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PRINCIPAL COFIBRATIONS AND GENERALIZED CO-H-SPACES

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Abstract. For a map $p : X \to A$, there are concepts of co-H^p-spaces, co-T^p-spaces, which are generalized ones of co-H-spaces [17,18]. For a principal cofibration $i_r : X \to C_r$ induced by $r : X' \to X$ from $\iota : X' \to cX'$, we obtain some sufficient conditions to having extensions co-$H^{\bar{p}}$-structures and co-$T^{\bar{p}}$-structures on C_r of co-H^p-structures and co-T^p-structures on X respectively. We can also obtain some results about co-H^p-spaces and co-T^p-spaces in homology decompositions for spaces, which are generalizations of Golasinski and Klein's result about co-H-spaces.

1. Introduction

A map $f : X \to B$ is *cocyclic* [13] if there is a map $\theta : X \to X \vee B$ such that $j\theta \sim (1 \times f)\Delta$, where $j : X \vee B \to X \times B$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal. It is clear that a space X is a co-H-space if and only if the identity map 1_X of X is cocyclic. We called a space X as a *co-H^p-space for a map $p : X \to A$* [17] if there is a cocyclic map $p : X \to A$, that is, there is a co-H^p-structure $\theta : X \to X \vee A$ such that $j\theta \sim (1 \times p)\Delta$, where $j : X \vee A \to X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal. It is clear that if X is a co-H-space, then for any map $p : X \to A$, X is a co-H^p-space. In Example 2.4, there is a space Q_p which is a co-H^δ-space, but not a co-H-space. Let τ be the adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$. The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$and $\tau(1_{\Sigma X})$ respectively. In [1], Aguade introduced a T-space as a space X having the property that the evaluation fibration $\Omega X \to X^{S^1} \to X$ is fibre homotopically trivial. It is well known [1] that a space X is a T-space if and only if the evaluating

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map $e : \Sigma\Omega X \to X$ is cyclic. As a dual space of T-space, we introduced [14] that a space X is a co-T-space if $e' : X \to \Omega\Sigma X$ is cocyclic. A space X is called [5] a G'-space if $G^n(X) = H^n(X)$ for all n. It is clear that any co-H-space is a co-T-space, and any co-T-space is a G'-space. It is known [14] that $\mathbb{R}P^2$ is a G'-space, but not a co-T-space. We called a space X as a *co-T^p-space for a map* $p : X \to A$ [18] if $e' : X \to \Omega\Sigma X$ is p-cocyclic, that is, there is a co-T^p-structure $\theta : X \to \Omega\Sigma X \vee A$ such that $j\theta \sim (e' \times p)\Delta$, where $j : \Omega\Sigma X \vee A \to \Omega\Sigma X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. It is shown [18] that X is a co-T-space if and only if for any space A and any map $p : X \to A$, X is a co-T^p-space for a map $p : X \to A$. We called a space X as a *G'_p-space for a map* $p : X \to A$ [19] if $e' : X \to \Omega\Sigma X$ is weakly p-cocyclic, that is, $e^*(H^n(\Omega\Sigma X)) \subset G^n(X, p, A)$ for all n. For a map $p : X \to A$, there are concepts of co-H^p-spaces, co-T^p-spaces and G'_p-spaces which are generalized ones of co-H-spaces. In general, any co-H-space is a co-H^p-space, any co-H^p-space is a co-T^p-space and any co-T^p-space is a G'_p-space. In [19], we already studied about some properties of G'_p-spaces for maps and their homology decompositions.

In this paper, we study about relationships between co-H^p-spaces, co-T^p-spaces and their homology decompositions respectively. For a principal cofibration $i_r : X \to C_r$ induced by $r : X' \to X$ from $\iota_{X'} : X' \to cX'$, we obtain some sufficient conditions to having extendings co-$H^{\bar{p}}$-structures and co-$T^{\bar{p}}$-structures on C_r of H^p-structures and T^p-structures on X respectively. Let X and A be rational spaces and $p : X \to A$ a map, and $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ homology decompositions for X and A respectively. Then we can obtain that X is a co-H^p-space for a map $p : X \to A$ if and only if for each n, X_n is a co-H^{p_n}-space and the all pair of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are co-H^{p_n}-primitive. Thus we have, as a corollary, that Golasinski and Klein's result about co-H-spaces. We can also obtain that X is a co-T^p-space for a map $p : X \to A$ if and only if for each n, X_n is a co-T^{p_n}-space and the all pair of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are co-T^{p_n}-primitive.

