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ROBUST DUALITY FOR NONSMOOTH MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. In this paper, we consider a nonsmooth multiobjective robust optimization problem with more than two locally Lipschitz objective functions and locally Lipschitz constraint functions in the face of data uncertainty. We prove a nonsmooth sufficient optimality theorem for a weakly robust efficient solution of the problem. We formulate a Wolfe type dual problem for the problem, and establish duality theorems which hold between the problem and its Wolfe type dual problem.

1. Introduction and preliminaries

Let X be a Banach space, and let functions $f_i, g_i : X \to \mathbb{R}, i = 1, \cdots, p, j = 1, \cdots, m$ be given. Consider the following multiobjective optimization problem with inequality constraints:

$$\begin{array}{lll} \text{(MP)} & \text{Minimize} & (f_1(x), \cdots, f_p(x)) \\ & \text{subject to} & g_j(x) \leq 0,\ j = 1, \cdots, m. \end{array}$$

This problem in the face of data uncertainty in the constraints can be written by the multiobjective optimization problem:

$$\begin{array}{lll} \text{(UMP)} & \text{Minimize} & (f_1(x, u_1), \cdots, f_p(x, u_p)) \\ & \text{subject to} & g_j(x, v_j) \leq 0,\ j = 1, \cdots, m, \end{array}$$

where u_i, v_j are uncertain parameters, and $u_i \in U_i$, $v_j \in V_j$ for some sequentially compact topological space U_i and V_j, $i = 1, \cdots, p, j =$

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$1, \cdots, m$ and $f_i : X \times U_i \to \mathbb{R}$, $g_j : X \times V_j \to \mathbb{R}, i = 1, \cdots, p, j = 1, \cdots, m$ are functions.

In this paper, we treat the robust approach for (UMP), which is the worst case approach for (UMP). Now we associate with (UMP) its robust counterpart:

$$\begin{array}{lll} \text{(RMP)} & \text{Minimize} & (\max\limits_{u_1 \in U_1} f_1(x, u_1), \cdots, \max\limits_{u_p \in U_p} f_p(x, u_p)) \\ & \text{subject to} & g_j(x, v_j) \leq 0, \ \forall v_j \in V_j, j = 1, \cdots, m, \end{array}$$

where the uncertain objective functions and constraints are enforced for every possible valued of the parameters within their prescribed uncertainty sets $U_i, i = 1, \cdots, p$ and $V_j, j = 1, \cdots, m$. The problem (RMP) can be understood as the robust case (the worst case) of (UMP). So, optimizing (UMP) with (RMP) can be regarded as the robust approach (worst approach) for (UMP).

Robust optimization, which has emerged as a powerful deterministic approach for studying mathematical programming under uncertainty ([1]-[11]).

Kuroiwa and Lee [8, 9] studied scalarizations and optimality theorems for (RMP) when involved functions are convex.

Recently, Lee and Lee [10] proved nonsmooth optimality theorems for weakly robust efficient solutions and properly robust efficient solutions for (RMP).

In this paper, we prove a nonsmooth sufficient optimality theorem for weakly robust efficient solutions for (RMP). We formulate a Wolfe type dual problem for (RMP), and establish the weak duality theorem and the strong duality theorem which hold between (RMP) and its Wolfe type dual problem. The works in this paper is a continuation of ones in [10].

Let a function $f : X \to \mathbb{R}$ be given. We shall suppose that f is locally Lipschitz, that is, for each $x \in X$, there exist an open neighborhood U and a constant $L > 0$ such that for all y and z in U,

$$|f(y) - f(z)| \leq L\|y - z\|.$$

DEFINITION 1.1. For each $d \in X$, the generalized directional derivative of f at x in the direction d, denoted $f^0(x; d)$, is given by

$$f^0(x; d) = \limsup_{h \to 0, t \to 0+} \frac{f(x + h + td) - f(x + h)}{t}.$$

We also denote the usual one-side directional derivative of f at x by $f'(x;d)$. Thus

$$f'(x;d) = \lim_{t\to 0+} \frac{f(x+td)-f(x)}{t},$$

whenever this limit exists.

In the sequel, X^* denotes the (continuous) dual space of X and $\langle\cdot,\cdot\rangle$ is the duality pairing between X and X^*. The norm of an element ξ of X^*, denoted $\|\xi\|^*$, is given by

$$\|\xi\|^* := \sup\{\langle \xi, d\rangle \mid d \in X,\ \|d\| \leqq 1\}.$$

However, all statements involving a topology on X^* are with respect to the weak* topology, unless otherwise stated.

Definition 1.2. The generalized gradient of f at x, denoted by $\partial f(x)$, is the (nonempty) set of all ξ in X^* satisfying the following condition:

$$f^0(x;d) \geqq \langle \xi, d\rangle \quad \text{for all} \quad d \in X.$$

We summarize some fundamental results in the calculus of generalized gradients:

(1) $\partial f(x)$ is a nonempty, convex, weak* compact subset of X^*.
(2) The function $d \mapsto f^0(x;d)$ is convex.
(3) For every d in X, one has

$$f^0(x;d) = \max\{\langle \xi, d\rangle \mid \xi \in \partial f(x)\}.$$

In 1981, Hanson [4] introduced the invexity of differentiable functions and established the Kuhn-Tucker sufficient optimality, the weak and strong duality for a nonlinear optimization problem involving differentiable invex function.

