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# ON SOME TWISTED COHOMOLOGY OF THE RING OF INTEGERS

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Abstract. As an analogy of Poincaré series in the space of modular forms, T. Ono associated a module $M_c/P_c$ for $\gamma = [c] \in H^1(G, R^\times)$ where finite group $G$ is acting on a ring $R$. $M_c/P_c$ is regarded as the 0-dimensional twisted Tate cohomology $\widehat{H}^0(G, R^+)_\gamma$. In the case that $G$ is the Galois group of a Galois extension $K$ of a number field $k$ and $R$ is the ring of integers of $K$, the vanishing properties of $M_c/P_c$ are related to the ramification of $K/k$. We generalize this to arbitrary $n$-dimensional twisted cohomology of the ring of integers and obtain the extended version of theorems. Moreover, some explicit examples on quadratic and biquadratic number fields are given.

## 1. Introduction

T. Ono developed an algebraic analogy of the space of modular forms and the subspace of Poincaré series in the cohomological point of view, as a form of quotient module. Let $G$ be a finite group acting on a ring $R$ with the unit 1 and denote the action by $(s, a) \mapsto {}^s a$, $a \in R$, $s \in G$. Since the group $G$ acts on the unit group $R^\times$ of $R$, we can consider the 1-cocycle set

$$Z^1(G, R^\times) = \{c : G \to R^\times;\ c_{st} = c_s\, {}^s c_t,\ s, t \in G\},$$

where $c_s = c(s)$ for a cocycle $c$ and $s \in G$. Two cocycles $c, c'$ are equivalent (cohomologous), denoted by $c \sim c'$, if there is a unit element $u \in R^\times$ such that $c'_s = u^{-1} c_s\, {}^s u,\ s \in G$. The cohomology set is defined by

$$H^1(G, R^\times) = Z^1(G, R^\times)/\sim .$$

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For a 1-cocycle $c \in Z^1(G, R^\times)$, T. Ono defined the modules $M_c$ and $P_c$, named after modular form and Poincaré series, given by

$$M_c = \{a \in R;\ c_s\,{}^s a = a,\ s \in G\}, \text{ and}$$
$$P_c = \{p_c(a) = \sum_{t \in G} c_t\,{}^t a;\ a \in R\},$$

where $p_c(a)$ is called the *Poincaré sum* [4, 5, 6]. The module structure of $M_c/P_c$ depends only on the cohomology class $\gamma = [c] \in H^1(G, R^\times)$ containing $c$. In particular, if $c \sim 1$, the module $M_c/P_c$ is equal to the 0-dimensional Tate cohomology $\widehat{H}^0(G, R)$. For a general cocycle $c$, $M_c/P_c$ can be considered as the twisted 0-Tate cohomology $\widehat{H}^0(G, R)_\gamma$ with the new action $a \mapsto c_s\,{}^s a$. The case that $G$ is the Galois group of a Galois extension $K/k$ of number fields and $R$ is the ring of integers $\mathcal{O}_K$ of $K$ was studied in [4, 5, 6, 7]. T. Ono proved in [7] the following theorems;

THEOREM 1.1. *Let $K/k$ be a finite Galois extension of number fields. If $K/k$ is unramified or tamely ramified, then $M_c = P_c$ for all cocycle $c \in Z^1(\mathrm{Gal}(K/k), {\mathcal{O}_K}^\times)$.*

THEOREM 1.2. *Let $K/k$ be a finite Galois extension of number fields and $G = \mathrm{Gal}(K/k)$. For a cocycle $c \in Z^1(G, {\mathcal{O}_K}^\times)$ denote by $c_{\mathfrak{P}}$ the cocycle induced from $c$ by localization at $\mathfrak{P}$. Then we have the product relation $(M_c : P_c) = \prod_{\mathfrak{p}} (M_{c_{\mathfrak{P}}} : P_{c_{\mathfrak{P}}})$ where for each $\mathfrak{p}$ we choose one $\mathfrak{P}$ dividing $\mathfrak{p}$.*

In this work, these theorems are generalized as follows: First, we define $n$-dimensional twisted cohomology $H^n(G, \mathcal{O}_K)_c$ in section 2, then we show that if $K/k$ is tamely ramified then the twisted $n$-cohomology vanishes for any integer $n$ and for any cocycle $c$ (Theorem 3.1), and that

$$H^n(G, \mathcal{O}_K)_c \cong \prod_{\mathfrak{p}} H^n(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}},$$

where for each $\mathfrak{p}$ we choose one $\mathfrak{P}$ dividing $\mathfrak{p}$ (Theorem 5.1).

On the other hand, Yokoi [9, 10, 11] and Lee and Madan [3] studied $n$-dimensional Galois cohomology of the ring of integers. We may consider this paper as a generalization of their works [3] and [9, 10, 11] into the twisted cohomology as well.

Moreover, in Theorem 3.2 it is shown that for the cyclic Galois extension, we have $|H^n(G, \mathcal{O}_K)_c| = |M_c/P_c|$ for all positive integer $n$ and for all 1-cocycle $c \in Z^1(G, \mathcal{O}_K^\times)$. This does not hold for a non-cyclic extension, e.g. biquadratic fields (Section 7).

## 2. $n$-dimensional twisted cohomology

For a 1-cocycle $c \in Z^1(G, R^\times)$, we define the twisted action $(s, a) \mapsto c_s\, {}^s a$ of $G$ on the additive group $R$. We can consider $R$ as the $G$-module defined by the twisted action and let us call it a $c$-twisted $G$-module. If $A$ is a $G$-module derived from the additive group $R$ which admits the twisted action by $c$, then also we can consider $A$ as a $c$-twisted $G$-module.

For a $c$-twisted $G$-module $A$, we have $n$-dimensional twisted cohomology group $H^n(G, R)_c$ for each nonnegative integer $n$ as follows: Let $C^n(G, A)_c$, the set of $n$-chains, be the set of all maps of $G^n$ to $A$ for $n > 0$ and let $C^0(G, A)_c = \{1_G\}$. The coboundary map $d_{n+1} : C^n(G, A)_c \to C^{n+1}(G, A)_c$ is defined by

$$\begin{aligned}(d_{n+1}f)(s_1, \ldots, s_{n+1}) =& c_{s_1}\, {}^{s_1}f(s_2, \ldots, s_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(s_1, \ldots, s_i s_{i+1}, \ldots, s_{n+1}) \\ &+ (-1)^{n+1} f(s_1, \ldots, s_n)\end{aligned}$$

then $\operatorname{im} d_n \subset \ker d_{n+1}$. Denote by $H^n(G, R)_c = Z^n(G, R)_c / B^n(G, R)_c$ where $Z^n(G, R)_c = \ker d_{n+1}$ is the the twisted $n$-cocycle and $B^n(G, R)_c = \operatorname{im} d_n$ is the twisted $n$-coboundary. Here we have $H^0(G, R)_c = M_c$ and $\widehat{H}^0(G, R)_c = M_c / P_c$. The twisted 1-cocycle in $A$ is the function $d : G \to A$ such that

$$d_{st} = d_s + c_s\, {}^s d_t$$

and the twisted 1-coboundary is $d_s = c_s\, {}^s b - b$ for some $b \in A$.

A homomorphism between $c$-twisted $G$-modules is well defined, and induces a group homomorphism between cohomologies. As usual, if

$$0 \to A \to B \to C \to 0$$

is an exact sequence of $c$-twisted $G$-modules, then we have the long exact sequence of twisted cohomology groups:

$$\begin{aligned}0 &\to H^0(G, A)_c \to H^0(G, B)_c \to H^0(G, C)_c \to H^1(G, A)_c \\ &\to H^1(G, B)_c \to H^1(G, C)_c \to H^2(G, A)_c \to H^2(G, B)_c \\ &\to H^2(G, C)_c \to \cdots\end{aligned}$$

Now assume that the group $G$ is cyclic with the generator $s$ of order $n$. Define two endomorphisms $\Delta$ and $N$ of $c$-twisted $G$-module $A$ such that

$$\Delta = 1 - c_s s, \qquad N = \sum_{i=0}^{n-1} c_{s^i} s^i.$$

Note that $c_{s^i} = c_s\, {}^{s}c_s\, {}^{s^2}c_s \cdots {}^{s^{i-1}}c_s$. Then we have $\Delta N = N\Delta = 0$ and this means $\operatorname{im} N \subset \ker \Delta$ and $\operatorname{im} \Delta \subset \ker N$. It turns out to be

$$\widehat{H}^0(G, A)_c = \ker \Delta / \operatorname{im} N, \qquad H^1(G, A)_c = \ker N / \operatorname{im} \Delta$$

and Herbrand theorem holds: If

$$0 \to A \to B \to C \to 0$$

is an exact sequence of $c$-twisted $G$-modules, then the twisted cohomology exact sequence is an exact hexagon:

$$\begin{array}{ccccc} & \widehat{H}^0(G,A)_c & \longrightarrow & \widehat{H}^0(G,B)_c & \\ \nearrow & & & & \searrow \\ H^1(G,C)_c & & & & \widehat{H}^0(G,C)_c \\ \nwarrow & & & & \swarrow \\ & H^1(G,B)_c & \longleftarrow & H^1(G,A)_c & \end{array}$$

We denote by $h_c(A) = |\widehat{H}^0(G, A)_c|/|H^1(G, A)_c|$ the Herbrand quotient of the twisted cohomology. If two of three Herbrand quotients $h_c(A)$, $h_c(B)$, $h_c(C)$ are defined, so is the third and we have

$$h_c(B) = h_c(A)h_c(B).$$

Now we will verify that for $n \geq 1$, the structure of the twisted cohomology $H^n(G, R)_c$ twisted by $c$ depends only on the cohomology class $\gamma = [c] \in H^1(G, R^\times)$. Let $c \sim c'$, i.e. for some $u \in R^\times$,

$$c'_s = u^{-1} c_s\, {}^{s}u \quad \text{for all } s \in G.$$

Then

$$\begin{aligned} f \in Z^n(G,R)_{c'} \Leftrightarrow\ & f : G^n \to R \text{ such that , for } s_1,\dots,s_{n+1} \in G, \\ & c'_{s_1}\,{}^{s_1}f(s_2,\dots,s_{n+1}) \\ & = \sum_{i=1}^{n}(-1)^{i+1} f(s_1,\dots,s_i s_{i+1},\dots,s_{n+1}) \\ & + (-1)^n f(s_1,\dots,s_n) \\ & \Leftrightarrow c_{s_1}\,{}^{s_1}(uf(s_2,\dots,s_{n+1})) \\ & = \sum_{i=1}^{n}(-1)^{i+1} uf(s_1,\dots,s_i s_{i+1},\dots,s_{n+1}) \\ & + (-1)^n uf(s_1,\dots,s_n) \\ & \Leftrightarrow uf \in Z^n(G,R)_c. \end{aligned}$$

And

$$\begin{aligned} g \in B^n(G,R)_{c'} \Leftrightarrow\ & g : G^n \to R \text{ such that there exists } f : G^{n-1} \to R, \\ & g(s_1,\dots,s_n) = c'_{s_1}\,{}^{s_1}f(s_2,\dots,s_n) \\ & \qquad + \sum_{i=1}^{n-1}(-1)^i f(s_1,\dots,s_i s_{i+1},\dots,s_n) \\ & \qquad + (-1)^{n+1} f(s_1,\dots,s_{n-1}) \\ \Leftrightarrow\ & ug(s_1,\dots,s_n) = c_{s_1}\,{}^{s_1}(uf(s_2,\dots,s_n)) \\ & \qquad + \sum_{i=1}^{n-1}(-1)^i uf(s_1,\dots,s_i s_{i+1},\dots,s_n) \\ & \qquad + (-1)^{n+1} uf(s_1,\dots,s_{n-1}) \\ \Leftrightarrow\ & ug \in B^n(G,R)_c. \end{aligned}$$

Hence we have

$$\begin{aligned} H^n(G,R)_c &= Z^n(G,R)_c / B^n(G,R)_c \\ &= uZ^n(G,R)_{c'} / uB^n(G,R)_{c'} \cong H^n(G,R)_{c'}. \end{aligned}$$

We state this result in the following proposition.