2. Dual Gottlieb sets for maps and generalized co-H-spaces

Let $p : X \to A$ be a map. A based map $f : X \to B$ is called *p-cocyclic* [10] if there is a map $\theta : X \to A \vee B$ such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} X & \xrightarrow{\theta} & A \vee B \\ {\scriptstyle\Delta}\downarrow & & {\scriptstyle j}\downarrow \\ X \times X & \xrightarrow{(p\times f)} & A \times B, \end{array}$$

where $j : A \vee B \to A \times B$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. We call such a map θ a *coassociated map* of a p-cocyclic map f.

In the case $p = 1_X : X \to X$, $f : X \to B$ is called *cocyclic* [13]. Clearly any cocyclic map is a p-cocyclic map and also $f : X \to B$ is p-cocyclic iff $p : X \to A$ is f-cocyclic. The *dual Gottlieb set* $DG(X, p, A; B)$ *for a map* $p : X \to A$ [16] is the set of all homotopy classes of p-cocyclic maps from X to B. In the case $p = 1_X : X \to X$, we called such a set $DG(X, 1, X; B)$ the *dual Gottlieb set* [13] denoted $DG(X; B)$, that is, the dual Gottlieb set is exactly same with the dual Gottlieb set for the identity map. We denote $DG(X, p, A; K(\pi, n))$ by $G^n(X, p, A; \pi)$ and $DG(X, p, A; K(\mathbb{Z}, n))$ by $G^n(X, p, A)$, $DG(X; K(\mathbb{Z}, n))$ by $G^n(X)$. Haslam [5] introduced and studied the *coevaluation subgroups* $G^n(X; \pi)$ of $H^n(X; \pi)$. $G^n(X; \pi)$ is defined to be the set of all homotopy classes of cocyclic maps from X to $K(\pi, n)$. A space X is called [5] a G'-space if $G^n(X) = H^n(X)$ for all n.

In general, $DG(X; B) \subset DG(X, p, A; B) \subset [X, B]$ for any map $p : X \to B$ and any space B. It is known [16] that for any n, $G^n(S^n \times S^n; \mathbb{Z}) \neq G^n(S^n \times S^n, p_1, S^n; \mathbb{Z}) \neq H^n(S^n \times S^n; \mathbb{Z})$.

The next proposition is an immediate consequence from the definition.

Proposition 2.1. [17]

(1) *For any maps* $g : X \to A$, $h : A \to B$ *and any space* C, $DG(X, g, A; C) \subset DG(X, hg, B; C)$.

(2) $DG(X, B) = DG(X, 1_X, X; B) \subset DG(X, g, A; B) \subset DG(X, *, A; B) = [X, B]$ *for any spaces* X, A *and* B.

(3) $DG(X, B) = \cap\{DG(X, g, A; B) | g : X \to A$ *is a map and* A *is a space*$\}$.

(4) *If* $h : A \to B$ *is a homotopy equivalence, then* $DG(X, g, A; C) = DG(X, hg, B; c)$.

(5) *For any map* $k : Y \to X$, $k^*(DG(X, g, A; B)) \subset DG(Y, gk, A; B)$.

(6) *For any map* $k : Y \to X$, $k^*(DG(X; B)) \subset DG(Y, k, X; B)$.

(7) *For any map* $s : B \to C$, $s_*(DG(X, g, A; B)) \subset DG(X, g, A; C)$.

Proposition 2.2.
(1) [9] *X is a co-H-space* $\Longleftrightarrow$ $DG(X,B)=[X,B]$ *for any space* B.
(2) [14] *X is a co-T-space* $\Longleftrightarrow$ $DG(X,\Omega C)=[X,\Omega C]$ *for any space* C.
(3) [5] *X is a G'-space* $\Longleftrightarrow$ $G^n(X)=H^n(X)$ *for all* n.

It is clear that any co-H-space is a co-T-space and any co-T-space is a G'-space. It is known [14] that $\mathbb{R}P^2$ is a G'-space, but not a co-H-space and co-T-space.