The invexity conception has been extended to the nonsmooth cases by many authors. In particular, Mishra and Giorgi [12] defined the nonsmooth invexity of Lipschitz functions for the finite dimensional space.

Now we defined the nonsmooth invexity of Lipschitz functions as follows:

Definition 1.3. Let $g : X \to \mathbb{R}$ be a locally Lipschitz function. Then g is invex on X if there exists a function $\eta : X \times X \to X$ such that, $\forall x, y \in X$,

$$g(y) - g(x) \geqq g^0(x; \eta(y,x)).$$

Equivalently, $\forall x, y \in X,\ \forall \xi \in \partial_x g(x)$,

$$g(y) - g(x) \geqq \xi(\eta(y, x)).$$

Let V be a sequentially compact topological space and let $g : X \times V \to \mathbb{R}$ be a given function. Now, we will assume that the following conditions hold:

(C1) $g(x, v)$ is upper semicontinuous in (x, v).

(C2) g is locally Lipschitz in x, uniformly for v in V, that is, for each $x \in X$, there exist and open neighborhood U of x and a constant $L > 0$ such that for all y and z in U, and $v \in V$,

$$|g(y, v) - g(z, v)| \leqq L\|y - z\|.$$

(C3) $g_x^0(x, v; \cdot) = g_x'(x, v; \cdot)$, the derivatives being with respect to x.

(C4) the generalized gradient $\partial_x g(x, v)$ with respect to x is weak* upper semicontinuous in (x, v).

We define a function $\psi : X \to \mathbb{R}$ via

$$\psi(x) := \max\{g(x, v) \mid v \in V\},$$

and we observe that our conditions (C1)-(C2) imply that ψ is defined and finite (with the maximum defining ψ attained) on X.

$$V(x) := \{v \in V \mid g(x, v) = \psi(x)\}.$$

It is easy to see that $V(x)$ is nonempty and closed for each x in X.

The following lemma, which is a nonsmooth version of Danskin's theorem [3] for max-functions, makes connection between the functions $\psi'(x; d)$ and $g^0(x, v; d)$.

Lemma 1.4. [11] *Under the conditions (C1)-(C4), the usual one-sided directional derivative $\psi'(x; d)$ exists, and satisfies*

$$\begin{aligned} \psi'(x; d) = \psi^0(x; d) &= \max\{g_x^0(x, v; d) \mid v \in V(x)\} \\ &= \max\{\langle \xi, d\rangle \mid \xi \in \partial_x g(x, v),\ v \in V(x)\}. \end{aligned}$$

Lemma 1.5. [11] *In addition to the basic conditions (C1)-(C4), suppose that V is convex, and that $g(x, \cdot)$ is concave on V, for each $x \in U$. Then the following statements hold:*

(i) *The set $V(x)$ is convex and sequentially compact.*

(ii) *The set*

$$\partial_x g(x, V(x)) := \{\xi \mid \exists v \in V(x) \text{ such that } \xi \in \partial_x g(x, v)\}$$

is convex and weak* compact.

(iii) $\partial\psi(x) = \{\xi \mid \exists v \in V(x)$ *such that* $\xi \in \partial_x g(x, v)\}$.

2. Sufficient optimality theorems

Let X be a Banach space. Recall the robust counterpart (RMP) of (UMP):

$$\text{(RMP)} \qquad \text{Minimize} \qquad (\max_{u_1 \in U_1} f_1(x, u_1), \cdots, \max_{u_p \in U_p} f_p(x, u_p))$$
$$\text{subject to} \qquad g_j(x, v_j) \leq 0, \ \forall v_j \in V_j, j = 1, \cdots, m.$$

We assume that $f_i : X \times U_i \to \mathbb{R}$, $g_j : X \times V_j \to \mathbb{R}, i = 1, \cdots, p, j = 1, \cdots, m$ are locally Lipschitz functions, and that, $U_i, i = 1, \cdots, p$ and $V_j, j = 1, \cdots, m$ are sequentially compact topological spaces.

We recall the set of robust feasible solutions of (UMP):

$$C := \{x \in X \mid g_j(x, v_j) \leq 0, \ \ \forall v_j \in V_j, \ j = 1, \cdots, m\}.$$

We say that $\bar{x} \in C$ is called a weakly robust efficient solution of (RMP) if there does not exist a robust feasible solution x of (RMP) such that

$$\max_{u_i \in U_i} f_i(x, u_i) < \max_{u_i \in U_i} f_i(\bar{x}, u_i), \ i = 1, \cdots, p.$$

Define $\phi_i(x) := \max_{u_i \in U_i} f_i(x, u_i)$ for each $i = 1, \cdots, p$ and $\psi_j(x) := \max_{v_j \in V_j} g_j(x, v_j)$ for each $j = 1, \cdots, m$. Then if f_i and g_j satisfy the conditions (C1) and (C2), $\phi_i, \psi_j : X \to \mathbb{R}$, $i = 1, \cdots, p$ and $j = 1, \cdots, m$, are locally Lipschitz functions.

