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THE ARCSINE LAW IN THE GENERALIZED ANALOGUE OF WIENER SPACE

Kun Sik Ryu*

Abstract. In this note, we prove the theorems in the generalized analogue of Wiener space corresponding to the second and the third arcsine laws in either concrete or analogue of Wiener space [1, 2, 7] and we show that our results are exactly same to either the concrete or the analogue of Wiener case when the initial condition gives either the Dirac measure at the origin or the probability Borel measure.

1. Introduction

In 1940, Levy proved a beautiful Theorem, say the first arcsine law in the concrete Wiener space [3], that proportion of time $m_{\delta_0}(T_+(x) \le t) = \frac{2}{\pi}\text{arcsin}\sqrt{t}$ for $t \in [0,1]$, where $T_+(x) = m_L(\{t \in [0,1] | x(t) > 0\})$. Since then, one proved the second and the third arcsin laws in the concrete Wiener space that $m_{\delta_0}(L(x) \le t) = m_{\delta_0}(M_1(x) \le t) = \frac{2}{\pi}\text{arcsin}\sqrt{t}$ for $t \in [0,1]$ where $L(x) = \sup_{x(s)=0} s$ and $M_1(x) = \sup_{0 \le s \le 1} x(s)$[1, 2, 4].

In 2002, the author and Dr.Im presented the definition of the analogue of Wiener space, a kind of the generalization of the concrete Wiener space, and its properties[5]. In 2010, the author introduce the definition of the generalized analogue of Wiener space, a kind of the more generalization of the analogue of Wiener space, and its properites[6]. In [7], the author proved the second and the third arcsine laws in the analogue of Wiener space as following; letting $\alpha = \frac{tu_0}{s+t}$ and $\sigma = \sqrt{\frac{st}{s+t}}$ for $0 < s < T$, $m_\phi(L(x) \le s) = \int_{-\infty}^{+\infty}\int_{T-s}^{+\infty} \frac{1}{\pi\sqrt{st^3}}(\sigma^2 \text{exp}(-\frac{\alpha^2}{2\sigma^2}) +$

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$\alpha\sigma\int_0^{\alpha/\sigma} \exp(-\frac{u^2}{2})du)\exp(-\frac{{u_0}^2}{2(s+t)})dtd\phi(u_0)$ and $m_\phi(\theta_t(x) \le s) = \int_{-\infty}^{+\infty}$ $(\int_0^s \frac{1}{\pi\sqrt{r(t-r)}}\exp(-\frac{{u_0}^2}{2r})dr)d\phi(u_0)$.

Here, if $\phi = \delta_0$, our results are exactly the same to the results in the concrete Wiener case. In this note, we will prove the theorems in the generalized analogue of Wiener space, corresponding to the second and the third arcsine laws in the concrete Wiener space.

2. The definitions and the basics properties of generalized analogue of Wiener space

In this section, we introduce the definitions of the generalized analogue of Wiener space and investigate the basic properties of it which are needed to understand the next section. Throughout in this note, let T be a positive real number, let $C[0,T]$ be the space of all continuous functions on a closed interval $[0,T]$ with the supremum norm $||x||_\infty = \sup_{t\in[0,T]} |x(t)|$, let ϕ be a probability Borel measure on $\mathbb{R}$, let m_L be the Lebesgue measure and let $\alpha, \beta : [0,T] \to$ be continous functions such that β is non-negative strictly increasing. We define the generalized analogue of Wiener measure $m_{\alpha,\beta;\phi}$ on $C[0,T]$ as follows. Let $\vec{t} = (t_0, t_1, t_2, \cdots, t_n)$ with $0 = t_0 < t_1 < t_2 < \cdots < t_n \le T$. Let $J_{\vec{t}} : C[0,T] \to \mathbb{R}^{n+1}$ be a function with $J_{\vec{t}}(x) = (x(t_0), x(t_1), x(t_2), \cdots, x(t_n))$ and let B_j $(j = 0, 1, 2, \cdots, n)$ be in $\mathcal{B}(\mathbb{R})$. The subsets $J_{\vec{t}}^{-1}(\prod_{j=1}^n (B_j)$ of $C[0,T]$ is called an interval and let $\mathcal{M}$ be the smallest σ-algebra contains all intervals. For an interval $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$, let $\omega_\phi(J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)) = \int_{B_0}\int_{B_1}\cdots\int_{B_n} W(n+1;\vec{t};\vec{u};\alpha,\beta)dm_L(u_n)dm_L(u_{n-1})\cdots dm_L(u_1)d\phi(u_0)$ where $W(n+1;\vec{t};\vec{u};\alpha,\beta)$
$= (\prod_{j=1}^n \frac{1}{\sqrt{2\pi(\beta(t_j)-\beta(t_{j-1})}})\exp(-\frac{1}{2}\sum_{j=1}^n \frac{(u_j-\alpha(t_j)-u_{j-1}+\alpha(t_{j-1}))^2}{\beta(t_j)-\beta(t_{j-1})})$.
Let $m_{\alpha,\beta;\phi}$ be the Borel measure on $C[0,T]$ such that for all I in $\mathcal{M}$, $\omega_\phi(I) = m_{\alpha,\beta;\phi}(I)$, this measure is called the generalized analogue of Wiener measure on $C[0,T]$.