Proposition 2.3. *Let* $p: X \to A$ *be a map. Then*
(1) [17] *X is a co-H^p-space* $\Longleftrightarrow$ $DG(X,p,A;B)=[X,B]$ *for any space* B.
(2) [18] *X is a co-T^p-space* $\Longleftrightarrow$ $DG(X,p,A;\Omega C)=[X,\Omega C]$ *for any space* C.
(3) [19] *X is a G'_p-space* $\Longleftrightarrow$ $G^n(X,p,A)=H^n(X)$ *for all* n.

Thus we know that any co-H-space is a co-H^p-space, any co-H^p-space is a co-T^p-space and any co-T^p-space is a G'_p-space for any map $p: X \to A$.

The following example says that there is a space which is a co-H^p-space, but not a co-H-space.

Example 2.4. *For any odd prime p, let $[f]$ be the generator of p-primary summand of $\pi_{4p-3}(S^2)$ which is isomorphic $\mathbb{Z}/p\mathbb{Z}$. Then it is known [7] that for $Q_p = S^2 \cup_f e^{4p-2}$, cat $Q_p = 2$. It is also well known fact that a space X is a co-H-space if and only if cat $X \leq 1$. Thus we know that Q_p is not a co-H-space. It is also known [6, Proposition 15.8] that for a cofibration sequence $S^{4p-3} \xrightarrow{f} S^2 \xrightarrow{i} Q_p \xrightarrow{\delta} S^{4p-2} \to \cdots$, $\delta: Q_p \to S^{4p-2}$ is a cocyclic map. Moreover, it is known [16] that $p: X \to A$ is a cocyclic map if and only if $DG(X,p,A;B)=[X,B]$ for any space B. Thus we know that Q_p is a co-H^δ-space.*

3. Principal cofibrations and generalized co-H-spaces

Given maps $p: X \to A$, $p': X' \to A'$, let $(s,r): p' \to p$ be a map from p' to p, that is, the following diagram is commutative;

$$\begin{array}{ccc} X' & \xrightarrow{p'} & A' \\ r\downarrow & & s\downarrow \\ X & \xrightarrow{p} & A. \end{array}$$

It is a well known fact that $Y \xrightarrow{\iota} cY \to \Sigma Y$ is a cofibration, where $\iota(y) = [y, 1]$. Let $i_r : X \to C_r$ be the cofibration induced by $r : X' \to X$ from $\iota_{X'} : X' \to cX'$. Let $i_s : A \to C_s$ be the cofibration induced by $s : A' \to A$ from $\iota_{A'} : A' \to cA'$. Then there is a map $\bar{p} : C_t \to C_s$ such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{p} & A \\ {\scriptstyle i_r}\downarrow & & {\scriptstyle i_s}\downarrow \\ C_r & \xrightarrow{\bar{p}} & C_s, \end{array}$$

where $C_r = cX' \amalg X/[x', 1] \sim r(x')$, and $C_s = cA' \amalg A/[a', 1] \sim s(a')$, $\bar{p} : C_r \to C_s$ is given by $\bar{p}([x', t]) = [p'(x'), t]$ if $[x', t] \in cX'$ and $\bar{p}(x) = p(x)$ if $x \in X$, $i_r(x) = x$, $i_s(a) = a$.

Definition 3.1. *Let X be a co-H^p-space for a map $p : X \to A$. A map $(s, r) : p' \to p$ is called a co-H^p-primitive if there is a map $\theta : X \to A \vee X$ such that $j\theta \sim (p \times 1)\Delta$ and $(i_s \vee i_r)\theta r \sim * : X' \to C_s \vee C_r$, where $j : A \vee X \to A \times X$ is the inclusion.*

Definition 3.2. *Let X be a co-T^p-space for a map $p : X \to A$. A map $(s, r) : p' \to p$ is called a co-T^p-primitive if there is a map $\theta : X \to A \vee \Omega\Sigma X$ such that $j\theta \sim (p \times e')\Delta$ and $(i_s \vee \Omega\Sigma i_r)\theta r \sim * : X' \to C_s \vee \Omega\Sigma C_r$, where $j : A \vee \Omega\Sigma X \to A \times \Omega\Sigma X$ is the inclusion.*

Definition 3.3. [19] *Let X be a G'_p-space for a map $p : X \to A$. A map $(s, r) : p' \to p$ is called a G'_p-primitive if for each map $g : \Omega\Sigma X \to K(\mathbb{Z}, m)$, m arbitrary, there is a map $G : X \to A \vee K(\mathbb{Z}, m)$ such that $jG \sim (p \times g \circ e'_X)\Delta$ and $(i_s \vee 1)Gr \sim * : X' \to C_s \vee K(\mathbb{Z}, m)$, where $j : A \vee K(\mathbb{Z}, m) \to A \times K(\mathbb{Z}, m)$ is the inclusion and $e'_X : X \to \Omega\Sigma X$ is the adjoint functor image, $\tau(1_{\Sigma X})$, of $1_{\Sigma X}$.*

Proposition 3.4.