PROPOSITION 2.1. *If $c$ and $c'$ are cohomologous in $Z^1(G,\mathcal{O}_K^\times)$, then we have*

$$H^n(G,R)_c \cong H^n(G,R)_{c'}.$$

## 3. Galois extension of number fields

Let $K/k$ be a finite Galois extension of number fields. Through this paper, we will take the group $G$ as the Galois group $\mathrm{Gal}(K/k)$ of the extension and the ring $R$ as the ring of integers $\mathcal{O}_K$ of $K$. However, firstly we consider the case $R = K$. By Hilbert's Theorem 90, any cocycle $c \in Z^1(G, K^\times)$ is a coboundary, that implies, $H^n(G,K)_c \cong H^n(G,K)$, which vanishes for all dimension $n$. Now we take $R = \mathcal{O}_K$. A cocycle $c \in Z^1(G, \mathcal{O}_K^\times)$ can be regarded as a cocycle into $K^\times$, then $\mathcal{O}_K$ can be dealt as a twisted submodule of $K$.

We have the generalization of Theorem 1.1 on $n$-dimensional twisted cohomology.

Theorem 3.1. *If the finite Galois extension $K/k$ of number fields is tamely ramified, then $H^n(G, \mathcal{O}_K)_c$ vanishes for all dimension $n \geq 1$ and for all cocycle $c \in Z^1(G, {\mathcal{O}_K}^\times)$.*

*Proof.* Let $F$ be an intermediate field of $K/k$ and $\Gamma$ be the Galois group of $K$ over $F$. Since $\Gamma$ is a subgroup of $G$, any $c$-twisted $G$-module is $c$-twisted $\Gamma$-module, and twisted cohomology of $\Gamma$ makes sense. We will verify that for any intermediate field $F$ of $K/k$ with the Galois group $\Gamma = \mathrm{Gal}(K/F)$, the 0- and 1-dimensional twisted Tate cohomology groups vanish. Then the result follows.

(1) For $c \in Z^1(G, {\mathcal{O}_K}^\times)$, let $c'$ be the restriction of $c$ on $\Gamma$ then $c' \in Z^1(\Gamma, {\mathcal{O}_K}^\times)$. Note that $\widehat{H}^0(\Gamma, \mathcal{O}_K)_c = \widehat{H}^0(\Gamma, \mathcal{O}_K)_{c'}$. Since $K/k$ is tamely ramified, $K/F$ is also tamely ramified. By Theorem 1.1 and the above assertion, we have $\widehat{H}^0(\Gamma, \mathcal{O}_K)_c = 0$.

(2) Since $K/F$ is tamely ramified, the trace map $Tr_{K/F} : \mathcal{O}_K \to \mathcal{O}_F$ is surjective so there exists an element $\alpha \in \mathcal{O}_K$ such that $Tr_{K/F}(\alpha) = 1$. Let $c \in Z^1(G, {\mathcal{O}_K}^\times)$ be a 1-cocycle. For any twisted 1-cocycle $d \in Z^1(\Gamma, \mathcal{O}_K)_c$, define $\beta = \sum_{t \in \Gamma} d_t\, {}^t\alpha \in \mathcal{O}_K$ *the Poincaré sum for the twisted cocycle* $d$ then we have

$$\begin{aligned} \beta - c_s\, {}^s\beta &= \sum_{t\in\Gamma} d_t\, {}^t\alpha - \sum_{t\in\Gamma} c_s\, {}^s d_t\, {}^{st}\alpha \\ &= \sum_{t\in\Gamma} d_t\, {}^t\alpha - \sum_{t\in\Gamma} (d_{st} - d_s)\, {}^{st}\alpha \\ &= \sum_{t\in\Gamma} d_t\, {}^t\alpha - \sum_{t\in\Gamma} d_{st}\, {}^{st}\alpha + d_s \sum_{t\in\Gamma} {}^{st}\alpha \\ &= d_s \cdot Tr_{K/F}(\alpha) = d_s. \end{aligned} \tag{3.1}$$

This implies that any twisted 1-cocycle is a twisted 1-coboundary.

□

THEOREM 3.2. *Let $K/k$ be a finite cyclic Galois extension of number fields with the Galois group $G = \mathrm{Gal}(K/k)$, and let $c \in Z^1(G, \mathcal{O}_K{}^{\times})$ be a 1-cocycle of $G$ in the unit group of $\mathcal{O}_K$. Then the twisted cohomology group $H^n(G, \mathcal{O}_K)_c$ has the same order with $\widehat{H}^0(G, \mathcal{O}_K)_c = M_c/P_c$ for every dimension $n \geq 1$.*

*Proof.* Let $s$ be the generator of the group $G$ of order $m$. We see $M_c$ as a twisted $G$-module. Since $M_c$ makes an ideal of $\mathcal{O}_k$ by multiplied by a proper element of $\mathcal{O}_K$ as in [7], the rank of $M_c$ is same as $\mathcal{O}_k$. On the other hand, let $\beta \in \mathcal{O}_K$ be the element generating the normal basis for $K/k$. Then $\{c_{s^1}\, {}^{s^1}\beta, c_{s^2}\, {}^{s^2}\beta, \ldots, c_{s^m}\, {}^{s^m}\beta\}$ is also a basis of $K/k$ since the basis change matrix is diagonal with entries $c_{s^i}$, the unit elements. Define

$$M = M_c\, c_{s^1}\, {}^{s^1}\beta + M_c\, c_{s^2}\, {}^{s^2}\beta + \cdots + M_c\, c_{s^m}\, {}^{s^m}\beta$$

then we have that the index $[\mathcal{O}_K : M]$ is finite and that $M$ is $G$-regular as a twisted $G$-module since each $M_c c_{s^i}\, {}^{s^i}\beta$ forms a direct summand of $M$. Let $N = \mathcal{O}_K/M$ then $N$ is a finite twisted $G$-module. We have an exact sequence of twisted modules:

$$0 \to M \to \mathcal{O}_K \to N \to 0$$

Since $G$ is cyclic, we have $H^{2l}(G, \mathcal{O}_K)_c$
$\cong \widehat{H}^0(G, \mathcal{O}_K)_c$ and $H^{2l-1}(G, \mathcal{O}_K)_c \cong H^1(G, \mathcal{O}_K)_c$ for any positive integer $l$. $H^n(G, M)_c$ vanishes for any $n$ because $M$ is a twisted $G$-regular module. Hence we have $\widehat{H}^0(G, \mathcal{O}_K)_c \cong \widehat{H}^0(G, N)_c$ and $H^1(G, \mathcal{O}_K)_c \cong H^1(G, N)_c$. Also, since $N$ is a finite module, we have that the Herbrand quotient $h_c(N)$ of the twisted cohomology of $N$ is equal to 1, which concludes the theorem. □

## 4. 1-dimensional twisted cohomology

In this section, we study the structure of 1-dimensional twisted cohomology. We give an analogy of the group $H_{K/k}(\mathfrak{a})$ suggested in [10] into the case of twisted cohomology. Let a 1-cocycle $c \in Z^1(G, \mathcal{O}_K{}^{\times})$ and a nonzero integer $a \in Tr_{K/k}(\mathcal{O}_K)$ in $k$ be fixed. For a twisted 1-cocycle $d \in Z^1(G, \mathcal{O}_K)_c$, we can choose an integer $\beta$ in $\mathcal{O}_K$ such that

$$\beta - c_s\, {}^s\beta = a \cdot d_s \qquad \text{for all } s \in G. \tag{4.1}$$

Indeed, let $\alpha$ be the element in $\mathcal{O}_K$ with the trace value $a$ and let $\beta = \sum_{t\in G} d_t\, {}^t\alpha \in \mathcal{O}_K$ be of form of the Poincaré sum for the twisted cocycle $d$ then we get the equation (4.1) with the similar assertion to (3.1). From the equation (4.1) we have that

$$\beta \equiv c_s\, {}^s\beta \bmod a\mathcal{O}_K. \tag{4.2}$$

Conversely, if $\beta \in \mathcal{O}_K$ satisfies the congruence relation (4.2) then $\beta$ defines a twisted 1-cocycle $d \in Z^1(G, \mathcal{O}_K)_c$. Indeed, for each $s \in G$, let $d_s \in \mathcal{O}_K$ such that $\beta - c_s\, {}^s\beta = a \cdot d_s$ then $d_s$ satisfies the condition of twisted 1-cocycle $d_{st} = d_s + c_s\, {}^s d_t$. Now we obtain the following proposition.

PROPOSITION 4.1. *Denote by*

$$Z_c(a) = \{\beta \in \mathcal{O}_K\ ;\ \beta \equiv c_s\, {}^s\beta \ mod\ a\mathcal{O}_K\}$$

*then we have*

$$Z^1(G, \mathcal{O}_K)_c \cong Z_c(a)/M_c.$$

*Proof.* Define the map $\varphi : Z_c(a) \to Z^1(G, \mathcal{O}_K)_c$ as above: $\beta \mapsto d$ such that $\beta - c_s\, {}^s\beta = a \cdot d_s$. Then $\varphi$ is surjective by the above assertion and trivially $\varphi$ is a homomorphism. One sees that $\ker\varphi = M_c$ easily. □

Now, for the twisted 1-coboundary, we have

PROPOSITION 4.2. *Denote by*

$$B_c(a) = \{\beta \in \mathcal{O}_K\ ;\ \beta \equiv b \ mod\ a\mathcal{O}_K \ for\ some\ b \in M_c\}$$

*then we have*

$$B^1(G, \mathcal{O}_K)_c \cong B_c(a)/M_c.$$

*Proof.* First note that clearly $B_c(A) \subset Z_c(a)$. The homomorphism $\varphi : Z_c(a) \to Z^1(G, \mathcal{O}_K)_c$ in the Proposition 4.1 induces an epimorphism $\varphi : B_c(a) \to B^1(G, \mathcal{O}_K)_c$ since

$$\begin{aligned}
\beta \in B_c(a) &\Leftrightarrow \beta \equiv b \bmod a\mathcal{O}_K \text{ for some } b \in M_c\\
&\Leftrightarrow \beta - a\gamma \in M_c \text{ for some } \gamma \in \mathcal{O}_K\\
&\Leftrightarrow \beta - a\gamma = c_s\, {}^s(\beta - a\gamma) \text{ for some } \gamma \in \mathcal{O}_K\\
&\Leftrightarrow \beta - c_s\, {}^s\beta = a(\gamma - c_s\, {}^s\gamma) \text{ for some } \gamma \in \mathcal{O}_K
\end{aligned}$$

and $\beta \mapsto \gamma - c_s\, {}^s\gamma \in B^1(G, \mathcal{O}_K)_c$. It is easy to see that the kernel is also $M_c$. □

As a corollary of Proposition 4.1 and Proposition 4.2, we have

Proposition 4.3. *Let $a$ be a nonzero element of $Tr_{K/k}(\mathcal{O}_K)$. Then the 1-dimensional twisted Galois cohomology group $H^1(G, \mathcal{O}_K)_c$ of $\mathcal{O}_K$ is isomorphic to the factor group*

$$H^1(G, \mathcal{O}_K)_c \cong Z_c(a)/B_c(a).$$

Now we define for an ideal $\mathfrak{a}$ of $\mathcal{O}_k$,

$$Z_c(\mathfrak{a}) := \{\alpha \in \mathcal{O}_K \ ; \ \alpha \equiv c_s\, {}^s\alpha \bmod \mathfrak{a}\mathcal{O}_K\}$$
$$B_c(\mathfrak{a}) := \{\alpha \in \mathcal{O}_K \ ; \ \alpha \equiv b \bmod \mathfrak{a}\mathcal{O}_K \text{ for some } b \in M_c\} = M_c + \mathfrak{a}\mathcal{O}_K$$

and

$$H_c(\mathfrak{a}) := Z_c(\mathfrak{a})/B_c(\mathfrak{a}).$$

If $\alpha \in Z_c(\mathfrak{a})$ then $p_c(\alpha) = \sum_{s\in G} c_s\, {}^s\alpha \equiv |G|\alpha \bmod \mathfrak{a}\mathcal{O}_K$ so that we have $|G|Z_c(\mathfrak{a}) \subset B_c(\mathfrak{a}) \subset Z_c(\mathfrak{a})$. In the case of principal ideal $(a)$ of $k$, let us denote $H_c(a) = H_c((a))$ just for convenience. Proposition 4.3 shows that $H^1(G, \mathcal{O}_K)_c \cong H_c(a)$ for any trace image $a \in T_{K/k}(\mathcal{O}_K)$.