When α is a zero function and β is an identity funtion, $m_{\alpha,\beta;\phi}$ is the analogue of Wiener measure, when ϕ is a Dirac measure δ_0 at the origin , α is a zero function and β is an identity function, $m_{\alpha,\beta;\phi}$ is the concrete Wiener measure.

From [6], we can find the following lemma.

LEMMA 2.1. *Under the notations in above, if $f:\mathbb{R}^{n+1}\to\mathbb{R}$ is a Borel measureable fuction then the following equality holds.*

$$\int_{C[0,T]} f(x(t_0),x(t_1),x(t_2),\cdots,x(t_n))dm_{\alpha,\beta,\phi}(x)$$
$$=\int_{\mathbb{R}}^{n+1} f(u_0,u_1,u_2,\cdots,u_n)W(n+1;\vec{t};\vec{u};\alpha,\beta)$$
$$d(((\prod_{j=1}^{n} m_L)\times\phi)((u_1,u_2,\cdots,u_n),u_0))$$

where if one side integral exists, the both sides integral exist and the two values are the same.

REMARK 2.2. (1) For a Borel subset B of $C[0,T]$, $m_{\alpha,\beta,\phi}(B)=\int m_{\alpha,\beta,\delta_u}(B)d\phi(u)$.
(2) m_ϕ has no atoms.

For x in $C[0,T]$, t in $[0,T]$ and a real number b, we let $T_b(x)=\inf_{x(t)=b}t$, $L(x)=\sup_{x(t)=0}t$, $M_t(x)=\sup_{0\le s\le t}x(s)$ and $\theta_t(x)=\sup_{x(s)=M_t(x)}s$. Here, T_b is called the first time hits b, L is called the last time of the last zero before T and M_t is called the running maximum.

A random variable X is said to have the arcsine distribution if it is supported on $[0,T]$ with the cumulative density function $F(t)=\frac{2}{\pi}\arcsin\sqrt{\frac{t}{T}}$(or probability density function $f(t)=F'(t)=\frac{1}{\pi\sqrt{t(T-t)}}$).

LEMMA 2.3. *(The reflection principle in the generalized analogue of Wiener space) For a real number a and t in $(0,T]$, $m_{\alpha,\beta;\phi}(x(t)<a)=\frac{1}{\sqrt{2\pi(\beta(t)-\beta(0))}}\int_{-\infty}^{+\infty}\int_{2(u_0-\alpha(0)+\alpha(t))-a}^{+\infty}\exp(-\frac{(u-\alpha(t)-u_0+\alpha(0))^2}{2(\beta(t)-\beta(0))})dud\phi(u_0)$.*

Proof. From the change of variables theorem, we have

$$m_{\alpha,\beta;\delta_{u_0}}(x(t)<a)=\frac{1}{\sqrt{2\pi(\beta(t)-\beta(0))}}\int_{-\infty}^{a}\exp(-\frac{(u-\alpha(t)-u_0+\alpha(0))^2}{2(\beta(t)-\beta(0))})du$$
$$=\frac{1}{\sqrt{2\pi(\beta(t)-\beta(0))}}\int_{2(u_0-\alpha(0)+\alpha(t))-a}^{+\infty}\exp(-\frac{(u-\alpha(t)-u_0+\alpha(0))^2}{2(\beta(t)-\beta(0))})du$$
$$=m_{\alpha,\beta;\delta_{u_0}}(x(t)>2(u_0-\alpha(0)+\alpha(t))-a).$$