(1) *If X is a co-H^p-space for a map $p : X \to A$ and $(s, r) : p' \to p$ is a co-H^p-primitive, then $(s, r) : p' \to p$ is a co-T^p-primitive.*

(2) *If X is a co-T^p-space for a map $p : X \to A$ and $(s, r) : p' \to p$ is a co-T^p-primitive, then $(s, r) : p' \to p$ is a G'_p-primitive.*

Proof.

(1) Since $(s, r) : p' \to p$ is a co-H^p-primitive, there is a map $\theta : X \to A \vee X$ such that $j\theta \sim (p \times 1)\Delta$ and $(i_s \vee i_r)\theta r \sim * : X' \to C_s \vee C_r$, where $j : A \vee X \to A \times X$ is the inclusion. Let $\theta' = (1 \vee e')\theta : X \to A \vee \Omega\Sigma X$. Then $j'\theta' \sim (1 \times e')j\theta \sim (1 \times e')(p \times 1)\Delta = (p \times e')\Delta$, where $j' : A \vee \Omega\Sigma X \to A \times \Omega\Sigma X$ is the inclusion. Moreover, since

$(i_s \vee \Omega\Sigma i_r)(1 \vee e'_X) \sim (1 \vee e'_{C_r})(i_s \vee i_r) : A \vee X \to C_s \vee \Omega\Sigma C_r$, we have that $(i_s \vee \Omega\Sigma i_r)\theta' r \sim (i_s \vee \Omega\Sigma i_r)(1 \vee e'_X)\theta r \sim (1 \vee e'_{C_r})(i_s \vee i_r)\theta r \sim (1 \vee e'_{C_r})* \sim *$. Thus $(s, r) : p' \to p$ is a co-T^p-primitive.

(2) Since $(s, r) : p' \to p$ is a co-T^p-primitive, there is a map $\theta : X \to A \vee \Omega\Sigma X$ such that $j\theta \sim (p \times e')\Delta$ and $(i_s \vee \Omega\Sigma i_r)\theta r \sim * : X' \to C_s \vee \Omega\Sigma C_r$, where $j : A \vee \Omega\Sigma X \to A \times \Omega\Sigma X$ is the inclusion. For any m, each $g : \Omega\Sigma X \to K(\mathbb{Z}, m)$, let $\theta' = (1 \vee g)\theta : X \to A \vee K(\mathbb{Z}, m)$. Then $j'\theta' \sim (1 \times g)j\theta \sim (1 \times g)(p \times e')\Delta = (p \times ge')\Delta$, where $j' : A \vee K(\mathbb{Z}, m) \to A \times K(\mathbb{Z}, m)$ is the inclusion. Moreover, since $(1 \vee \Omega\Sigma g)(i_s \vee \Omega\Sigma i_r) \sim (i_s \vee 1)(1 \vee g) : A \vee \Omega\Sigma X \to C_s \vee \Omega\Sigma K(\mathbb{Z}, m)$, we have that $(i_s \vee 1)\theta' r = (i_s \vee 1)(1 \vee g)\theta r \sim (1 \vee \Omega\Sigma g)(i_s \vee \Omega\Sigma i_r)\theta r \sim (1 \vee \Omega\Sigma g)* \sim *$. Thus $(s, r) : p' \to p$ is a G'_p-primitive. □

Lemma 3.5.

(1) *A map $f : X \to B$ can be extended to a map $h : C_r \to B$ with $hi_r = f$ if and only if $fr \sim *$.*

(2) [15] *Given maps $g_t : C_r \to B_t (t = 1, 2)$ and $g : C_r \to B_1 \vee B_2$ satisfying $p_t j g i_r \sim g_t i_r (t = 1, 2)$, then there is a map $h : C_r \to B_1 \vee B_2$ such that $gi_r = hi_r$ and $p_t jh \sim g_t (t = 1, 2)$, where $j : B_1 \vee B_2 \to B_1 \times B_2$ is the inclusion and $p_t : B_1 \times B_2 \to B_t, t = 1, 2$ are projections.*

Theorem 3.6.