Proposition 4.4. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be mutually prime ideals of $\mathcal{O}_k$. Then we have*

$$H_c(\mathfrak{ab}) \cong H_c(\mathfrak{a}) \oplus H_c(\mathfrak{b}).$$

*Proof.* Let $\gamma \in Z_c(\mathfrak{ab})$. Since $\gamma - c_s\, {}^s\gamma \in \mathfrak{ab}\mathcal{O}_k \subset \mathfrak{a}\mathcal{O}_k \cap \mathfrak{b}\mathcal{O}_k$, we have $\gamma \in Z_c(\mathfrak{a})$ and $\gamma \in Z_c(\mathfrak{b})$ so that the natural homomorphism

$$Z_c(\mathfrak{ab}) \longrightarrow Z_c(\mathfrak{a})/B_c(\mathfrak{a}) \oplus Z_c(\mathfrak{b})/B_c(\mathfrak{b})$$

is induced. The kernel of the homomorphism is $B_c(\mathfrak{ab})$. Indeed, let $\gamma \in Z_c(\mathfrak{ab})$ such that $\gamma \in B_c(\mathfrak{a}) \cap B_c(\mathfrak{b})$, say, $\gamma - a_0 = \alpha \in \mathfrak{a}\mathcal{O}_K$ and $\gamma - b_0 = \beta \in \mathfrak{b}\mathcal{O}_K$ for some $a_0, b_0 \in M_c$. Since $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime, there are elements $a$ and $b$ in $\mathfrak{a}$ and $\mathfrak{b}$, respectively, such that $a + b = 1$. Then $ab_0 + ba_0 \in M_c$ and

$$\gamma - (ab_0 + ba_0) = (a + b)\gamma - (ab_0 + ba_0) = a\beta + b\alpha \in \mathfrak{ab}\mathcal{O}_K$$

which means $\gamma \in B_c(\mathfrak{ab})$.

Now, it remains to show that the homomorphism is surjective: let $\alpha \in Z_c(\mathfrak{a})$ and $\beta \in Z_c(\mathfrak{b})$. Define $\gamma = a\beta + b\alpha$ where $a \in \mathfrak{a}$, $b \in \mathfrak{b}$ are fixed, with the property $a+b = 1$. Then we have $\gamma - c_s\, {}^s\gamma = a(\beta - c_s\, {}^s\beta) + b(\alpha - c_s\, {}^s\alpha) \in \mathfrak{ab}\mathcal{O}_K$ so that $\gamma \in Z_c(\mathfrak{ab})$. Since $a \in B_c(\mathfrak{a}) = M_c + \mathfrak{a}\mathcal{O}_K$ and $b \in B_c(\mathfrak{b}) = M_c + \mathfrak{b}\mathcal{O}_K$, we have $\gamma = a\beta + b\alpha \equiv \alpha \bmod B_c(\mathfrak{a})$ and $\gamma \equiv \beta \bmod B_c(\mathfrak{b})$. □

Proposition 4.5. *For any ideal $\mathfrak{a}$ of $\mathcal{O}_k$, the group $H_c(\mathfrak{a})$ is finite. More precisely, we have*

$$[B_c(\mathfrak{a}) : \mathfrak{a}\mathcal{O}_K] = N(\mathfrak{a}),$$

*and the order of $H_c(\mathfrak{a})$ divides $N(\mathfrak{a})^{n-1}$ where $n$ is the degree of the extension $K/k$. In particular, for a prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$, the order $|H_c(\mathfrak{p}^m)|$ is a power of a prime integer $p$ (possibly $1 = p^0$) where $\mathfrak{p} \mid p$.*

*Proof.* For any ideal $\mathfrak{a}$ of $\mathcal{O}_k$, we have $\mathfrak{a}\mathcal{O}_K \subset B_c(\mathfrak{a}) \subset Z_c(\mathfrak{a}) \subset \mathcal{O}_K$ as additive groups. So we have that $|H_c(\mathfrak{a})| = [Z_c(\mathfrak{a}) : B_c(\mathfrak{a})]$ is finite. On the other hand, it is known that there exists a nonzero integer $\xi$ in $\mathcal{O}_K$ such that $\xi M_c = \mathfrak{b} = \xi\mathcal{O}_K \cap \mathcal{O}_k$ is an ideal of $\mathcal{O}_k$, and $\xi\mathcal{O}_K$ is an ambiguous ideal of $\mathcal{O}_K$. Then $\xi\mathfrak{a}\mathcal{O}_K \cap \mathcal{O}_k = \mathfrak{a}\mathfrak{b}$ and we have

$$\frac{B_c(\mathfrak{a})}{\mathfrak{a}\mathcal{O}_K} \cong \frac{\mathfrak{b} + \xi\mathfrak{a}\mathcal{O}_K}{\xi\mathfrak{a}\mathcal{O}_K} \cong \frac{\mathfrak{b}}{\mathfrak{b} \cap \xi\mathfrak{a}\mathcal{O}_K} = \frac{\mathfrak{b}}{\mathfrak{a}\mathfrak{b}}.$$

Hence $[B_c(\mathfrak{a}) : \mathfrak{a}\mathcal{O}_K] = N(\mathfrak{a})$. We obtain that $|H_c(\mathfrak{a})| = [Z_c(\mathfrak{a}) : B_c(\mathfrak{a})]$ is a divisor of $[\mathcal{O}_K : B_c(\mathfrak{a})] = N(\mathfrak{a}\mathcal{O}_K)/N(\mathfrak{a}) = N(\mathfrak{a})^{n-1}$. Note that the norm of the ideal $\mathfrak{p}^m$ is a power of the prime integer $p \in \mathbf{Z}$ such that $\mathfrak{p} \mid p$. □

Theorem 4.6. *For the ideal $Tr_{K/k}(\mathcal{O}_K)$ of $\mathcal{O}_k$, we have*

$$H^1(G, \mathcal{O}_K)_c \cong H_c(Tr_{K/k}(\mathcal{O}_K)).$$

*Proof.* We borrow the idea used in the proof of Theorem 2 in [10]. We will show that $H_c(\mathfrak{b}) = 0$ for any ideal $\mathfrak{b}$ in $\mathcal{O}_k$ which is relatively prime to $Tr_{K/k}(\mathcal{O}_K)$. Let us denote by the ideal $Tr_{K/k}(\mathcal{O}_K)$ as $\mathfrak{a}$. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_k$ with $(\mathfrak{a}, \mathfrak{p}) = 1$. We may choose an ideal $\mathfrak{b}$ in $\mathcal{O}_k$ such that $(\mathfrak{a}\mathfrak{p}^m, \mathfrak{b}) = 1$ and $\mathfrak{a}\mathfrak{p}^m\mathfrak{b}$ is a principal ideal $(a)$ in $\mathcal{O}_k$. Also there exists an ideal $\mathfrak{c}$ in $\mathcal{O}_k$ such that $(\mathfrak{a}\mathfrak{p}^m\mathfrak{b}, \mathfrak{c}) = 1$ and $\mathfrak{a}\mathfrak{c}$ is a principal ideal $(a')$. By Proposition 4.3, we have

$$H_c(a) \cong H_c(a') \cong H^1(G, \mathcal{O}_K)_c.$$

Then by Proposition 4.4, we have

$$H_c(\mathfrak{a}) \oplus H_c(\mathfrak{p}^m) \oplus H_c(\mathfrak{b}) \cong H_c(\mathfrak{a}) \oplus H_c(\mathfrak{c}) \cong H^1(G, \mathcal{O}_K)_c.$$

We have $|H_c(\mathfrak{p}^m)| = p^i$ divides $|H_c(\mathfrak{c})|$ for some $i$. But since $(a, \mathfrak{c}) = 1$, whence $p \mid a$, we have $|H_c(\mathfrak{p}^m)|$ and $|H_c(\mathfrak{c})|$ are mutually prime. Therefore we obtain $H_c(\mathfrak{p}^m) = 0$. This happens for arbitrary prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ prime to $\mathfrak{a}$ and for any positive integer $m$, hence we also get $H_c(\mathfrak{b}) = 0$ and finally we have $H^1(G, \mathcal{O}_K)_c \cong H_c(\mathfrak{a})$. □

REMARK 4.7. Note that $H_c(\mathcal{O}_k) = 0$ since $Z_c(\mathcal{O}_k) = B_c(\mathcal{O}_k) = \mathcal{O}_K$. If $K/k$ is tamely ramified, we have $Tr_{K/k}(\mathcal{O}_K) = \mathcal{O}_k$ and we can also check $H^1(G, \mathcal{O}_K)_c = 0$ by Theorem 4.6.