So, we obtain our equality from Remark 2.2 (1) in above. □

REMARK 2.4. If α is a zero function and β is an identity function, $m_{\alpha,\beta;\phi}$ is the analogue of Wiener measure,

$$m_{\alpha,\beta;\phi}(x(t)<a)=\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}\int_{2u_0-a}^{+\infty}\exp(-\frac{(u-u_0)^2}{2t})dud\phi(u_0).$$

Which is exactly the same to the result in the analogue Wiener case [7]. When ϕ is a Dirac measure δ_0 at the origin in $\mathbb{R}$, α is a zero function,

$$m_{\alpha,\beta;\phi}(x(t) < a) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{a} \exp(-\frac{u^2}{2t}) du = \frac{1}{\sqrt{2\pi t}} \int_{-a}^{+\infty} \exp(-\frac{u^2}{2t}) du$$

which is exactly the same to the result in concrete Wiener measure [1].

LEMMA 2.5. *For a real number b and t in $(0,T]$,*

$$m_{\alpha,\beta,\phi}(T_b(x) < t) = \sqrt{\tfrac{2}{\pi}}\{\int_{-\infty}^{b}(\int_{(b-\alpha(t)-u_0+\alpha(0))/\sqrt{\beta(t)-\beta(0)}}^{+\infty} \exp(-\tfrac{v^2}{2})dv)d\phi(u_0)$$
$$+\int_{b}^{+\infty}(\int_{-\infty}^{(b-\alpha(t)-u_0+\alpha(0))/\sqrt{\beta(t)-\beta(0)}} \exp(-\tfrac{v^2}{2})dv)d\phi(u_0)\} + \phi(\{b\}).$$

Proof. When $b = u_0$, $m_{\delta_{u_0}}(T_b(x) < t) = m_{\delta_{u_0}}(x(0) = u_0) = 1$. If $b < u_0$, by the symmetry of Brownian motion and the intermediate value theorem, we have

$$\begin{aligned} & m_{\alpha,\beta;\delta_{u_0}}(T_b(x) < t) \\ &= m_{\alpha,\beta;\delta_{u_0}}(T_b(x) < t, x(t) > b) + m_{\alpha,\beta;\delta_{u_0}}(T_b(x) < t, x(t) < b) \\ &= 2m_{\alpha,\beta;\delta_{u_0}}(T_b(x) < t, x(t) < b) \\ &= 2m_{\alpha,\beta;\delta_{u_0}}(x(t) < b) \\ &= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{(b-\alpha(t)-u_0+\alpha(0))/\sqrt{\beta(t)-\beta(0)}} \exp(-\frac{v^2}{2})dv. \end{aligned}$$

By the essentially similar method in above, if $b > u_0$,

$$m_{\alpha,\beta;\delta_{u_0}}(T_b(x) < t) = \sqrt{\frac{2}{\pi}} \int_{(b-\alpha(t)-u_0+\alpha(0))/\sqrt{\beta(t)-\beta(0)}}^{+\infty} \exp(-\frac{v^2}{2})dv.$$

Hence, we have our conclusion from Remark 2.2 (1) in above. □

REMARK 2.6. (1) When α is a zero function, β is an identity function, $m_{\alpha,\beta;\phi}(T_b(x) < t) = \{\int_{-\infty}^{b}(\int_{(b-u_0)/\sqrt{t}}^{+\infty} \exp(-\frac{v^2}{2})dv)d\phi(u_0)$ $+ \int_{b}^{+\infty}(\int_{-\infty}^{(b-u_0)/\sqrt{t}} \exp(-\frac{v^2}{2})dv)\ d\phi(u_0)\} + \phi(\{b\})$. This is exactly the same to the results in the analogue of Wiener case. If α is a zero function, β is an identity function and $\phi = \delta_0$ then $m_{\alpha,\beta;\phi} = \sqrt{2}\pi \int_{b/\sqrt{t}}^{+\infty} \exp(-\frac{v^2}{2})dv$, $\frac{\partial}{\partial t} m_{\alpha,\beta;\phi}(T_b(x) < t = \frac{|b|}{\sqrt{2\pi t^3}} \exp(-\frac{b^2}{2t})$.

