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PERSISTENT ACTIONS ON COMPACT METRIC SPACES

Jiweon Ahn*, Keonhee Lee**, and Seunghee Lee***

Abstract. In this paper, we introduce the notion of persistent actions of finitely generated groups on compact metric spaces and give a necessary condition for a persistent dynamical system to be topologically stable.

1. Introduction

R. Bowen [1] introduced the concept of the pseudo-orbit-tracing-property and essentially showed that expansive homeomorphisms with this property are topologically stable. A. Morimoto [5] has proved that the topological stability implies the pseudo-orbit-tracing-property. K. Yano [6] showed that expansiveness condition is necessary in Bowen's result. Moreover J. Lewowicz [4] introduced the concept of persistence of a dynamical system which is weaker than that of topological stability.

Very recently, N. Chung and K. Lee [3] introduced the notion of topological stability for actions of finitely generated groups on compact metric spaces and proved an expansive action having the pseudo-orbit-tracing property is topologically stable. In this paper, we introduce the notion of persistent actions of finitely generated groups on compact metric spaces and give a necessary condition to be topologically stable.

Let X donote a compact metric space with a metric d. Let $Homeo(X)$ denote the space of all homeomorphisms of X to itself topologied by the C^0-metric

$$d_0(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

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Correspondence should be addressed to Seunghee Lee, shlee@nims.re.kr.
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To explain the main theorem of our paper, we recall some definitions for group actions which are introduced very recently in [3].

Let G be a finitely generated group and $Act(G, X)$ denote the set of all continuous actions of G on X. Let A be a finitely generating set of G. We define a metric d_A on $Act(G, X)$ by

$$d_A(T, S) = \sup\{d(T_a x, S_a x) : x \in X, a \in A\}$$

for every $T, S \in Act(G, X)$.

Definition 1.1. [3] An action $T \in Act(G, X)$ is said to be *A-topologically stable* if for every $\varepsilon > 0$, there is $\delta > 0$ such that if S is another continuous action of G on X with $d_A(T, S) < \delta$ then there exists a continuous map $f : X \to X$ with

- $T_g f = f S_g$ for every $g \in G$, and
- $d(f, 1_X) \leq \varepsilon$,

where 1_X is the identity map on X. The map f is called the *semiconjugacy* from S to T with respect to A.

We can see that topological stability does not depend on the choice of a symmetric finitely generating set. And we say that T is *topologically stable* if it is A-topologically stable for some symmetric finitely generating set A of G.

We say that an action $T \in Act(G, X)$ is *expansive* if there exists a constant $\eta > 0$ such that for every $x \neq y$, we have

$$\sup_{g \in G} d(T_g x, T_g y) > \eta.$$

Such number $\eta > 0$ is called *an expansive constant* of T.

Now we will introduce the definition of persistent actions of finitely generated groups by using symmetric finitely generating sets of the acting groups.

Definition 1.2. Let A be a symmetric finitely generating set of G, and let $T \in Act(G, X)$. We say that T is *α-persistent(or β-persistent)* with respect to A if for every $\varepsilon > 0$, there is $\delta > 0$ such that if $d_A(T, S) < \delta$ and $x \in X$, then there exists $y \in X$ satisfying

$$d(T_g y, S_g x) < \varepsilon \quad (\text{or} \quad d(T_g x, S_g y) < \varepsilon)$$

for all $g \in G$.

We can check that a persistent action does not depend on the choice of a symmetric finitely generating set A of G as following lemma.

LEMMA 1.3. *Let $T \in Act(G, X)$. Let A and B be two symmetric finitely generating sets of G. If T is α-persistent with respect to A, then it is also α-persistent with respect to B.*

Proof. Let d be a compatible metric on X. By the assumption, T is α-persistent with respect to A, for every $\varepsilon > 0$ there is $\delta > 0$ such that if $d_A(T, S) < \delta$ and $x \in X$, then there exists $y \in X$ satisfying $d(T_g y, S_g x) < \varepsilon$ for all $g \in G$. To prove this lemma, it is enough to claim that there is $\delta' > 0$ such that every $S \in Act(G, X)$ with $d_B(T, S) < \delta'$ satisfies $d_A(T, S) < \delta$.