COROLLARY 4.8. *We have*

$$H^1(G, \mathcal{O}_K)_c \cong \prod_{\mathfrak{p} | Tr_{K/k}(\mathcal{O}_K)} H_c\left(\mathfrak{p}^{\left\lfloor \frac{t_\mathfrak{p}}{e_\mathfrak{p}} \right\rfloor}\right)$$

*where* $e_\mathfrak{p}$ *is the ramification index of* $\mathfrak{p}$*, and* $\mathfrak{D}_{K/k} = \prod_\mathfrak{p} \left(\prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P}\right)^{t_\mathfrak{p}}$ *is the different of* $K$ *over* $k$*.*

*Proof.* Let $Tr_{K/k}(\mathcal{O}_K) = \prod_{\mathfrak{p}|Tr_{K/k}(\mathcal{O}_K)} \mathfrak{p}^{r_\mathfrak{p}}$. As in [7], we have $r_\mathfrak{p} = \lfloor t_\mathfrak{p}/e_\mathfrak{p} \rfloor$ since

$$\begin{aligned} \mathfrak{p}^r \mid Tr_{K/k}(\mathcal{O}_K) &\Leftrightarrow \mathfrak{p}^r \mid \mathfrak{D}_{K/k} \\ &\Leftrightarrow \prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P}^{e_\mathfrak{p} r} \mid (\prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P})^{t_\mathfrak{p}} \\ &\Leftrightarrow e_\mathfrak{p} r \leq t_\mathfrak{p} \\ &\Leftrightarrow r \leq \lfloor t_\mathfrak{p}/e_\mathfrak{p} \rfloor. \end{aligned}$$

□

## 5. Local and global

Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_k$ and $\mathfrak{P}$ be the prime ideal of $\mathcal{O}_K$ above $\mathfrak{p}$. We have completions $K_\mathfrak{P}$ and $k_\mathfrak{p}$ and the extension $K_\mathfrak{P}/k_\mathfrak{p}$ is Galois with the Galois group $G_\mathfrak{P}$ which is identified as the decomposition group at $\mathfrak{P}$ in $G$. The ring of integers $\mathcal{O}_K$, $\mathcal{O}_k$ are embedded in $\mathcal{O}_{K_\mathfrak{P}}$, $\mathcal{O}_{k_\mathfrak{p}}$, respectively. For 1-cocycle $c \in Z^1(G, \mathcal{O}_K^\times)$, the $c$-twisted $G_\mathfrak{P}$-module structure of $\mathcal{O}_K$ extends to $\mathcal{O}_{K_\mathfrak{P}}$ so that we can consider the twisted cohomology $H^n(G_\mathfrak{P}, \mathcal{O}_{K_\mathfrak{P}})_c$ of $G_\mathfrak{P}$ into $\mathcal{O}_{K_\mathfrak{P}}$. On the other hand, as $c$ induces a 1-cocycle $c_\mathfrak{P} \in Z^1(G_\mathfrak{P}, \mathcal{O}_{K_\mathfrak{P}}^\times)$ naturally through the embeddings

$$G_\mathfrak{P} \hookrightarrow G \xrightarrow{c} \mathcal{O}_K^\times \hookrightarrow \mathcal{O}_{K_\mathfrak{P}}^\times$$

so that we can regard $\mathcal{O}_{K_\mathfrak{P}}$ as $c_\mathfrak{P}$-twisted $G_\mathfrak{P}$-module and we can define the twisted cohomology $H^n(G_\mathfrak{P}, \mathcal{O}_{K_\mathfrak{P}})_{c_\mathfrak{P}}$ of $G_\mathfrak{P}$ into $\mathcal{O}_{K_\mathfrak{P}}$ twisted by $c_\mathfrak{P}$. Note that $c_\mathfrak{P}$-twisted action in $\mathcal{O}_{K_\mathfrak{P}}$ is exactly same as $c$-twisted action so that we have $H^n(G_\mathfrak{P}, \mathcal{O}_{K_\mathfrak{P}})_c \cong H^n(G_\mathfrak{P}, \mathcal{O}_{K_\mathfrak{P}})_{c_\mathfrak{P}}$.

Theorem 5.1. *For each dimension $n$ we have*

$$H^n(G,\mathcal{O}_K)_c \cong \prod_{\mathfrak{p}} H^n(G_{\mathfrak{P}},\mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}}$$

*where $\mathfrak{p}$ runs through the prime ideals in $\mathcal{O}_k$ and $\mathfrak{P}$ is one prime ideal of $\mathcal{O}_K$ dividing $\mathfrak{p}$.*

Remark 5.2. By the same assertion as Theorem 3.1, we have $H^n(G_{\mathfrak{P}},\mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}} = 0$ if $K_{\mathfrak{P}}/k_{\mathfrak{p}}$ is tamely ramified. Denote by $W$ the set of all prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$ which is wildly ramified in $\mathcal{O}_K$. Furthermore, since $Tr_{K/k}(\mathcal{O}_K) = \prod_{\mathfrak{p}} \mathfrak{p}^{\lfloor t_{\mathfrak{p}}/e_{\mathfrak{p}} \rfloor}$ by (4.3), we have

$$\mathfrak{p} \mid Tr_{K/k}(\mathcal{O}_K) \Leftrightarrow t_{\mathfrak{p}} > e_{\mathfrak{p}} - 1 \Leftrightarrow \mathfrak{p} \in W.$$

Hence the theorem states equivalently that

$$H^n(G,\mathcal{O}_K)_c \cong \prod_{\mathfrak{p}\in W} H^n(G_{\mathfrak{P}},\mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}} \cong \prod_{\mathfrak{p}\mid Tr_{K/k}(\mathcal{O}_K)} H^n(G_{\mathfrak{P}},\mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}}.$$

*Proof.* This proof is parallel to that of Theorem 1 in [3]. Let $\mathcal{O} = \prod_{\mathfrak{P}} \mathcal{O}_{K_{\mathfrak{P}}}$, and for each prime ideal $\mathfrak{p}$ in $\mathcal{O}_k$ let $\mathcal{O}(\mathfrak{p}) = \prod_{\mathfrak{P}\mid\mathfrak{p}} \mathcal{O}_{K_{\mathfrak{P}}}$. Then $O = \prod_{\mathfrak{p}} \mathcal{O}(\mathfrak{p})$, and we have that $\mathcal{O}$ and $\mathcal{O}(\mathfrak{p})$ are $c$-twisted $G$-modules. $\mathcal{O}_K$ is diagonally imbedded in $\mathcal{O}$ and $\mathcal{O}(\mathfrak{p})$. Since $c_s$ is unit in $\mathcal{O}_K$, we have

$$\prod_{s\in G/G_{\mathfrak{P}}} c_s\, {}^s\mathcal{O}_{K_{\mathfrak{P}}} = \prod_{s\in G/G_{\mathfrak{P}}} {}^s\mathcal{O}_{K_{\mathfrak{P}}} = \mathcal{O}(\mathfrak{p})$$

where in the product, $s$ varies over a system of representatives of the cosets in $G/G_{\mathfrak{P}}$. Therefore by Shapiro's lemma (of form of 34.1 in [1], p. 120), we have

$$H^n(G,\mathcal{O}(\mathfrak{p}))_c \cong H^n(G_{\mathfrak{P}},\mathcal{O}_{K_{\mathfrak{P}}})_c.$$

Now we claim that

$$H^n(G,\mathcal{O}_K)_c \cong H^n(G,\mathcal{O})_c, \tag{5.1}$$

which leads the theorem since

$$H^n(G,\mathcal{O}_K)_c \cong \prod_{\mathfrak{p}} H^n(G,\mathcal{O}(\mathfrak{p}))_c \cong \prod_{\mathfrak{p}} H^n(G_{\mathfrak{P}},\mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}}.$$

Let $\beta \in \mathcal{O}_K$ be chosen as a generator of normal basis of $K/k$. Define $M = \sum_{s\in G} M_c\, c_s\, {}^s\beta$ then we have that $M$ is twisted $G$-regular and $[\mathcal{O}_K : M]$ is finite, say $l$, as in the proof of Theorem 3.2. Also we define

$$\mathfrak{M} = \sum_{s\in G} \mathfrak{M}_c\, c_s\, {}^s\beta$$

where

$$\mathfrak{M}_c := \{\alpha \in \mathcal{O} \ ; \ c_s {}^s\alpha = \alpha \text{ for all } s \in G\}.$$

Then $\mathfrak{M}$ and $\mathfrak{M}_c$ are twisted $G$-modules, and we have $\mathcal{O}_K \cap \mathfrak{M} = M$ trivially.

On the other hand, we have $\mathcal{O} = \mathfrak{M} + \mathcal{O}_K$. Indeed, since $l\mathcal{O}_K \subset M$ and $M\mathcal{O} \subset \mathfrak{M}$, we have $l\mathcal{O}_K\mathcal{O} \subset \mathfrak{M}$. Denote by $\mathfrak{A} = l\mathcal{O}_K$ the ideal in $\mathcal{O}_K$ then we have $\mathfrak{A}\mathcal{O} + \mathcal{O}_K \subset \mathfrak{M} + \mathcal{O}_K \subset \mathcal{O}$. It is enough to show that $\mathcal{O} \subset \mathfrak{A}\mathcal{O} + \mathcal{O}_K$. Now let $\alpha = (a_{\mathfrak{P}})_{\mathfrak{P}} \in \mathcal{O}$. By the strong approximation theorem, there exists an element $b$ in $K$ such that $\nu_{\mathfrak{P}}(b - a_{\mathfrak{P}}) \geq \nu_{\mathfrak{P}}(\mathfrak{A})$ for prime ideals $\mathfrak{P} \mid \mathfrak{A}$ and $\nu_{\mathfrak{P}}(b) \geq 0$ for prime ideals $\mathfrak{P}$ not dividing $\mathfrak{A}$. Then $b \in \mathcal{O}_K$. Let $\gamma = b - \alpha \in \mathcal{O}$ then $\gamma \in \prod_{\mathfrak{P}} \mathfrak{P}^{\nu_{\mathfrak{P}}(\mathfrak{A})} \subset \mathfrak{A}\mathcal{O}$. So we have $\alpha = \gamma + b \in \mathfrak{A}\mathcal{O} + \mathcal{O}_K$ and hence $\mathcal{O} = \mathfrak{M} + \mathcal{O}_K$.

Therefore we have twisted $G$-module isomorphisms

$$\frac{\mathcal{O}}{\mathfrak{M}} \cong \frac{\mathfrak{M} + \mathcal{O}_K}{\mathfrak{M}} \cong \frac{\mathcal{O}_K}{\mathcal{O}_K \cap \mathfrak{M}} \cong \frac{\mathcal{O}_K}{M}. \tag{5.2}$$

Since $M$ and $\mathfrak{M}$ are twisted $G$-regular, we have $H^n(G, M)_c = 0$ and $H^n(G, \mathfrak{M})_c = 0$ so that $H^n(G, \mathcal{O}_K)_c \cong H^n(G, \mathcal{O}_K/M)_c$ and $H^n(G, \mathcal{O})_c \cong H^n(G, \mathcal{O}/\mathfrak{M})_c$ from the exact sequences $0 \to M \to \mathcal{O}_K \to \mathcal{O}_K/M \to 0$ and $0 \to \mathfrak{M} \to \mathcal{O} \to \mathcal{O}/\mathfrak{M} \to 0$. Combined with (5.2), we have $H^n(G, \mathcal{O}_K)_c \cong H^n(G, \mathcal{O})_c$. □

Now we consider the twisted 1-cohomology for the completion $K_{\mathfrak{P}}$ and $k_{\mathfrak{p}}$ at a prime $\mathfrak{P}$ which is wildly ramified over the extension $K/k$. Fix prime elements $\pi$ and $\Pi$ in $\mathcal{O}_{k_{\mathfrak{p}}}$ and $\mathcal{O}_{K_{\mathfrak{P}}}$, respectively, so that $\pi = \Pi^{e_{\mathfrak{p}}}$ and $\mathfrak{p} = (\pi)$ and $\mathfrak{P} = (\Pi)$ where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$. As in [6], the 1-cohomology of the unit group $H^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}}^{\times})$ is the cyclic group of order $e_{\mathfrak{p}}$, generated by the canonical cohomology class $\gamma_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}$ with a representative cocycle $c_{0s}$ such that ${}^s\Pi = \Pi c_{0s}$. Let $c_{\mathfrak{P}}$ be the 1-cocycle of $\mathcal{O}_{K_{\mathfrak{P}}}^{\times}$ induced by $c \in Z^1(G, \mathcal{O}_K)$ and $\gamma = [c_{\mathfrak{P}}] \in H^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})$. We have $\gamma = (\gamma_{K_{\mathfrak{P}}/k_{\mathfrak{p}}})^m$ for some $m$ such that $0 \leq m < e_{\mathfrak{p}}$. We may regard $c_{\mathfrak{P}}$ as $c_0^m$ by Proposition 2.1.