(2) When $b \neq u_0$, $\frac{\partial}{\partial b} m_{\alpha,\beta;\delta_{u_0}}(T_b(x) < t) = sgn(b - \alpha(t) + u_0)\sqrt{\frac{2}{\pi(\beta(t)-\beta(0))}}$ $\exp(-\frac{(b-\alpha(t)+u_0)^2}{2(\beta(t)-\beta(0))})$. If the Radon-Nikodym derivative $\frac{d\phi}{dm_L} \equiv h$ exists and is continuous, $\frac{\partial}{\partial b} m_{\alpha,\beta;\delta_{u_0}}(T_b(x) < t) = \sqrt{\frac{2}{\pi(\beta(t)-\beta(0))}}$ $\{-\int_{-\infty}^{b} \exp(-\frac{(b-\alpha(0)-u_0)^2}{2(\beta(t)-\beta(0))})d\phi(u_0) + \int_{b}^{+\infty} \exp(-\frac{(b-\alpha(0)-u_0)^2}{2(\beta(t)-\beta(0))})d\phi(u_0)\}$

$+\sqrt{\frac{2}{\pi(\beta(t)-\beta(0))}}\{\int_{-\alpha(t)/\sqrt{\beta(t)-\beta(0)}}^{+\infty}\exp(-\frac{u^2}{2})dm_L(u)-\int_{-\infty}^{\alpha(t)/\sqrt{\beta(t)-\beta(0)}}$ $\exp(-\frac{u^2}{2})dm_L(u)\}h(b)$.

(3) $\frac{\partial}{\partial t}m_{\alpha,\beta;\delta_{u_0}}(T_b(x)<t)=\frac{1}{\sqrt{2\pi(\beta(t)-\beta(0))^3}}|u_0-\alpha(0)-\alpha(t)-b|$ $\exp(-\frac{(b-\alpha(t)-u_0+\alpha(0))^2}{2(\beta(t)-\beta(0))})$ for t in $(0,T]$. So, $m_{\alpha,\beta;\delta_{u_0}}(T_b(x)>t)$ $=\int_t^{+\infty}\frac{1}{\sqrt{2\pi(\beta(s)-\beta(0))^3}}|u_0-\alpha(0)-\alpha(t)-b|\exp(-\frac{(b-u_0)+\alpha(0)}{2(\beta(s)-\beta(0))})ds$.
If the Radon-Nikodym derivative $\frac{d\phi}{dm_L}\equiv h$ exists and is continuous, $\frac{\partial}{\partial t}m_{\alpha,\beta,\phi}(T_b(x)<t)=\sqrt{\frac{1}{2\pi(\beta(t)-\beta(0))^3}}[\int_{-\infty}^{b}\{2\alpha'(t)(\beta(t)-\beta(0))+\beta'(b-\alpha(t)-u_0)\}\exp(-\frac{(b-\alpha(t)-u_0+\alpha(0))^2}{2(\beta(t)-\beta(0))})d\phi(u_0)-\int_b^{+\infty}\{2\alpha'(t)(\beta(t)-\beta(0))+\beta'(b-\alpha(t)-u_0)\}\exp(-\frac{(b-\alpha(t)-u_0+\alpha(0))^2}{2(\beta(t)-\beta(0))})d\phi(u_0)]$. When α is a zero function, β is an identity function and the Radon-Nikodym derivative $\frac{d\phi}{dm_L}\equiv h$ exists and is continuous, $\frac{\partial}{\partial t}m_{\alpha,\beta;\phi}(T_b(t)<t)=\sqrt{\frac{1}{2\pi t^3}}[\int_{-\infty}^{b}(b-u_0)\exp(-\frac{(b-u_0)^2}{2t})d\phi(u_0)-\int_b^{+\infty}(b-u_0)\exp(-\frac{(b-u_0)^2}{2t})d\phi(u_0)]$.
This is exactly same to the results in the concrete Wiener case. From [1, 2], we know that, in the concrete Wiener case, for $b>0$, $0<t<T$ and a Borel subset B in $\mathbb{R}$, $m_{\alpha,\beta;\delta_{u_0}}(T_b(x)\le t,x(t)\in B)=\frac{1}{\sqrt{2\pi t}}\int_{2b-B}\exp(-\frac{v^2}{2t})dv$. By the essentially similar method, we obtain, in the generalized analogue of Wiener case, if $b>u_0$,