(1) *If X is a co-H^p-space for a map $p : X \to A$ and $(s, r) : p' \to p$ is co-H^p-primitive, then C_r is a co-$H^{\bar{p}}$-space for a map $\bar{p} : C_r \to C_s$.*

(2) *If X is a co-T^p-space for a map $p : X \to A$ and $(s, r) : p' \to p$ is co-T^p-primitive, then C_r is a co-$T^{\bar{p}}$-space for a map $\bar{p} : C_r \to C_s$.*

Proof.

(1) Since $(s, r) : p' \to p$ is a co-H^p-primitive, there is a map $\theta : X \to A \vee X$ such that $j\theta \sim (p \times 1)\Delta$ and $(i_s \vee i_r)\theta r \sim * : X' \to C_s \vee C_r$, where $j : A \vee X \to A \times X$ is the inclusion. From Lemma 3.5(1), there is an extending $\theta' : C_r \to C_s \vee C_r$ of $(i_s \vee i_r) \circ \theta : X \to C_s \vee C_r$, that is, $\theta' \circ i_r = (i_s \vee i_r) \circ \theta$. Then we have that $p_1 j'\theta' i_r = p_1 j'(i_s \vee i_r)\theta = p_1(i_s \times i_r)j\theta \sim p_1(i_s \times i_r)(p \times 1)\Delta = i_s \circ p \sim \bar{p} \circ i_r$ and $p_2 j'\theta' i_r = p_2 j'(i_s \vee i_r)\theta = p_2(i_s \times i_r)j\theta \sim p_2(i_s \times i_r)(p \times 1)\Delta \sim i_r \sim 1_{C_r} \circ i_r$. Thus we have, from Lemma 3.5(2), that there is a map $\bar{\theta} : C_r \to C_s \vee C_r$ such that $\bar{\theta} i_r = \theta' i_r = (i_s \vee i_r)\theta$ and $p_1 j'\bar{\theta} \sim \bar{p}$ and $p_2 j'\bar{\theta} \sim 1_{C_r}$. Thus we know that $1 : C_r \to C_r$ is $\bar{p}$-cocyclic and C_r is a co-$H^{\bar{p}}$-space for a map $\bar{p} : C_r \to C_s$. This proves the theorem.

(2) Since $(s, r) : p' \to p$ is a co-T^p-primitive, there is a map $\theta : X \to A \vee \Omega\Sigma X$ such that $j\theta \sim (p \times e')\Delta$ and $(i_s \vee \Omega\Sigma i_r)\theta r \sim * : X' \to C_s \vee \Omega\Sigma C_r$, where $j : A \vee \Omega\Sigma X \to A \times \Omega\Sigma X$ is the inclusion. From Lemma

3.5(1), there is an extending $\theta' : C_r \to C_s \vee \Omega\Sigma C_r$ of $(i_s \vee \Omega\Sigma i_r) \circ \theta : X \to C_s \vee \Omega\Sigma C_r$, that is, $\theta' \circ i_r = (i_s \vee \Omega\Sigma i_r) \circ \theta$. Then we have that $p_1 j'\theta' i_r = p_1 j'(i_s \vee \Omega\Sigma i_r)\theta = p_1(i_s \times \Omega\Sigma i_r) j\theta \sim p_1(i_s \times \Omega\Sigma i_r)(p \times e')\Delta = i_s \circ p \sim \bar{p} \circ i_r$ and $p_2 j'\theta' i_r = p_2 j'(i_s \vee \Omega\Sigma i_r)\theta = p_2(i_s \times \Omega\Sigma i_r) j\theta \sim p_2(i_s \times \Omega\Sigma i_r)(p \times e')\Delta \sim \Omega\Sigma i_r e'_X \sim e'_{C_r} \circ i_r$. Thus we have, from Lemma 3.5(2), that there is a map $\bar{\theta} : C_r \to C_s \vee \Omega\Sigma C_r$ such that $\bar{\theta} i_r = \theta' i_r = (i_s \vee \Omega\Sigma i_r)\theta$ and $p_1 j'\bar{\theta} \sim \bar{p}$ and $p_2 j'\bar{\theta} \sim e'_{C_r}$. Thus we know that $e'_{C_r} : C_r \to \Omega\Sigma C_r$ is $\bar{p}$-cocyclic and C_r is a co-$T^{\bar{p}}$-space for a map $\bar{p} : C_r \to C_s$. This proves the theorem. $\square$