For an ideal $\mathfrak{a}$ of $\mathcal{O}_{k_{\mathfrak{p}}}$, let

$$Z_{c_{\mathfrak{P}}}(\mathfrak{a}) = \{\alpha \in \mathcal{O}_{K_{\mathfrak{P}}} \ ; \ \alpha \equiv c_{\mathfrak{P}_s} {}^s\alpha \bmod \mathfrak{a}\mathcal{O}_{K_{\mathfrak{P}}}\}$$

$$B_{c_{\mathfrak{P}}}(\mathfrak{a}) = \{\alpha \in \mathcal{O}_{K_{\mathfrak{P}}} \ ; \ \alpha \equiv b \bmod \mathfrak{a}\mathcal{O}_{K_{\mathfrak{P}}} \text{ for some } b \in M_{c_{\mathfrak{P}}}\} = M_{c_{\mathfrak{P}}} + \mathfrak{a}\mathcal{O}_{K_{\mathfrak{P}}}$$

and $H_{c_{\mathfrak{P}}}(\mathfrak{a}) = Z_{c_{\mathfrak{P}}}(\mathfrak{a})/B_{c_{\mathfrak{P}}}(\mathfrak{a})$ where $M_{c_{\mathfrak{P}}} = Z^0(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}}$. Then we have very similar arguments with propositions in section 4 for the local case.

Since the set of trace image $Tr_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\mathcal{O}_{K_{\mathfrak{P}}})$ is an ideal in $\mathcal{O}_{k_{\mathfrak{p}}}$, there is a nonnegative integer $r_{\mathfrak{p}}$ such that $Tr_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\mathcal{O}_{K_{\mathfrak{P}}}) = \mathfrak{p}^{r_{\mathfrak{p}}} = (\pi^{r_{\mathfrak{p}}})$. It is known that $r_{\mathfrak{p}} = \lfloor t_{\mathfrak{p}}/e_{\mathfrak{p}} \rfloor$ where $\mathfrak{P}^{t_{\mathfrak{p}}} = \mathfrak{D}_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}$, and $\mathfrak{D}_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}$ is the different of $K_{\mathfrak{P}}$ over $k_{\mathfrak{p}}$ because of the same argument with (4.3). We have $H^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}} \cong H_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) = H_{c_{\mathfrak{P}}}(Tr_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\mathcal{O}_{K_{\mathfrak{P}}}))$ because propositions 4.1, 4.2, and 4.3 hold naturally in the local case.

Since we have the tower

$$\pi^{r_{\mathfrak{p}}}\mathcal{O}_{K_{\mathfrak{P}}} = \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}} \subset B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) \subset Z_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) \subset \mathcal{O}_{K_{\mathfrak{P}}},$$

we have

$$[Z_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) : B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}})] \;\Big|\; N(\mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}})/[B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) : \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}]\,.$$

As in [6], we have $\Pi^m M_{c_{\mathfrak{P}}} = \mathfrak{p}$ if $m \neq 0$ and $M_{c_{\mathfrak{P}}} = \mathcal{O}_{k_{\mathfrak{p}}}$ if $m = 0$. Therefore we have $\Pi^m B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) = \Pi^m M_{c_{\mathfrak{P}}} + \Pi^m \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}} = \mathfrak{p} + \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}+m}$ if $m \neq 0$ and $B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) = \mathcal{O}_{k_{\mathfrak{p}}} + \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}$ if $m = 0$. Now

$$[B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) : \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}] = \left|\frac{\mathfrak{p} + \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}+m}}{\mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}+m}}\right| = \left|\frac{\mathfrak{p}}{\mathfrak{p} \cap \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}+m}}\right| = \left|\frac{\mathfrak{p}}{\mathfrak{p}^{r_{\mathfrak{p}}+1}}\right| = N(\mathfrak{p})^{r_{\mathfrak{p}}}$$

for $m \neq 0$, and

$$[B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) : \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}] = \left|\frac{\mathcal{O}_{k_{\mathfrak{p}}} + \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}}{\mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}}\right| = \left|\frac{\mathcal{O}_{k_{\mathfrak{p}}}}{\mathcal{O}_{k_{\mathfrak{p}}} \cap \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}}\right| = \left|\frac{\mathcal{O}_{k_{\mathfrak{p}}}}{\mathfrak{p}^{r_{\mathfrak{p}}}}\right| = N(\mathfrak{p})^{r_{\mathfrak{p}}}$$

for $m = 0$. Since $N(\mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}) = N(\mathfrak{p})^{e_{\mathfrak{p}}f_{\mathfrak{p}}r_{\mathfrak{p}}}$, finally we have $N(\mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}})/[B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) : \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}] = N(\mathfrak{p})^{e_{\mathfrak{p}}f_{\mathfrak{p}}r_{\mathfrak{p}}-r_{\mathfrak{p}}}$. We state the summary in the following proposition.

Proposition 5.3. *We have*

$$H^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}} \cong H_{c_{\mathfrak{P}}}(Tr_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\mathcal{O}_{K_{\mathfrak{P}}})) = H_{c_{\mathfrak{P}}}(\mathfrak{p}^{\lfloor t_{\mathfrak{p}}/e_{\mathfrak{p}} \rfloor})$$

*and* $[B_{c_{\mathfrak{P}}}(Tr_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\mathcal{O}_{K_{\mathfrak{P}}})) : Tr_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\mathcal{O}_{K_{\mathfrak{P}}})\mathcal{O}_{K_{\mathfrak{P}}}] = N(\mathfrak{p})^{\lfloor t_{\mathfrak{p}}/e_{\mathfrak{p}} \rfloor}$*. The order* $|H^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}}|$ *is a divisor of* $N(\mathfrak{p})^{(e_{\mathfrak{p}}f_{\mathfrak{p}}-1)\lfloor t_{\mathfrak{p}}/e_{\mathfrak{p}} \rfloor}$.

## 6. Quadratic fields

Let $K = \mathbf{Q}(\sqrt{m})$ be a quadratic number field where $m$ is square-free, and let $k = \mathbf{Q}$. Since the Galois group $G = \langle s \rangle$ is cyclic, we have $|H^n(G, \mathcal{O}_K)_c| = |M_c/P_c|$ for all integer $n$ and for all cocycle $c \in Z^1(G, \mathcal{O}_K^\times)$. Hence for all dimension $n$, we have $|H^n(G, \mathcal{O}_K)_c| = 1$ if and only if (a) $m$ is congruent to 1 modulo 4, or (b) $m \equiv 3 \pmod 4$, $m > 0$, $c \sim \pm\varepsilon$, and $v$ is odd, where $\varepsilon = u + v\sqrt{m}$ is the fundamental unit of norm 1 in $\mathcal{O}_K$, or equivalently (b′) $m \equiv 3 \pmod 4$, $m > 0$, $c \sim \pm\varepsilon$, and the central entry $a_{r/2}$ of the simple continued fraction expansion of $\sqrt{m} = [a_0; \overline{a_1, \ldots, a_r}]$ is odd (see [4]). In other words, for $m \equiv 3 \pmod 4$, $|H^n(G, \mathcal{O}_K)_c| = 1$ if $m_2$ is odd and $|H^n(G, \mathcal{O}_K)_c| = 2$ if $m_2$ is even, where $c \sim \frac{s\Pi^{m_2}}{\Pi^{m_2}}$ as 1-cocycle in $Z^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})$ in the completion at $\mathfrak{P} \mid 2$ (see [7]).

This is verified by computing the 1-twisted cohomology in two explicit ways. First, let $\Delta_c = 1 - c_s s$ and $N_c = 1 + c_s s$, and denote by $M^c = \ker N_c$ and $P^c = \operatorname{im} N$ then we have $M^c = M_{-c}$ and $P^c = P_{-c}$ clearly. On the other hand, as the use of theorem 4.6, we compute $H_c(Tr_{K/\mathbf{Q}}(\mathcal{O}_K))$: We may assume $m \equiv 2, 3 \pmod 4$ for considering the wild ramified case only. We have $B_c(Tr_{K/\mathbf{Q}}(\mathcal{O}_K)) = M_c + Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K = M_c + 2\mathcal{O}_K$, which satisfies

$$[B_c(Tr_{K/\mathbf{Q}}(\mathcal{O}_K)) : Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K] = 2$$

by Proposition 4.5. To determine $Z_c(Tr_{K/\mathbf{Q}}(\mathcal{O}_K))$ when $c \sim \pm\varepsilon$, let $x + y\sqrt{m}$ be satisfying $\pm\varepsilon(x - y\sqrt{m}) - (x + y\sqrt{m}) \in 2\mathcal{O}_K$, equivalently,

$$(6.1) \qquad 2 \mid x(u \pm 1) + yvm, \quad 2 \mid xv - y(u \pm 1).$$

If $v$ is even, the condition (6.1) gives nothing, so we have $Z_c(Tr_{K/\mathbf{Q}}(\mathcal{O}_K)) = \mathcal{O}_K$ and $|H^1(G, \mathcal{O}_K)_c| = 2$. In the case $v$ is odd, which happens when $m \equiv 3 \pmod 4$ only, we obtain $x \equiv y \pmod 2$ so we have $[Z_c(Tr_{K/\mathbf{Q}}(\mathcal{O}_K)) : Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K] = 2$ and hence $|H^1(G, \mathcal{O}_K)_c| = 1$. For $c \sim \pm 1$, it is trivial to see $Z_c(Tr_{K/\mathbf{Q}}(\mathcal{O}_K)) = \mathcal{O}_K$ and $|H^1(G, \mathcal{O}_K)_c| = 2$.

## 7. Biquadratic fields

Here we give an example of the twisted cohomology of a non-cyclic extension. Let $K = \mathbf{Q}(\sqrt{m_1}, \sqrt{m_2})$ be a biquadratic field where $m_1$, $m_2$ are distinct and squarefree. $K$ has three subfields $k_1 = \mathbf{Q}(\sqrt{m_1})$, $k_2 = \mathbf{Q}(\sqrt{m_2})$, and $k_3 = \mathbf{Q}(\sqrt{m_3})$ where $m_3 = \frac{m_1}{d}\frac{m_2}{d}$ and $d = \gcd(m_1, m_2)$. The Galois group of $K$ over $\mathbf{Q}$ is $G = \mathrm{Gal}(K/\mathbf{Q}) = \{1, s_1, s_2, s_3\}$ where $s_i s_j = s_k$ with $i, j, k$ distinct, $s_i^2 = 1$ for all $i$, and for each $i$ $s_i$ acts trivially on the subfield $k_i$, not on $k_j$ for $j \neq i$. It is known that any of biquadratic fields falls into the following three types:

**[Type I]** $m_1 \equiv m_2 \equiv m_3 \equiv 1 \pmod 4$ with

$$\mathcal{O}_K = \left[1, \frac{1 + \sqrt{m_1}}{2}, \frac{1 + \sqrt{m_2}}{2}, \left(\frac{1 + \sqrt{m_1}}{2}\right)\left(\frac{1 + \sqrt{m_2}}{2}\right)\right]_{\mathbf{Z}}.$$

**[Type II]** $m_1 \equiv 1$ and $m_2 \equiv m_3 \equiv 2, 3 \pmod 4$ with

$$\mathcal{O}_K = \left[1, \frac{1+\sqrt{m_1}}{2}, \sqrt{m_2}, \frac{\sqrt{m_2}+\sqrt{m_3}}{2}\right]_{\mathbf{Z}}.$$

**[Type III]** $m_1 \equiv 3$ and $m_2 \equiv m_3 \equiv 2 \pmod 4$ with

$$\mathcal{O}_K = \left[1, \sqrt{m_1}, \sqrt{m_2}, \frac{\sqrt{m_2}+\sqrt{m_3}}{2}\right]_{\mathbf{Z}}.$$

The unit group $\mathcal{O}_K^\times$ has one or three fundamental units if only one or three of $m_1, m_2, m_3$ are positive, respectively. Denote by $\varepsilon_1, \varepsilon_2, \varepsilon_3$ the fundamental units of $k_1, k_2, k_3$ respectively, with norm 1, if it is the case that ones exist. We also denote $\varepsilon_i = u_i + v_i\omega_i$ for $u_i, v_i \in \mathbf{Z}$ and $\omega_i = \frac{1+\sqrt{m_i}}{2}$ if $m_i \equiv 1 \pmod 4$, $\omega_i = \sqrt{m_i}$ if $m_i \equiv 2, 3 \pmod 4$.