$$\begin{aligned}&m_{\alpha,\beta;\delta_{u_0}}(T_b(x)<t,x(t)\le a)\\&=\sqrt{\frac{2}{\pi}}\int_{b-\alpha(t)-u_0-\alpha(0)/\sqrt{\beta(t)-\beta(0)}}^{+\infty}\exp(-\frac{v^2}{2})dv \text{ and if } u_0\le b,\\&m_{\alpha,\beta;\phi}(T_b(x)<t,x(t)\le a)\\&=\frac{2}{\pi}\int_{-\infty}^{(b-\alpha(t)-u_0+\alpha(0))/\sqrt{\beta(t)-\beta(0)}}\exp(-\frac{v^2}{2})dv.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&m_{\alpha,\beta;\phi}(T_b(x)\le t,x(t)\le a)\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a}\int_{2(b-\alpha(t)-u_0-\alpha(0))-a/\sqrt{\beta(t)-\beta(0)}}^{+\infty}\exp(-\frac{(u)^2}{2})dud\phi(u_0)\\&+\int_a^{+\infty}\int_{-\infty}^{2(b-\alpha(t)-u_0-\alpha(0))-a/\sqrt{\beta(t)-\beta(0)}}\exp(-\frac{u^2}{2})dud\phi(u_0).\end{aligned}$$

LEMMA 2.7. *Let a and b be two real numbers with $a \le b$ and $b > 0$.*

Then $m_{\alpha,\beta;\phi}(M_t(x) \ge b, x(t) < a)$

$$= \frac{1}{\sqrt{2\pi}}\{\int_b^{+\infty}\int_{-\infty}^{a-u_0} \exp(-\frac{u^2}{2})dud\phi(u_o)$$

$$+ \int_{-\infty}^{b}\int_{\{2(b-\alpha(t)-u_0-\alpha(0))-a\}/\sqrt{\beta(t)-\beta(0)}}^{+\infty} \exp(-\frac{u^2}{2t})dud\phi(u_o)\}$$

and $m_{\alpha,\beta;\phi}(M_t(x) \le b, x(t) < a)$

$$= \frac{1}{\sqrt{2\pi}}\{\int_{-\infty}^{b}\int_{-\infty}^{(a-u_0)/\sqrt{\beta(t)-\beta(0)}} \exp(-\frac{u^2}{2})dud\phi(u_o)$$

$$- \int_{-\infty}^{b}\int_{\{2(b-\alpha(t)-u_0-\alpha(0))-a\}/\sqrt{\beta(t)-\beta(0)}}^{+\infty} \exp(-\frac{u^2}{2})dud\phi(u_o)\}.$$

Proof. If $u_0 \ge b$, by the intermediate value theorem, $m_{\alpha,\beta,;\delta_{u_0}}(M_t(x) \ge b, x(t) < a) = m_{\delta_{\alpha,\beta,u_0}}(x(t) < a)$. If $u_0 < b$, $m_{\alpha,\beta,\delta_{u_0}}(M_t(x) \ge b, x(t) < a) = m_{\alpha,\beta;\delta_{u_0}}(T_b(x) \le b, x(t) < a)$. So, we obtain our equality. □

REMARK 2.8. (1) $m_{\alpha,\beta;\phi}(M_t(x) \le b, x(t) < a) = m_{\alpha,\beta,\phi}(x(t) < a) - m_{\alpha,\beta;\phi}(M_t(x) \ge b, x(t) < a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{b}\{\int_{-\infty}^{(a-u_0)/\sqrt{\beta(t)-\beta(0)}} \exp(-\frac{u^2}{2})du\}d\phi(u_0) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{b}\{\int_{\{2(b-\alpha(t)-u_0+\alpha(0))-a\}/\sqrt{\beta(t)-\beta(0)}}^{+\infty} \exp(-\frac{u^2}{2}du\}d\phi(u_0)$.