In 1959, Eckmann and Hilton [2] introduced a dual concept of Postnikov system as follows; *A homology decomposition of* X consists of a sequence of spaces and maps $\{X_n, q_n, i_n\}$ satisfying (1) $q_n : X_n \to X$ induces an isomorphism $(q_n)_* : H_i(X_n) \to H_i(X)$ for $i \leq n$ and $H_i(X_n) = 0$ for $i > n$, (2) $i_n : X_n \to X_{n+1}$ is a cofibration with cofiber $M(H_{n+1}(X), n)$(a Moore space of type $(H_{n+1}(X), n)$), (3) $q_n \sim q_{n+1} \circ i_n$. It is known by [6] that if X be a 1-connected space having the homotopy type of CW complex, then there is a homology decomposition $\{X_n, q_n, i_n\}$ of X such that $i_n : X_n \to X_{n+1}$ is the principal cofibration induced from $\iota : M(H_{n+1}(X), n) \to cM(H_{n+1}(X), n)$ by a map $\kappa'_n : M(H_{n+1}(X), n) \to X_n$ which is called the dual Postnikov invariants. A space X is called a *rational space* [11] if X is a 1-connected space having homotopy type of a CW-complex such that for each $n > 0$, $H_n(X, \mathbb{Z})$ is a finite dimensional vector space over $\mathbb{Q}$. It is well known [11] that if X and A are rational spaces and $p : X \to A$ is a based map, then there exist homology decompositions $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ for X and A respectively and induced maps $\{p_n : X_n \to A_n\}$ satisfying

(1) for each n, the following diagram is homotopy commutative

$$\begin{array}{ccc} M(H_{n+1}(X), n) & \xrightarrow{\tilde{p}_*} & M(H_{n+1}(A), n) \\ {\scriptstyle k'_n(X)}\downarrow & & \downarrow{\scriptstyle k'_n(A)} \\ X_n & \xrightarrow{p_n} & A_n \end{array}$$

, that is, $(k'_n(A), k'_n(X)) : \tilde{p}_\# \to p_n$ is a map,

(2) $p_{n+1} : X_{n+1} \to A_{n+1}$ given by $p_{n+1} = \bar{p_n}$ satisfying commute diagram

$$\begin{array}{ccc} X_n & \xrightarrow{p_n} & A_n \\ {\scriptstyle i_n(=\iota_{k'_n(X)})}\downarrow & & \downarrow{\scriptstyle i'_n(=\iota_{k'_n(A)})} \\ X_{n+1} & \xrightarrow{p_{n+1}} & A_{n+1}, \end{array}$$

(3) for each n, the following diagram is homotopy commutative

$$\begin{array}{ccc} X_n & \xrightarrow{p_n} & A_n \\ {\scriptstyle q_n}\downarrow & & \downarrow{\scriptstyle q'_n} \\ X & \xrightarrow{p} & A. \end{array}$$

THEOREM 3.7. *Let X and A be rational spaces and $p : X \to A$ a map, and $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ homology decompositions for X and A respectively.*

(1) *If X is a co-H^p-space for a map $p : X \to A$, then each X_n is a co-H^{p_n}-space and the all pair of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are co-H^{p_n}-primitive.*

(2) *If X_{n-1} is a co-$H^{p_{n-1}}$-space and the pair of k'-invariants $(k'_{n-1}(A), k'_{n-1}(X)) : \tilde{p}_* \to p_{n-1}$ is co-$H^{p_{n-1}}$-primitive, then X_n is a co-H^{p_n}-space.*