For a 1-cocycle $c \in Z^1(G, {\mathcal{O}_K}^\times)$, $M_c/P_c$ for the biquadratic fields is computed along the 2-adic property of $\gamma = [c]$ in [2] and we give an explicit computation of $M_c/P_c$ and $H^1(G, \mathcal{O}_K)_c$ globally for certain cocycles $c$ in $\mathcal{O}_K^\times$, which inherit cocycles in the unit group of quadratic subfields, and we compare with the case of the quadratic field. We may consider only the cases [Type II] and [Type III] because in [Type I], we have $Tr_{K/\mathbf{Q}}(\mathcal{O}_K) = \mathbf{Z}$ so that the extension is tamely ramified, hence all the twisted cohomologies vanish.

For $i = 1, 2, 3$, define $c_i : G \to {\mathcal{O}_K}^\times$ as $c_{i1} = 1$, $c_{is_i} = 1$ and $c_{is_j} = \varepsilon_i$, $j \neq i$ then $c_i$ is a 1-cocycle into ${\mathcal{O}_K}^\times$. Since the nontrivial action on $k_i$ by $G_i = \mathrm{Gal}(k_i/\mathbf{Q})$ is the same as ones by $s_j$ for $j \neq i$, we may regard in the quadratic field $k_i$ $s_j = s_k \in G_i$ and $c_i \in Z^1(G_i, {\mathcal{O}_{k_i}}^\times)$. Denote by

$$\begin{aligned}
M_{c_i}(\mathcal{O}_K) &= M_{c_i} = Z^0(G, \mathcal{O}_K)_{c_i},\\
M_{c_i}(\mathcal{O}_{k_i}) &= Z^0(G_i, \mathcal{O}_{k_i})_{c_i},\\
P_{c_i}(\mathcal{O}_K) &= P_{c_i} = \{p_{c_i\mathcal{O}_K}(\alpha) = \sum_{s\in G} c_s\, {}^s\alpha\ ;\ \alpha \in \mathcal{O}_K\}, \text{ and}\\
P_{c_i}(\mathcal{O}_{k_i}) &= \{p_{c_i\mathcal{O}_{k_i}}(\alpha) = \alpha + \varepsilon_i\, {}^{s_j}\alpha\ ;\ \alpha \in \mathcal{O}_{k_i}\}.
\end{aligned}$$

### 7.1. 0-dimensional twisted cohomology

Let $\alpha \in M_{c_i}(\mathcal{O}_K)$. Since ${}^{s_i}\alpha = \alpha$ if and only if $\alpha \in \mathcal{O}_{k_i}$, we have

$$M_{c_i}(\mathcal{O}_K) = M_{c_i}(\mathcal{O}_{k_i}). \tag{7.1}$$

We will compute $P_{c_i}(\mathcal{O}_K)$: First note that $\alpha + {}^{s_i}\alpha \in \mathcal{O}_{k_i}$ and

$$\begin{aligned} p_{c_i\mathcal{O}_K}(\alpha) &= \sum_{s\in G} c_s\, {}^{s}\alpha = \alpha + {}^{s_i}\alpha + \varepsilon_i\, {}^{s_j}\alpha + \varepsilon_i\, {}^{s_k}\alpha \\ &= (\alpha + {}^{s_i}\alpha) + \varepsilon_i\, {}^{s_j}(\alpha + {}^{s_i}\alpha) \\ &= p_{c_i\mathcal{O}_{k_i}}(\alpha + {}^{s_i}\alpha). \end{aligned}$$

**[Type II]** Let $\alpha = x + y\left(\frac{1+\sqrt{m_1}}{2}\right) + z\sqrt{m_2} + w\left(\frac{\sqrt{m_2}+\sqrt{m_3}}{2}\right)$ with $x, y, z, w \in \mathbf{Z}$. Then we have

$$\alpha + {}^{s_i}\alpha = \begin{cases} 2x + 2y\left(\frac{1+\sqrt{m_1}}{2}\right), & i = 1 \\ (2x+y) + (2z+w)\sqrt{m_2}, & i = 2 \\ (2x+y) + w\sqrt{m_3}, & i = 3. \end{cases}$$

So we have

$$P_{c_1}(\mathcal{O}_K) = 2P_{c_1}(\mathcal{O}_{k_1}), \quad P_{c_2}(\mathcal{O}_K) = P_{c_2}(\mathcal{O}_{k_2}), \quad P_{c_3}(\mathcal{O}_K) = P_{c_3}(\mathcal{O}_{k_3}).$$

Hence if $m_2 \equiv m_3 \equiv 3 \pmod 4$,

$$\left|\frac{M_{c_1}}{P_{c_1}}\right| = 2$$

$$\left|\frac{M_{c_2}}{P_{c_2}}\right| = \begin{cases} 1 & \text{if } v_2 \text{ is odd} \\ 2 & \text{if } v_2 \text{ is even} \end{cases}$$

$$\left|\frac{M_{c_3}}{P_{c_3}}\right| = \begin{cases} 1 & \text{if } v_3 \text{ is odd} \\ 2 & \text{if } v_3 \text{ is even} \end{cases}$$

while if $m_2 \equiv m_3 \equiv 2 \pmod 4$, it is shown in [2] that $\left|\frac{M_c}{P_c}\right| = 2$ for all cocycles $c$.

**[Type III]** Let $\alpha = x + y\sqrt{m_1} + z\sqrt{m_2} + w\left(\frac{\sqrt{m_2}+\sqrt{m_3}}{2}\right)$ with $x, y, z, w \in \mathbf{Z}$. Then

$$\alpha + {}^{s_i}\alpha = \begin{cases} 2x + 2y\sqrt{m_1}, & i = 1 \\ 2x + (2z+w)\sqrt{m_2}, & i = 2 \\ 2x + w\sqrt{m_3}, & i = 3. \end{cases}$$

We obtain that

$$P_{c_1}(\mathcal{O}_K) = 2P_{c_1}(\mathcal{O}_{k_1}), \text{ and } P_{c_i}(\mathcal{O}_K) = p_{c_i\mathcal{O}_{k_i}}([2, \sqrt{m_i}]_{\mathbf{Z}}) \subset P_{c_i}(\mathcal{O}_{k_i}),$$

for $i = 2, 3$. For $i = 2, 3$, let $\xi_i \in \mathcal{O}_{k_i}$ be an integer in $k_i$ such that $c_{is_j} = \frac{{}^{s_j}\xi_i}{\xi_i}$. We may choose $\xi_i = 1 + {}^{s_j}\varepsilon_i = (u_i+1) - v_i\sqrt{m_i}$. Then

$\xi_i P_{c_i}(\mathcal{O}_{k_i}) = Tr_{k_i/\mathbf{Q}}(\xi_i \mathcal{O}_{k_i})$ and
$\xi_i p_{c_i \mathcal{O}_{k_i}}([2, \sqrt{m_i}]_{\mathbf{Z}}) = Tr_{k_i/\mathbf{Q}}(\xi_i [2, \sqrt{m_i}]_{\mathbf{Z}})$. It is easy to see that

$$Tr_{k_i/\mathbf{Q}}(\xi_i \mathcal{O}_{k_i}) = \{2\left((u_i+1)x - y v_i m_i\right) \ ; \ x, y \in \mathbf{Z}\} \tag{7.2}$$
$$= 2\gcd(u_i+1, v_i m_i)\mathbf{Z} \tag{7.3}$$
$$Tr_{k_i/\mathbf{Q}}(\xi_i [2, \sqrt{m_i}]_{\mathbf{Z}}) = \{2\left(2(u_i+1)x - y v_i m_i\right) \ ; \ x, y \in \mathbf{Z}\} \tag{7.4}$$
$$= 2\gcd(2(u_i+1), v_i m_i)\mathbf{Z}. \tag{7.5}$$

Hence we have

$$p_{c_i \mathcal{O}_{k_i}}([2, \sqrt{m_i}]_{\mathbf{Z}}) = \begin{cases} P_{c_i}(\mathcal{O}_{k_i}) & \text{if } D_i \text{ is odd}, \\ 2P_{c_i}(\mathcal{O}_{k_i}) & \text{if } D_i \text{ is even}, \end{cases}$$

if we denote by

$$D_i = \frac{v_i}{\gcd(u_i+1, v_i)}. \tag{7.6}$$

As a result, we have

$$\left|\frac{M_{c_1}}{P_{c_1}}\right| = \begin{cases} 2 & \text{if } v_1 \text{ is odd} \\ 4 & \text{if } v_1 \text{ is even} \end{cases}$$

$$\left|\frac{M_{c_i}}{P_{c_i}}\right| = \begin{cases} 2 & \text{if } D_i \text{ is odd} \\ 4 & \text{if } D_i \text{ is even} \end{cases} \quad (i = 2, 3).$$

We can rewrite the condition for $i = 2, 3$ as

$$\left|\frac{M_{c_i}}{P_{c_i}}\right| = \begin{cases} 4 & \text{if } u_i \equiv 1, v_i \equiv 0 \pmod 4 \\ 2 & \text{otherwise} \end{cases} \quad (i = 2, 3).$$

Indeed, if $u_i \equiv 1, v_i \equiv 0 \pmod 4$, $\gcd(u_i+1, v_i) = 2$ so trivially $D_i$ is even. Also it is easy to see that if $v_i \equiv 2 \pmod 4$ then $D_i$ is odd for any odd $u_i$. The case remained is that $u_i \equiv 3, v_i \equiv 0 \pmod 4$. Let $\nu \geq 2$ be satisfying $2^\nu \parallel v_i$. By Lemma 7 in [4], we have $u_i + 1 \equiv 0$ or $2 \pmod{2^{\nu+1}}$, and hence we have that $D_i$ is odd.

Remark 7.1. For $m_i \equiv 2 \pmod 4$, squarefree, with the norm of fundamental unit $N(\varepsilon_i) = 1$, the both parity of $D_i$ occur, while the odd case appears more frequently. Up to 500, we have $m_i$'s with odd $D_i$:
6, 14, 22, 30, 34, 38, 42, 46, 62, 70, 78, 86, 94, 102, 110, 118, 134, 138, 142, 154, 158, 166, 174, 182, 186, 190, 194, 206, 210, 214, 222, 230, 238, 246, 254, 262, 266, 278, 282, 286, 302, 210, 218, 322, 326,

330, 334, 358, 366, 374, 382, 386, 390, 398, 406, 422, 426, 430, 434, 438, 446, 454, 462, 470, 474, 478, 482, 494,

and $m_i$'s with even $D_i$ (that is, $u_i \equiv 1, v_i \equiv 0 \pmod 4$): 66, 114, 146, 178, 258, 354, 402, 410, 418, 466, 498.