Put $k := \max_{a \in A} l_B(a)$, where l_B is the word length metric on G induced by B. Choose $\delta_1 > 0$ such that $k\delta_1 < \delta$. Since X is compact, A and B are finite and T is a continuous action, there exists $\delta' > 0$ such that $d(T_{g'} x, T_{g'} y) < \delta_1$ for any x and y with $d(x, y) < \delta'$ and $g' \in G$ with $l_B(g') \leq k$. For every $a \in A$, we write a as $b_1 \cdots b_{l(a)}$, where $l(a) = l_B(a) \leq k, b_i \in B$, and $i = 1, \cdots, l(a)$. Then for every $S \in Act(G, X)$ with $d_B(T, S) < \delta'$, we get the following conclusion;

$$\begin{aligned}
d_A(T, S) &= d(T_a x, S_a x) \\
&= d(T_{b_1 \cdots b_{l(a)}} x, S_{b_1 \cdots b_{l(a)}} x) \\
&\leq d(T_{b_1 \cdots b_{l(a)-1}} T_{b_{l(a)}} x, T_{b_1 \cdots b_{l(a)-1}} S_{b_{l(a)}} x) \\
&+ d(T_{b_1 \cdots b_{l(a)-2}} T_{b_{l(a)-1}} S_{b_{l(a)}} x, T_{b_1 \cdots b_{l(a)-1}} S_{b_{l(a)}-1} S_{b_{l(a)}} x) \\
&+ \cdots + d(T_{b_1} T_{b_2} S_{b_3 \cdots b_{l(a)}} x, T_{b_1} S_{b_2} S_{b_3 \cdots b_{l(a)-1}} x) \\
&+ d(T_{b_1} S_{b_2 \cdots b_{l(a)}} x, S_{b_1 \cdots b_{l(a)}} x) \\
&\leq k\delta' < \delta.
\end{aligned}$$

□

We say that T is *α-persistent* if it is α-persistent with respect to some symmetric finitely generating set A of G. Throughout this paper, a persistent action means both α and β-persistent.

LEMMA 1.4. *A topologically stable dynamical system is persistent for group action.*

Proof. It is straightforward. □

The following is our main theorem which gives a necessary condition for a persistent action to be topologically stable.

Theorem A. A persistent action is topologically stable if it is expansive.

2. Proof of Theorem A

Let G and X be as before. Let $T, S \in Act(G, X)$. If $d_A(T, S) < \delta$, then each S-orbit $\{S_g x\}$ of $x \in X$ is nearly a T-orbit in the sense that $d(T_a S_g x, S_{ag} x) < \delta$ for every $a \in A, g \in G$. To prove Theorem A, we need the following lemmas.

LEMMA 2.1. *Assume that an expansive action T of a finitely generated group G on a compact metric space (X, d) is α-persistent with respect to some finitely generating set A of G. Let $\varepsilon < \eta/2$ and δ correponds to ε as in Definition 1.2, where η is an expansive constant of the action T. Then for every S-orbit $\{S_g x\}_{g \in G}$ of $x \in X$ with $d(T, S) < \delta$ there exists a unique point in X satisfying α-persistentness.*

Proof. Let $\{S_g x\}_{g \in G}$ be a S-orbit of x with $d(T, S) < \delta$ and let y and z be two points which is $\{T_g y\}_{g \in G}$ and $\{T_g z\}_{g \in G}$ are two orbits such that $d(T_g y, S_g x) < \varepsilon$ and $d(T_g z, S_g x) < \varepsilon$ for all $g \in G$. Then we can certify

$$d(T_g y, T_g z) \leq d(T_g y, S_g x) + d(S_g x, T_g z) < 2\varepsilon < \eta$$

for every $g \in G$. This fact means $y = z$, since T is expansive. □

LEMMA 2.2. [3] *Let T be an expansive action of G on a compact metric space (X, d) and let η be an expansive constant of the action. Then for every $\varepsilon > 0$ there exists a non-empty finite subet F of G such that whenever $\sup_{g \in F} d(T_g x, T_g y) \leq \eta$, one has $d(x, y) < \varepsilon$.*

Proof. See Lemma 1.18 in [3]. □

Proof of Theorem A. Let $\eta > 0$ be an expansive constant of T and $\varepsilon < \eta/3$. Let A be a finite generating set of G. Choose δ corresponding to ε as in Definition 1.2. Let S be another continuous action of G on (X, d) with $d_A(T, S) < \delta$. Then by the persistentness, there exists $y \in X$ such that

$$(*) \qquad d(T_g y, S_g x) < \varepsilon \quad \text{and} \quad d(T_g x, S_g y) < \varepsilon$$

for all $g \in G$ and $x \in X$.