Proof. (1) Since X is a co-H^p-space for a map $p : X \to A$, there is a map $\theta : X \to A \vee X$ such that $j\theta \sim (p \times 1)\Delta$, where $j : A \vee X \to A \times X$ is the inclusion. Then $\{A_n \vee X_n, q'_n \vee q_n, i'_n \vee i_n\}$ is a homology decomposition for $A \vee X$. Then we have, by Toomer's result [12,Theorem 4], that there are families of maps $p_n : X_n \to A_n$ and $\theta_n : X_n \to A_n \vee X_n$ such that $i'_n p_n = p_{n+1} i_n$ and $q'_n p_n \sim p q_n$, and $(i'_n \vee i_n)\theta_n = \theta_{n+1} i_n$ and $(q'_n \vee q_n)\theta_n \sim \theta q_n$ for $n = 2, 3, \cdots$ respectively, and $k'_n(A)\tilde{p}_* \sim p_n k'_n(X) : M(H_{n+1}(X), n) \to A_n$ and $(k'_n(A) \vee k'_n(X)\tilde{\theta}_* \sim \theta_n k'_n(X) : M(H_{n+1}(X), n) \to A_n \vee X_n$, where $k'_n(A) : M(H_{n+1}(A), n) \to A_n$, $k'_n(X) : M(H_{n+1}(X), n) \to X_n$ are k'-invariants of A and X respectively, $\tilde{p}_* : M(H_{n+1}(X), n) \to M(H_{n+1}(A), n)$ and $\tilde{\theta}_* : M(H_{n+1}(X), n) \to M(H_{n+1}(A \vee X), n) \approx M(H_{n+1}(A) \oplus H_{n+1}(X), n) \approx M(H_{n+1}(A), n) \vee M(H_{n+1}(X), n)$ are the induced maps by $p : X \to A$ and $\theta : X \to A \vee X$ respectively. It is known [12] that the homology decomposition of a rational space is well defined up to homotopy type. Thus we know that if $f \sim g : X \to A$, then $f_n \sim g_n : X_n \to A_n$. Since $p_1 j\theta \sim p$ and $p_2 j\theta \sim 1$, we know that $p_1 j_n \theta_n \sim p_n$ and $p_2 j_n \theta_n \sim 1$. Thus for each n, there exists a co-H^p-structure $\theta_n : X_n \to A_n \vee X_n$ such that $j_n\theta_n \sim (p_n \times 1)\Delta$, where $j_n : A_n \vee X_n \to A_n \times X_n$ is the inclusion and $p_n : X_n \to A_n$ is an induced map from $p : X \to A$, and X_n is a co-H^{p_n}-space. Moreover, since there is an extension $\theta_{n+1} : X_{n+1} \to A_{n+1} \vee X_{n+1}$ of θ_n such that $\theta_{n+1} i_n = (i'_n \vee i_n)\theta_n$, we know, from Lemma , that $(i'_n \vee i_n)\theta_n k'_n(X) \sim *$ and all the pair of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are co-H^{p_n}-primitive.

(2) It follows from Theorem 3.6 (1).

□

Observe that X and A are homotopy types of the direct limits $\varinjlim X_n$ and $\varinjlim A_n$ respectively. Moreover, since each X_n a co-H^{p_n}-space and all pair of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are co-H^{p_n}-primitive, we see that X admit a co-H^p-structure. Thus we have the following corollary.

Corollary 3.8. *Let X and A be rational spaces and $p : X \to A$ a map, and $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ homology decompositions for X and A respectively. Then X is a co-H^p-space for a map $p : X \to A$ if and only if for each n, X_n is a co-H^{p_n}-space and the all pair of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are co-H^{p_n}-primitive.*

Taking $p = 1_{X_n}$, $p' = 1_{M(H_{n+1}(X),n)}$, $r = s = k'_n(X)$, we can obtain, from the fact that $i_n : X_n \to X_{n+1}$ is a co-H-map if and only if $(k'_n(X), k'_n(X)) : 1 \to 1_{X_n}$ is co-H-primitive and the above corollary, the following corollary given by Golasinski and Klein [3] for rational spaces.

Corollary 3.9. [3] *Let X be a rational space and $\{X_n, q_n, i_n\}$ a homology decomposition for X. Then X is n co-H-space if and only if for each X_n there exists such a co-H-structure that $i_n : X_n \to X_{n+1}$ is a co-H-map.*

Theorem 3.10. *Let X and A be rational spaces and $p : X \to A$ a map, and $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ homology decompositions for X and A respectively.*

(1) *If X is a co-T^p-space for a map $p : X \to A$, then each X_n is a co-T^{p_n}-space and the all pair of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are co-T^{p_n}-primitive.*

(2) *If X_{n-1} is a co-$T^{p_{n-1}}$-space and the pair of k'-invariants $(k'_{n-1}(A), k'_{n-1}(X)) : \tilde{p}_* \to p_{n-1}$ is co-$T^{p_{n-1}}$-primitive, then X_n is a co-T^{p_n}-space.*