### 7.2. 1-dimensional twisted cohomology

We compute $Z_{c_i}(Tr_{K/\mathbf{Q}}(\mathcal{O}_K))$ in order to get the index $[Z_{c_i}(Tr_{K/\mathbf{Q}}(\mathcal{O}_K)) : Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K]$, while

$$|H^1(G, \mathcal{O}_K)_c| = [Z_{c_i}(Tr_{K/\mathbf{Q}}(\mathcal{O}_K)) : Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K]/|\mathcal{O}_k/Tr_{K/\mathbf{Q}}(\mathcal{O}_K)|$$

by Proposition 4.5. For the condition to $\alpha \in Z_{c_i}(Tr_{K/\mathbf{Q}}(\mathcal{O}_K))$, note that for distinct $i$, $j$, and $k$, when $c_{is_i}\,{}^{s_i}\alpha - \alpha = {}^{s_i}\alpha - \alpha \in Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K$, since $\varepsilon_i\,{}^{s_j}\alpha - \alpha = {}^{s_i}(\varepsilon_i\,{}^{s_k}\alpha - \alpha) + {}^{s_i}\alpha - \alpha$, we have

$$c_{is_j}\,{}^{s_j}\alpha - \alpha = \varepsilon_i\,{}^{s_j}\alpha - \alpha \in Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K$$

if and only if

$$c_{is_k}\,{}^{s_k}\alpha - \alpha = \varepsilon_i\,{}^{s_k}\alpha - \alpha \in Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K,$$

hence it is enough to check cocycle images of only two automorphisms, one with the same index $i$ and one any other.

**[Type II]** Note that $Tr_{K/\mathbf{Q}}(\mathcal{O}_K) = 2\mathbf{Z}$ and $[B_c(Tr_{K/\mathbf{Q}}(\mathcal{O}_K)) : Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K] = 2$ for all 1-cocycles $c$ by Proposition 4.5. Denote by the integral basis elements $\eta_1 = 1$, $\eta_2 = \frac{1+\sqrt{m_1}}{2}$, $\eta_3 = \sqrt{m_2}$, $\eta_4 = \frac{\sqrt{m_2}+\sqrt{m_3}}{2}$ so that $\mathcal{O}_K = [\eta_1, \eta_2, \eta_3, \eta_4]_{\mathbf{Z}}$.

(a) $c = c_1$.

Let $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4 \in Z_{c_1}(2\mathcal{O}_K)$. The condition ${}^{s_1}\alpha - \alpha \in 2\mathcal{O}_K$ holds for all integer in $\mathcal{O}_K$. Since

$$\begin{aligned}
\varepsilon_1\,{}^{s_2}\alpha - \alpha &= \left[x(u_1-1) + y\left\{u_1 + v_1\left(\frac{1-m_1}{4}\right)\right\}\right]\eta_1 \\
&+ [xv_1 - y(u_1+1)]\eta_2 \\
&+ \left[z\left\{u_1 - 1 - v_1\left(\frac{d-1}{2}\right)\right\}\right. \\
&+ \left. w\left\{u_1 + v_1\left(\frac{2-m_1/d-d}{4}\right)\right\}\right]\eta_3 \\
&+ \left[zdv_1 + w\left\{v_1\left(\frac{d-1}{2}\right) - u_1 - 1\right\}\right]\eta_4,
\end{aligned}$$

where $\frac{1-m_1}{4}$, $\frac{d-1}{2}$, and $\frac{2-m_1/d-d}{4}$ are integers, we have the system of congruence equations

$$x(u_1-1)+y(u_1+v_1(1-m_1)/4)\equiv 0 \pmod 2 \tag{7.7}$$

$$xv_1-y(u_1+1)\equiv 0 \pmod 2 \tag{7.8}$$

$$z(u_1-1-v_1(d-1)/2)+w(u_1+v_1(2-m_1/d-d)/4)\equiv 0 \pmod 2 \tag{7.9}$$

$$zv_1+w(v_1(d-1)/2-u_1-1)\equiv 0 \pmod 2 \tag{7.10}$$

for $d$ is odd as a divisor of $m_1$. We separate the cases along to the parity of $u_1$ and $v_1$:

(i) Suppose $v_1$ is even. Then $u_1$ is odd and the system of equations turns out to be $y\equiv w\equiv 0 \pmod 2$. We have $[Z_{c_1}(2\mathcal{O}_K):2\mathcal{O}_K]=4$ and hence $|H^1(G,\mathcal{O}_K)_{c_1}|=[Z_{c_1}(2\mathcal{O}_K):B_{c_1}(2\mathcal{O}_K)]=2$.

(ii) Suppose $v_1$ is odd and $u_1$ is even. Note the fact that $v_1$ is odd implies $m_1\equiv 5 \pmod 8$ since if we assume $m_1\equiv 1 \pmod 8$ then $\frac{m_1-1}{4}$ is even and $1=N(\varepsilon_1)=u_1(u_1+v_1)-v_1^2\frac{m_1-1}{4}\equiv u_1(u_1+1)\equiv 0 \pmod 2$ makes a contradiction. Now since $m_1\equiv 5 \pmod 8$, we have that $\frac{m_1-1}{4}$ is odd and $\frac{2-m_1/d-d}{4}\equiv\frac{d+1}{2} \pmod 2$. Indeed, since $d$ is odd we have $d^2\equiv 1 \pmod 8$ so $m_1/d\equiv m_1 d \pmod 8$ and

$$\begin{aligned}2-m_1/d-d &\equiv 2-5d-d \pmod 8\\ &\equiv 2-6(d+1)+6 \pmod 8\\ &\equiv 8-12\left(\frac{d+1}{2}\right) \pmod 8\\ &\equiv 4\left(\frac{d+1}{2}\right) \pmod 8.\end{aligned}$$

The system of equations (7.7)–(7.10) becomes

$$x+y\equiv 0 \pmod 2 \tag{7.11}$$

$$z\,(1-(d-1)/2)+w(d+1)/2\equiv 0 \pmod 2 \tag{7.12}$$

$$z+w((d-1)/2-1)\equiv 0 \pmod 2. \tag{7.13}$$

First suppose that $(d-1)/2$ is odd and $(d+1)/2$ is even. The system of equation turns to $x\equiv y \pmod 2$ and $z\equiv 0 \pmod 2$. On the other hand, if $(d-1)/2$ is even and $(d+1)/2$ is odd, then the system of equation turns to $x\equiv y \pmod 2$ and $z\equiv w \pmod 2$. In any case, the index of $Z_{c_1}(2\mathcal{O}_K)$ from $2\mathcal{O}_K$ is equal to 4 and hence $|H^1(G, \mathcal{O}_K)_{c_1}| = 2$.

(iii) Suppose $v_1$ and $u_1$ are odd. We have that $\frac{m_1-1}{4}$is odd and $\frac{2-m_1/d-d}{4} \equiv \frac{d+1}{2}$. The system of equations (7.7)–(7.10) turns to

$$x \equiv 0 \pmod 2 \tag{7.14}$$
$$-z(d-1)/2 + w(1 + (d+1)/2) \equiv 0 \pmod 2 \tag{7.15}$$
$$z + w(d-1)/2 \equiv 0 \pmod 2. \tag{7.16}$$

If $d \equiv 1 \pmod 4$, since $(d-1)/2$ is even and $(d+1)/2$ is odd, we obtain $x \equiv 0 \pmod 2$ and $z \equiv 0 \pmod 2$. If $d \equiv 3 \pmod 4$ then we obtain $x \equiv 0 \pmod 2$ and $z \equiv w \pmod 2$. Either of the cases, we have $[Z_{c_1}(2\mathcal{O}_K) : 2\mathcal{O}_K] = 4$ and therefore we have $|H^1(G, \mathcal{O}_K)_{c_1}| = [Z_{c_1}(2\mathcal{O}_K) : B_{c_1}(2\mathcal{O}_K)] = 2$.

(b) $c = c_2$.
Let $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4 \in Z_{c_2}(2\mathcal{O}_K)$. From ${}^{s_2}\alpha - \alpha = y\eta_1 - 2y\eta_2 + w\eta_3 - 2w\eta_4 \in 2\mathcal{O}_K$, we obtain that $y, w$ are even. Now we have

$$\begin{aligned}\varepsilon_2\, {}^{s_1}\alpha - \alpha = &\left[x(u_2-1) - zv_2m_2 - wv_2\left(\frac{m_2 - m_2/d}{2}\right)\right]\eta_1 \\ &+ \left[y(u_2-1) - wv_2\frac{m_2}{d}\right]\eta_2 \\ &+ \left[xv_2 - yv_2\left(\frac{d-1}{2}\right) - z(u_2+1)\right]\eta_3 \\ &+ \left[yv_2d - w(u_2+1)\right]\eta_4 \\ &\in 2\mathcal{O}_K\end{aligned}$$

which yields the system of equations

$$x(u_2-1) - zv_2m_2 \equiv 0 \pmod 2 \tag{7.17}$$
$$xv_2 - z(u_2+1) \equiv 0 \pmod 2 \tag{7.18}$$

for $y, w \equiv 0 \pmod 2$. Since we have that $u_2$ and $v_2$ have distinct parity, there are two cases:

(i) Suppose $v_2$ is odd. Then $m_2$ is odd and $u_2$ is even. the system of equations (7.17)–(7.18) turns to be $x \equiv z \pmod 2$. So $Z_{c_2}(2\mathcal{O}_K)$ is the set of elements $x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4$ with $x \equiv z, y \equiv w \equiv 0 \pmod 2$ and has index 2 from $2\mathcal{O}_K$. Therefore we have $|H^1(G, \mathcal{O}_K)_{c_2}| = 1$.

(ii) Suppose $v_2$ is even. The system of equations (7.17)–(7.18) does not give any restricting conditions. So we have $Z_{c_2}(2\mathcal{O}_K)$ is the set of elements $x\eta_1+y\eta_2+z\eta_3+w\eta_4$ with $y \equiv w \equiv 0 \pmod 2$ with index 4 from $2\mathcal{O}_K$ and hence $|H^1(G,\mathcal{O}_K)_{c_2}| = 2$.

(c) $c = c_3$.
Let $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4 \in Z_{c_3}(2\mathcal{O}_K)$. From ${}^{s_3}\alpha - \alpha = y\eta_1 - 2y\eta_2 - (2z+w)\eta_3 \in 2\mathcal{O}_K$ we have $y \equiv w \equiv 0 \pmod 2$. Now

$$\begin{aligned}\varepsilon_3\, {}^{s_1}\alpha - \alpha = &\left[x(u_3-1) + zv_3\left(\frac{m_2}{d}\right) + wv_3\left(\frac{m_2/d - m_3}{2}\right)\right]\eta_1 \\ &+ \left[y(u_3-1) - 2zv_3\left(\frac{m_2}{d}\right) - wv_3\left(\frac{m_2}{d}\right)\right]\eta_2 \\ &+ \left[-xv_3 + yv_3\left(\frac{m_1/d-1}{2}\right) - z(u_3+1)\right]\eta_3 \\ &+ \left[2xv_3 + yv_3 - w(u_3+1)\right]\eta_4\end{aligned}$$

where $\frac{m_2}{d}$, $\frac{m_2/d-m_3}{2}$, $\frac{m_1/d-1}{2}$ are integers. Since $y \equiv w \equiv 0 \pmod 2$, we obtain the system of congruence equations

$$x(u_3-1) + zv_3(m_2/d) \equiv 0 \pmod 2 \tag{7.19}$$

$$xv_3 + z(u_3+1) \equiv 0 \pmod 2 \tag{7.20}$$

(i) Suppose $v_3$ is odd. Then $m_3$ and $m_2$ are odd, and we obtain from the system of equations (7.19)–(7.20) that $x + z \equiv 0 \pmod 2$. So we have $Z_{c_3}(2\mathcal{O}_K)$ is the set of elements $x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4$ with $x \equiv z$, $y \equiv w \equiv 0 \pmod 2$ with index 2 from $2\mathcal{O}_K$ and hence $|H^1(G,\mathcal{O}_K)_{c_3}| = 1$.