(2) For $b > 0$, α be a zero function and β be an identity function. $m_{\alpha,\beta;\delta_0}(M_t(x) \le b, x(t) < a) = \frac{1}{\sqrt{2\pi}}(\int_{-\infty}^{a/\sqrt{t}} \exp(-\frac{u^2}{2})du - \int_{(2b-a)/\sqrt{t}}^{+\infty} \exp(-\frac{u^2}{2})du)$. So, $\frac{\partial}{\partial b}m_{\alpha,\beta,\delta_0}(M_t(x) \le b, x(t) < a) = \sqrt{\frac{2}{\pi t}}\exp(-\frac{(2b-a)^2}{2t})$ and $\frac{\partial^2}{\partial b\partial a}m_{\alpha,\beta;\delta_0}(M_t(x) \le b, x(t) < a) = \frac{\sqrt{2}(2b-a)}{\sqrt{\pi t^3}}\exp(-\frac{(2b-a)^2}{2t})$. This is exactly same to the result in the concrete Wiener case.

(3) If $u_0 > b$, then $m_{\alpha,\beta;\delta_{u_0}}(M_t(x) \le b, x(t) < a) = 0$. So, $\frac{\partial^2}{\partial b\partial a}m_{\alpha,\beta,\delta_{u_0}}(M_t(x) \le b, x(t) < a) = 0$ and if $u_0 \le b$ then from Remark(2) in above, $\frac{\partial^2}{\partial b\partial a}m_{\alpha,\beta;\delta_{u_0}}(M_t(x) \le b, x(t) < a) = \frac{\sqrt{2}(2b-\alpha(t)-u_0-\alpha(0))-a}{\sqrt{\pi(\beta(t)-\beta(0))^3}} \exp(-\frac{\{(2b-\alpha(t)-u_0+\alpha(0))-a\}^2}{2(\beta(t)-\beta(0))})$.

3. The second and the third arcsine laws in generalized analogue of Wiener space

In this section, we prove the second and the third arcsine laws in generalized analogue of Wiener space which are main theorems in this notes.

Let $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp(-\frac{u^2}{2})du$. Then the following two equalities are known facts

$$\int_0^{+\infty} \exp(-\tfrac{(u+m)^2}{2\sigma^2})du = \sqrt{2\pi}\sigma\Phi(-\tfrac{m}{\sigma})$$

and

$$\int_0^{+\infty} u\exp(-\tfrac{(u+m)^2}{2\sigma^2})du = \sigma^2\exp(-\tfrac{m^2}{2\sigma^2}) - \sqrt{2\pi}\alpha\sigma\Phi(-\tfrac{m}{\sigma}).$$

Theorem 3.1. *(The seond arcsine Laws in generalized analogue of Wiener space) Let* $v = \frac{(\beta(t)-\beta(0))(u_0-\alpha(0))}{\beta(s)+\beta(t)-2\beta(0)}$ *and* $\sigma = \sqrt{\frac{(\beta(t)-\beta(0))(u_0-\alpha(0))}{\beta(s)+\beta(t)-2\beta(0)}}$. *For* $0 < s < T$,

$$\begin{aligned}
&m_{\alpha,\beta;\phi}(L(x) \le s)\\
&= \int_{-\infty}^{+\infty}\int_{T-s}^{+\infty}[\frac{1}{\sqrt{(\beta(s)-\beta(0))(\beta(t)-\beta(0))^3}}(\sigma^2\exp(-\frac{v^2}{2\sigma^2})\\
&+v\sigma\int_0^{v/\sigma}\exp(-\frac{u^2}{2})du\exp(-\frac{(u_0-\alpha(0))^2}{2(\beta(s)+\beta(t)-2\beta(0))})dtd\phi(u_0).
\end{aligned}$$