Define a map $f : X \to X$ by $f(x) = y$. By Lemma 2.1, f is well-defined. In particular, we have $d(f(x), x) < \varepsilon$ for every $x \in X$, and hence $d(f, Id_X) \leq \varepsilon$.

Now we will prove that f is continuous. Let $\varepsilon_1 > 0$. By Lemma 2.2, there is a non-empty finite subset F of G such that whenever $\sup_{g \in F} d(T_g x, T_g y) \leq \eta$ one has $d(x, y) < \varepsilon_1$. Choose $\delta_1 > 0$ such that for every $x, y \in X$ with $d(x, y) < \delta_1$, one has $d(S_g x, S_g y) < \eta/3$ for any $g \in F$. Then, for every $x, y \in X$ with $d(x, y) < \delta_1$ and for every $g \in F$, we get

$$\begin{aligned} d(T_g f(x), T_g f(y)) &= d(f S_g(x), f S_g(y)) \\ &\leq d(f S_g(x), S_g(x)) + d(S_g(x), S_g(y)) + d(S_g(y), f S_g(y)) \\ &< \varepsilon + \eta/3 + \varepsilon < \eta. \end{aligned}$$

Thus $d(f(x), f(y)) < \varepsilon_1$ for every $x, y \in X$ with $d(x, y) < \delta_1$ and hence f is continuous.

Next we will show that $T_g f(x) = f S_g(x)$ for every $x \in X$ and $g \in G$. Then

$$d(T_{g'} f(S_g x), S_{g'g} x) = d(T_{g'} f(S_g x), S_{g'} S_g x) < \varepsilon$$

for every $g' \in G$. One the other hand, applying $(*)$, we obtain

$$d(T_{g'} T_g f(x), S_{g'g} x) = d(T_{g'g} f(x), S_{g'g} x) < \varepsilon.$$

Then by Lemma 2.1, we get $T_g f(x) = f S_g(x)$. This completes the proof of Theorem A. □

Let $T, S \in \mathrm{Act}(G, X)$. We say that T is topologically conjugate to S if there exists $h \in \mathrm{Homeo}(X)$ satisfying $hS = Th$, and the homeomorphism h is called *a topological conjugacy* between T and S. We can see that a persistence action is invariant under a topological conjugacy.

THEOREM 2.3. *A dynamical system which is topologically conjugate to a persistent action is persistent.*

Proof. Suppose that a persistent action $T \in Act(G, X)$ is topologically conjugate to a dynamical system S. Then we have a topological conjugacy $f \in Homeo(X)$ between T and S. Let $\varepsilon > 0$ be given, and choose $0 < \varepsilon' < \varepsilon$ such that if $d(a, b) < \varepsilon'$ then $d(f^{-1}(a), f^{-1}(b)) < \varepsilon$ for $a, b \in X$.

Since T is persistent, there is $\delta' > 0$ such that if $d_A(T, T') < \delta'$ for some symmetric finitely generating set A of G then for any $x \in X$, there exists $y \in X$ satisfying

$$d(T_g(y), T'_g(x)) < \varepsilon'$$

for all $g \in G$.

We can choose $0 < \delta < \delta'$ such that if $d(a, b) < \delta$, then

$$d(f(a), f(b)) < \delta'.$$

Let $S_0 \in Act(G, X)$ be such that $d_A(S, S_0) < \delta$, and put $T_0 = f \circ S_0 \circ f^{-1}$. Since

$$d(f(S(x)), f(S_0(x))) = d(T(f(x)), T_0(f(x))) < \delta'$$

for any $x \in X$, we have $d_A(T, T_0) < \delta'$. Since T is persistent, there is $f(y) \in X$ such that

$$d(T_g(f(y)), T'_g(f(x))) = d(f(S_g(y)), f(S'_g(x))) < \varepsilon'$$

for all $g \in G$. Therefore we have $d(S_g(y), S'_g(x)) < \varepsilon'$ for all $g \in G$. This completes the proof. □

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Department of Mathematics
Chungnam National University
Daejeon 34134, Republic of Korea
E-mail: jwahn@cnu.ac.kr

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Department of Mathematics
Chungnam National University
Daejeon 34134, Republic of Korea
E-mail: khlee@cnu.ac.kr

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Development of Mathematical Data-Analytics and Its Applications
National Institute for Mathematical Sciences
Daejeon 34047, Republic of Korea
E-mail: shlee@cnu.ac.kr