \lim_{n\to\infty} \frac{1}{k^{3n}} f(k^n x)$ for all $x \in X$. By lemma 2.7, we also have

$$d(f, T) \leq \frac{1}{1-L} d(Jf, f) \leq \frac{1}{2k^3(1-L)}.$$

It follows from (2.13) and (2.14) that

$$\begin{aligned} d_H\big(2kT(x+ky) &\oplus 2T(kx-y), (k^3+k)T(x+y) \oplus (k^3+k)T(x-y) \\ &\oplus 2(k^4-1)T(y)\big) \leq \lim_{n\to\infty} \frac{1}{2k^{3n}} \phi(k^n x, k^n y) = 0 \end{aligned}$$

for all $x, y \in X$. Therefore $T : X \to (CBC(Y), d_H)$ is a unique cubic set-valued mapping of type (EL). $\square$

Remark 2.9. Let $0 < p < 3$ and $\theta \geq 0$ be real numbers. Let $f : X \to (CBC(Y), d_H)$ with $f(0) = \{0\}$ be a mapping satisfying

$$\begin{aligned} d_H\big(2kf(x+ky) &\oplus 2f(kx-y), (k^3+k)f(x+y) \oplus (k^3+k)f(x-y) \\ &\oplus 2(k^4-1)f(y)\big) \leq \theta(\|x\|^p + \|y\|^p) \end{aligned}$$

for all $x, y \in X$. Then there exists a unique cubic set-valued mapping of type (EL) $T : X \to (CBC(Y), d_H)$ such that $d_H(f(x), T(x)) \leq \frac{\theta}{2(k^3-k^p)}\|x\|^p$ for all $x \in X$. Using a similar method, we get the same result for the case $p > 3$.

References

[1] T. Aoki, *On the stability of the linear transformation in the Banach space*, J. Math. Soc. Jap. **2** (1950), 64-66.

[2] H. Y. Chu, A. Kim, and S. K. Yoo, *On the stability of generalized cubic set-valued functional equation*, Appl. Math. Lett. **37** (2014), 7-14.

[3] H. Y. Chu and S. K. Yoo, *On the stability of an additive set-valued functional equation*, J. Chungcheong Math. Soc. **27** (2014), 455-467.

[4] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. **27** (1941), 222-224.

[5] K. W. Jun and H. M. Kim, *The generalized Hyer-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. **274** (2002), 867-878.

[6] K. W. Jun, H. M. Kim and I. S. Chang, *On the Hyers-Ulam stability of an Euler-Lagrange type cubic functional equation*, J. Comput. Anal. Appl. **7** (2005), 21-33.

[7] H. A. Kenary, H. Rezaei, Y. Gheisari and C. Park, *On the stability of set-valued functional equations with the fixed point alternative*, Fixed Point Theory and Appl. **2012** 2012:81, 17pp.

[8] G. Lu and C. Park, *Hyers-Ulam stability of additive set-valued functional euqtions*, Appl. Math. Lett. **24** (2011), 1312-1316.

[9] B. Margolis and J. B. Diaz, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **126** (1968), 305-309.

[10] A. Najati and F. Moradlou, *Stability of an Euler-Lagrange type cubic functional equation*, Turk. J. Math. **33** (2009), 65-73.

[11] C. Park, D. O'Regan and R. Saadati, *Stability of some set-valued functional equations*, Appl. Math. Lett. **24** (2011), 1910-1914.

[12] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300.

[13] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

*
Department of Mathematics
Chungnam National University
Daejon 34134, Republic of Korea
E-mail: hychu@cnu.ac.kr

**
Department of Mathematics
Chungnam National University
Daejeon 34134, Republic of Korea
E-mail: jpark@cnu.ac.kr

***
Department of Mathematics
Chungnam National University
Daejeon 34134, Republic of Korea
E-mail: skyoo@cnu.ac.kr