Proof. From Remark 2.6 (3), we have

$$\begin{aligned}
&m_{\alpha,\beta,\delta_{u_0}}(T_b(x) > t)\\
&= \int_t^{+\infty}\frac{1}{\sqrt{2\pi(\beta(s)-\beta(0))^3}}|u_0-\alpha(0)-b|\exp(-\frac{(b-u_0+\alpha(0))^2}{2(\beta(s)-\beta(0))})ds.
\end{aligned}$$

Hence, by the Fubini Theorem, for u_0 in $\mathbb{R}$,

$$\begin{aligned}
&m_{\delta_{u_0}}(L(x) \le s)\\
&= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi(\beta(s)-\beta(0))}}\exp(-\frac{(u_1-\alpha(s)-u_0+\alpha(0))^2}{2(\beta(s)-\beta(0))})\\
&m_{\delta_{u_1}}(T_0(x) > T-s)du_1\\
&= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi(\beta(s)-\beta(0))}}\exp(-\frac{(u_1-\alpha(s)-u_0+\alpha(0))^2}{2(\beta(s)-\beta(0))})\\
&\{\int_{T-s}^{+\infty}\frac{1}{\sqrt{2\pi(\beta(t)-\beta(0))^3}}|u_1-\alpha(t)|\exp(-\frac{(u_1-\alpha(t))^2}{2t})dt\}du_1\\
&= \int_{T-s}^{+\infty}\frac{1}{2\pi\sqrt{(\beta(s)-\beta(0))(\beta(t)-\beta(0))^3}}(\int_{-\infty}^{+\infty}|u_1-\alpha(t)|\exp(-\frac{(u_1-v)^2}{2\sigma})\\
&\exp(-\frac{({u_0}^2-\alpha(0))^2}{2(\beta(s)+\beta(t)-2\beta(0))})du_1)dt.
\end{aligned}$$

Using the equality in above of this theorem,

$$m_{\delta_{u_0}}(L(x)\le s)$$
$$= \int_{T-s}^{+\infty} \frac{1}{\pi\sqrt{(\beta(s)-\beta(0))(\beta(t)-\beta(0))^3}}(\sigma^2 \exp(-\frac{v^2}{2\sigma^2}))$$
$$+v\sigma(\int_0^{v/\sigma} \exp(-\frac{u^2}{2})du)\exp(-\frac{(u_0-\alpha(0))^2}{2(\beta(s)+\beta(t)-2\beta(0))})dt.$$

Therefore, we have our equality. □

REMARK 3.2. If $\phi=\delta_0$, α is a zero function and β be an identity function then putting $u=\frac{s}{s+t}$,

$$\begin{aligned} m_\phi(L(x)\le s) &= \int_{T-s}^{+\infty}\frac{\sigma^2}{\pi\sqrt{st^3}} \\ &= \frac{\sqrt{s}}{\pi}\int_{T-s}^{+\infty}\frac{1}{\sqrt{t}(s+t)}dt \\ &= \frac{1}{\pi}\int_0^{s/T}\frac{1}{\sqrt{u(1-u)}}du \\ &= \frac{2}{\pi}\arcsin\sqrt{\frac{s}{T}}. \end{aligned}$$

This is exactly same to the results in the concrete Wiener case.

THEOREM 3.3. *(The third arcsine laws in generalized analogue of Wiener space) Let $0<s<t<T$ with $t+s<T$. Then*

$$m_\phi(\theta_t(x)\le s) = \int_{-\infty}^{+\infty}\int_0^s \frac{1}{\pi\sqrt{(\beta(t)-\beta(r))(\beta(r)-\beta(0))}}$$
$$\exp(-\frac{(b-u_0+\alpha(0))^2}{2(\beta(r)-\beta(0))})drd\phi(u_0).$$