Proof. (1) Since X is a co-T^p-space for a map $p : X \to A$, there is a map $\theta : X \to A \vee \Omega\Sigma X$ such that $j\theta \sim (p \times e')\Delta$, where $j : A \vee \Omega\Sigma X \to A \times \Omega\Sigma X$ is the inclusion. Then $\{A_n \vee \Omega\Sigma X_n, q'_n \vee \Omega\Sigma q_n, i'_n \vee \Omega\Sigma i_n\}$ is a homology decomposition for $A \vee \Omega\Sigma X$. Then we have, by Toomer's result [15,Theorem 4], that there are families of maps $p_n : X_n \to A_n$ and $\theta_n : X_n \to A_n \vee \Omega\Sigma X_n$ such that $i'_n p_n = p_{n+1} i_n$ and $q'_n p_n \sim p q_n$, and $(i'_n \vee \Omega\Sigma i_n)\theta_n = \theta_{n+1} i_n$ and $(q'_n \vee \Omega\Sigma q_n)\theta_n \sim \theta q_n$ for $n = 2, 3, \cdots$ respectively, and $k'_n(A)\tilde{p}_* \sim p_n k'_n(X) : M(H_{n+1}(X), n) \to A_n$ and $(k'_n(A) \vee$

$k'_n(\Omega\Sigma X)\tilde{\theta}_* \sim \theta_n k'_n(X) : M(H_{n+1}(X), n) \to A_n \vee \Omega\Sigma X_n$, where $k'_n(A) : M(H_{n+1}(A), n) \to A_n$, $k'_n(X) : M(H_{n+1}(X), n) \to X_n$ are k'-invariants of A and X respectively, $\tilde{p}_* : M(H_{n+1}(X), n) \to M(H_{n+1}(A), n)$ and $\tilde{\theta}_* : M(H_{n+1}(X), n) \to M(H_{n+1}(A \vee \Omega\Sigma X), n) \approx M(H_{n+1}(A) \oplus H_{n+1}(\Omega\Sigma X), n) \approx M(H_{n+1}(A), n) \vee M(H_{n+1}(\Omega\Sigma X), n)$ are the induced maps by $p : X \to A$ and $\theta : X \to A \vee \Omega\Sigma X$ respectively. It is known [12] that the homology decomposition of a rational space is well defined up to homotopy type. Thus we know that if $f \sim g : X \to A$, then $f_n \sim g_n : X_n \to A_n$. Since $p_1 j\theta \sim p$ and $p_2 j\theta \sim e'$, we know that $p_1 j_n \theta_n \sim p_n$ and $p_2 j_n \theta_n \sim e'_{X_n}$. Thus for each n, there exists a co-T^p-structure $\theta_n : X_n \to A_n \vee \Omega\Sigma X_n$ such that $j_n\theta_n \sim (p_n \times e'_{X_n})\Delta$, where $j_n : A_n \vee \Omega\Sigma X_n \to A_n \times \Omega\Sigma X_n$ is the inclusion and $p_n : X_n \to A_n$ is an induced map from $p : X \to A$, and X_n is a co-T^{p_n}-space. Moreover, since there is an extension $\theta_{n+1} : X_{n+1} \to A_{n+1} \vee \Omega\Sigma X_{n+1}$ of θ_n such that $\theta_{n+1} i_n = (i'_n \vee \Omega\Sigma i_n)\theta_n$, we know, from Lemma , that $(i'_n \vee \Omega\Sigma i_n)\theta_n k'_n(X) \sim *$ and all the pair of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are co-T^{p_n}-primitive.

(2) It follows from Theorem 3.6(2).

□

Observe that X and A are homotopy types of the direct limits $\varinjlim X_n$ and $\varinjlim A_n$ respectively. Moreover, since each X_n a co-T^{p_n}-space and all pair of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are co-T^{p_n}-primitive, we see that X admit a co-T^p-structure. Thus we have the following corollary.

Corollary 3.11. *Let X and A be rational spaces and $p : X \to A$ a map, and $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ homology decompositions for X and A respectively. Then X is a co-T^p-space for a map $p : X \to A$ if and only if for each n, X_n is a co-T^{p_n}-space and the all pair of k' invariants $(k'_n(A), k'_n(X)) : \tilde{p}_* \to p_n$ are co-T^{p_n}-primitive.*

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