(ii) Suppose $v_3$ is even. We obtain no more conditions other than $y \equiv w \equiv 0 \pmod 2$ from the system of equations (7.19)–(7.20). $|H^1(G,\mathcal{O}_K)_{c_3}| = 2$.

**[Type III]** Note that $Tr_{K/\mathbf{Q}}(\mathcal{O}_K) = 4\mathbf{Z}$ and we have $[B_c(4\mathcal{O}_K) : 4\mathcal{O}_K] = 4$ for all cocycles $c$ by Proposition 4.5. Denote by the integral basis elements $\eta_1 = 1$, $\eta_2 = \sqrt{m_1}$, $\eta_3 = \sqrt{m_2}$, $\eta_4 = \frac{\sqrt{m_2}+\sqrt{m_3}}{2}$ so that $\mathcal{O}_K = [\eta_1,\eta_2,\eta_3,\eta_4]_{\mathbf{Z}}$.

(a) $c = c_1$.
Let $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4 \in Z_{c_1}(4\mathcal{O}_K)$. From ${}^{s_1}\alpha - \alpha = -2z\eta_3 - 2w\eta_4 \in 4\mathcal{O}_K$ we have $z \equiv w \equiv 0 \pmod 2$. On the

other hand, we have that

$$\begin{aligned}\varepsilon_1{}^{s_2}\alpha - \alpha = &[x(u_1-1) - yv_1m_1]\,\eta_1\\ &+ [xv_1 - y(u_1+1)]\,\eta_2\\ &+ \left[z(u_1-1-v_1d) + w\left\{u_1 - v_1\left(\frac{m_1/d+d}{2}\right)\right\}\right]\eta_3\\ &+ [2zv_1d + w(-u_1-1+v_1d)]\,\eta_4\\ \in &4\mathcal{O}_K\end{aligned}$$

where $\left(\frac{m_1/d+d}{2}\right)$ is an integer, which yields the system of congruence equations

$$x(u_1-1) + yv_1 \equiv 0 \pmod 4 \tag{7.21}$$

$$xv_1 - y(u_1+1) \equiv 0 \pmod 4 \tag{7.22}$$

$$wu_1 \equiv 0 \pmod 4 \tag{7.23}$$

since $m_1 \equiv 3 \pmod 4$, $u_1 \not\equiv v_1 \pmod 2$, $z \equiv w \equiv 0 \pmod 2$, $d \equiv 1 \pmod 2$, and $\frac{m_1/d+d}{2} \equiv 0 \pmod 2$.

(i) Suppose $v_1$ is odd and so $u_1$ is even. Then equations (7.21) and (7.22) are equivalent and equation (7.23) is trivial. We obtain $x \equiv y \pmod 4$ if $u_1 + 1 \equiv v_1 \pmod 4$ and we obtain $x \equiv -y \pmod 4$ if $u_1 - 1 \equiv v_1 \pmod 4$. Together with the condition $z \equiv w \equiv 0 \pmod 2$, we have that $[Z_{c_1}(4\mathcal{O}_K) : 4\mathcal{O}_K] = 16$. Hence $|H^1(G, \mathcal{O}_K)_{c_1}| = 4$, which is the case that $|\widehat{H}^0(G, \mathcal{O}_K)_c| \neq |H^1(G, \mathcal{O}_K)_c|$.

(ii) Suppose $v_1$ is even. Note that $v_1 \equiv 0 \pmod 4$. Indeed, since $u_1$ is odd, we have $\left(\frac{u_1-1}{2}\right)\left(\frac{u_1+1}{2}\right) = m_1\left(\frac{v_1}{2}\right)^2$ with integer terms. As the multiple of consecutive numbers, the left hand side is even, so $v_1/2$ is even. Now, equation (7.23) gives $w \equiv 0 \pmod 4$. Equations (7.21) and (7.22) gives that if $u_1 \equiv 1 \pmod 4$ then $y \equiv 0 \pmod 2$ and if $u_1 \equiv 3 \pmod 4$ then $x \equiv 0 \pmod 2$. Together with the conditions $z \equiv 0 \pmod 2$ and $w \equiv 0 \pmod 4$, we have that $[Z_{c_1}(4\mathcal{O}_K) : 4\mathcal{O}_K] = 16$ and $|H^1(G, \mathcal{O}_K)_{c_1}| = 4$.

(b) $c = c_2$.
Let $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4 \in Z_{c_2}(4\mathcal{O}_K)$. Since ${}^{s_2}\alpha - \alpha = -2y\eta_2 + w\eta_3 - 2w\eta_4 \in 4\mathcal{O}_K$, we have $y \equiv 0 \pmod 2$ and $w \equiv 0 \pmod 4$.

Now from the condition

$$\begin{aligned}\varepsilon_2\,{}^{s_1}\alpha - \alpha = &\left[x(u_2-1) - zv_2m_2 - wv_2\left(\frac{m_2}{2}\right)\right]\eta_1 \\ &+\left[y(u_2-1) - wv_2\left(\frac{m_2}{2d}\right)\right]\eta_2 \\ &+\left[xv_2 - yv_2d - z(u_2+1)\right]\eta_3 \\ &+\left[2yv_2d - w(u_2+1)\right]\eta_4 \\ \in &4\mathcal{O}_K\end{aligned}$$

we obtain the system of congruence equations

$$x(u_2-1) \equiv 0 \pmod 4 \tag{7.24}$$

$$xv_2 - z(u_2+1) \equiv 0 \pmod 4 \tag{7.25}$$

since $m_2 \equiv 2 \pmod 4$, $u_2 \equiv 1 \pmod 2$, and $v_2 \equiv 0 \pmod 2$. If $u_2 \equiv 3 \pmod 4$, the system of congruence equations turns to $x \equiv 0 \pmod 2$. When $u_2 \equiv 1 \pmod 4$, if $v_2 \equiv 0 \pmod 4$ then the system of equations implies that $z \equiv 0 \pmod 2$, and if $v_2 \equiv 2 \pmod 4$, one does that $x \equiv z \pmod 2$. Therefore, we have $Z_{c_2}(4\mathcal{O}_K)$ is the set of elements $x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4$ with $x \equiv y \equiv 0 \pmod 2$ and $w \equiv 0 \pmod 4$ if $u_2 \equiv 3 \pmod 4$, the set of elements $x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4$ with $y \equiv z \equiv 0 \pmod 2$ and $w \equiv 0 \pmod 4$ if $u_2 \equiv 1 \pmod 4$ and $v_2 \equiv 0 \pmod 4$, and the set of elements defined by the conditions $x \equiv z \pmod 2$, $y \equiv 0 \pmod 2$, and $w \equiv 0 \pmod 4$ if $u_2 \equiv 1 \pmod 4$ and $v_2 \equiv 2 \pmod 4$. In any cases the index of $Z_{c_2}(4\mathcal{O}_K)$ from $4\mathcal{O}_K$ is 16 and we have $|H^1(G, \mathcal{O}_K)_{c_2}| = 4$. Comparing to the order of 0-Tate twisted cohomology, we have $|\widehat{H}^0(G, \mathcal{O}_K)_{c_2}| \neq |H^1(G, \mathcal{O}_K)_{c_2}|$ if $u_2 \equiv 3 \pmod 4$, or if $u_2 \equiv 1 \pmod 4$ and $v_2 \equiv 2 \pmod 4$.

(c) $c = c_3$.
Let $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4 \in Z_{c_3}(4\mathcal{O}_K)$. Since ${}^{s_3}\alpha - \alpha = -2y\eta_2 - (w+2z)\eta_3 \in 4\mathcal{O}_K$, we have $y \equiv 0 \pmod 2$ and $w + 2z \equiv 0 \pmod 4$. The latter condition shows that $w$ is even, and that $w \equiv 0 \pmod 4$ if and only if $z \equiv 0 \pmod 2$.

Now from the condition

$$
\begin{aligned}
{\varepsilon_3}^{s_1}\alpha - \alpha = & \left[x(u_3-1) - wv_3\left(\frac{m_3}{2}\right)\right]\eta_1 \\
& + \left[y(u_3-1) - zv_3\left(\frac{m_2}{d}\right) - wv_3\left(\frac{m_2}{2d}\right)\right]\eta_2 \\
& + \left[-xv_3 + yv_3\left(\frac{m_1}{d}\right) - z(u_3+1)\right]\eta_3 \\
& + \left[2xv_3 - w(u_3+1)\right]\eta_4 \\
& \in 4\mathcal{O}_K,
\end{aligned}
$$

we obtain the system of congruence equations

$$x(u_3-1) \equiv 0 \pmod 4 \tag{7.26}$$

$$-xv_3 - z(u_3+1) \equiv 0 \pmod 4, \tag{7.27}$$

and it turns to be following equations due to $u_3$, $v_3$ modulo 4: $z \equiv 0 \pmod 2$ and $w \equiv 0 \pmod 4$ if $u_3 \equiv 1 \pmod 4$ and $v_3 \equiv 0 \pmod 4$; $x \equiv z \pmod 2$ if $u_3 \equiv 1 \pmod 4$ and $v_3 \equiv 2 \pmod 4$; and $x \equiv 0 \pmod 2$ if $u_3 \equiv 3 \pmod 4$.
Hence $Z_{c_3}(4\mathcal{O}_K)$ is the set of elements $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4$ satisfying

$$\left\{\begin{array}{l} y \equiv z \equiv 0 \pmod 2 \\ w \equiv 0 \pmod 4 \end{array}\right\} \quad \text{if } u_3 \equiv 1 \pmod 4 \text{ and } v_3 \equiv 0 \pmod 4$$

$$\left\{\begin{array}{l} x \equiv z \pmod 2 \\ y \equiv 0 \pmod 2 \\ w \equiv 2z \pmod 4 \end{array}\right\} \quad \text{if } u_3 \equiv 1 \pmod 4 \text{ and } v_3 \equiv 2 \pmod 4$$

$$\left\{\begin{array}{l} x \equiv y \equiv 0 \pmod 2 \\ w \equiv 2z \pmod 4 \end{array}\right\} \quad \text{if } u_3 \equiv 3 \pmod 4$$

and we have that $[Z_{c_3}(4\mathcal{O}_K) : 4\mathcal{O}_K] = 16$ and $|H^1(G,\mathcal{O}_K)_{c_3}| = 4$ in any cases. Also we have $|\widehat{H}^0(G,\mathcal{O}_K)_{c_3}| \neq |H^1(G,\mathcal{O}_K)_{c_3}|$ if $u_3 \equiv 3 \pmod 4$ or if $u_3 \equiv 1$, $v_3 \equiv 2 \pmod 4$.

Remark 7.2. It is easy to see that $|H^1(G,\mathcal{O}_K)| = |\widehat{H}^1(G,\mathcal{O}_K)| = 2$ or 4 for Type II or Type III, respectively. As a summary, we have $|H^1(G,\mathcal{O}_K)_c| \neq |\widehat{H}^1(G,\mathcal{O}_K)_c|$ in Type III, when

(a) $c = c_1$ and $v_1$ is odd
(b) $c = c_i$ and $u_i \equiv 3 \pmod 4$, for $i = 2, 3$
(c) $c = c_i$, $u_i \equiv 1 \pmod 4$, and $v_i \equiv 2 \pmod 4$, for $i = 2, 3$.

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