Proof. Let $X_t(x)=x(t+s)-x(s)$ and $N_u(x)=\max_{0\le v\le u}X(v)$ for $0<u\le T$. Then

$$N_{t-s}(x) = \max_{0\le u\le t-s} x(u+s)-x(s) = \max_{s\le u\le t} x(u)-x(s)$$

and the following (a), (b) and (c) are equivalent :
(a) $\theta_t(x)\le s$,
(b) $M_s(x)=M_t(x)$,
(c) $M_s(x)\ge N_{t-s}(x)+x(s)$.
Hence, for u_0 in , $b\ge 0$ and $0<s<t<T$ with $t+s<T$, by Remark 2.2 (3),

$$m_{\delta_{u_0}}(M_t(x)\le b,\theta_t(x)\le s) = m_{\delta_{u_0}}(M_s(x)\le b, M_s(x)-x(s)\ge N_{t-s}(x))$$
$$= m_{\delta_{u_0}}(M_s(x)-u_0\le b-u_0, M_s(x)-x(s)>c, c\ge N_{t-s}(x))\frac{m_{\delta_{u_0}}(c>N_{t-s}(x))}{m_{\delta_{u_0}}(c>N_{t-s}(x))}$$
$$= m_{\delta_{u_0}}(M_s(x)\le b, M_s(x)-x(s)>c)m_{\delta_{u_0}}(c>N_{t-s}(x)).$$

Therefore, from Lemma 2.4 and the Fubini theorem,

$$
\begin{aligned}
& m_{\delta_{u_0}}(M_t(x) \le b, \theta_t(x) \le s) \\
&= \int_0^{+\infty} [\int_0^{+\infty} \{\int_0^{+\infty} \frac{\partial^2}{\partial b \partial h} m_{\delta_{u_0}}(M_s(x) \le b, N_s(x) - x(s) > h) \\
&\quad \frac{\partial}{\partial c} m_{\delta_{u_0}}(c > N_{t-s}(x) dh\} dc] db \\
&= \int_0^{+\infty} [\int_0^{+\infty} \{\int_c^{+\infty} \frac{2(b+h-u_0+\alpha(0))}{\pi\sqrt{(\beta(s)-\beta(0))^3(\beta(t)-\beta(s))}} \\
&\quad \exp(-\frac{(b+h-u_0+\alpha(0))^2}{2(\beta(s)-\beta(0))} - \frac{c^2}{2(\beta(t)-\beta(s))}) dh\} dc] db \\
&= \int_0^{+\infty} [\int_0^{+\infty} \frac{2}{\pi\sqrt{\beta(s)(\beta(t)-\beta(s))}} \\
&\quad \exp(-\frac{(b+c-u_0+\alpha(0))^2}{2(\beta(s)-\beta(0))} - \frac{c^2}{2(\beta(t)-\beta(s))}) dc] db \\
&= (\int_0^{+\infty} \frac{2}{\pi\sqrt{\beta(t)-\beta(0)}} \exp(-\frac{(u-u_0+\alpha(0))^2}{2(\beta(t)-\beta(0))}) du). \\
&\quad (\int_{(b-u_0+\alpha(0))\sqrt{(\beta(t)-\beta(s))[(\beta(s)-\beta(0))(\beta(t)-\beta(0))]}}^{+\infty} \exp(-\frac{u^2}{2}) du).
\end{aligned}
$$

So,

$$
\begin{aligned}
& m_{\delta_{u_0}}(\theta_t(x) \le s) \\
&= \int_0^s (\int_0^{+\infty} \frac{\partial^2}{\partial b \partial r} m_{\delta_{u_0}}(M_t(x) \le b, \theta_t(x) \le r) db) dr \\
&= \int_0^s \frac{1}{\pi\sqrt{(\beta(t)-\beta(r))(\beta(r)-\beta(0))}} \exp(-\frac{(b-u_0+\alpha(0))^2}{2(\beta(r)-\beta(0))}) dr \\
&= \int_0^s \frac{1}{\pi\sqrt{(t-r)r}} \exp(-\frac{{u_0}^2}{2r}) dr.
\end{aligned}
$$

Therefore, we obtain our equality □

REMARK 3.4. If $\phi = \delta_0$, α is a zero function and β is an identity function then

$$
m_\phi(\theta_t(x) \le s) = \int_0^s \frac{1}{\pi\sqrt{r(t-r)}} \exp(-\frac{{u_0}^2}{2r}) dr = \frac{2}{\pi} \arcsin\sqrt{\frac{s}{t}}.
$$

This is exactly same to the results in the concrete Wiener case.

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Department of Mathematics Education
Hannam University
Daejon 306-791, Republic of Korea
E-mail: ksr@hannam.ac.kr