Let $x \in C$ and let us decompose $J := \{1, \cdots, m\}$ into two index sets $J = J_1(x) \cup J_2(x)$, where $J_1(x) = \{j \in J \mid \psi_j(x) = 0\}$ and $J_2(x) = J \setminus J_1(x)$. We put for each $i = 1, \cdots, p$,

$$U_i(x) := \{u_i \in U_i \mid f_i(x, u_i) = \phi_i(x)\},$$

and for each $j \in J_1(x)$,

$$V_j(x) := \{v_j \in V_j \mid g_j(x, v_j) = \psi_j(x)\}.$$

Definition 2.1. We define an Extended Nonsmooth Mangasarian-Fromovitz constraint qualification (ENMFCQ) at $x \in C$ as follows:

$$\exists d \in X \text{ such that } g^0_{jx}(x, v_j; d) < 0, \ \ \forall v_j \in V_j(x), \ \ \forall j \in J_1(x),$$

where $g^0_{jx}(x, v_j; d)$ denotes the generalized directional derivative of g_j with respect to x.

Now we present a necessary optimality theorem for a weakly robust efficient solution of (RMP).

THEOREM 2.2. [10] *Assume that* $f_i, i = 1, \cdots, p$ *and* $g_j, j = 1, \cdots, m$ *satisfy the conditions (C1)–(C4). Suppose that for each* $x \in X$, $f_i(x, \cdot), i = 1, \cdots, p$, *are concave on* $U_i, i = 1, \cdots, p$ *and* $g_j(x, \cdot)$ *are concave on* $V_j, j = 1, \cdots, m$. *Let* $x^* \in C$ *be a weakly robust efficient solution of (RMP). Then there exist* $\mu_i \geqq 0, i = 1, \cdots, p, \lambda_j \geqq 0, j = 1, \cdots, m$, *not all zero, and* $u_i^* \in U_i(x^*), i = 1, \cdots, p, v_j^* \in V_j(x^*), j = 1, \cdots, m$ *such that*

$$\begin{aligned} &0 \in \sum_{i=1}^{p} \mu_i \partial_x f_i(x^*, u_i^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*), \\ &\lambda_j g_j(x^*, v_j^*) = 0, \ j = 1, \cdots, m. \end{aligned}$$

Moreover, if we further assume that the Extended Nonsmooth Mangasarian-Fromovitz constraint qualification (ENMFCQ) holds, then there exist $\mu_i \geqq 0, i = 1, \cdots, p$, *not all zero, and* $u_i^* \in U_i(x^*), i = 1, \cdots, p, \lambda_j \geqq 0$ *and* $v_j^* \in V_j(x^*), j = 1, \cdots, m$ *such that*

$$\begin{aligned} &0 \in \sum_{i=1}^{p} \mu_i \partial_x f_i(x^*, u_i^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*), \\ &\lambda_j g_j(x^*, v_j^*) = 0, \ j = 1, \cdots, m. \end{aligned}$$

Now we give a sufficient optimality theorem for weakly robust efficient solutions for (RMP):

THEOREM 2.3. *Let* x^* *be a robust feaible solution of (UMP). Suppose that there exist* $\mu_i \geqq 0, i = 1, \cdots, p$, *not all zero, and* $u_i^* \in U_i(x^*), i = 1, \cdots, p, \lambda_j \geqq 0$ *and* $v_j^* \in V_j(x^*), j = 1, \cdots, m$ *such that*

$$\begin{aligned} &0 \in \sum_{i=1}^{p} \mu_i \partial_x f_i(x^*, u_i^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*), \\ &\lambda_j g_j(x^*, v_j^*) = 0, \ j = 1, \cdots, m. \end{aligned}$$

If each $f_i(\cdot, u_i^*), i = 1, \cdots, p$ *and* $g_j(\cdot, v_j^*), j = 1, \cdots, m$ *are invex at* x^* *with respect to the same* η, *then* x^* *is a weakly robust efficient solution of (RMP).*

Proof. Let x^* be a robust feasible solution of (RMP). Suppose that there exist $\mu_i \geqq 0, i = 1, \cdots, p$, not all zero, and $u_i^* \in U_i(x^*), i = 1, \cdots, p, \lambda_j \geqq 0$ and $v_j^* \in V_j(x^*), j = 1, \cdots, m$ such that

$$0 \in \sum_{i=1}^{p} \mu_i \partial_x f_i(x^*, u_i^*) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(x^*, v_j^*).$$

Then there exists $\xi_i^* \in \partial_x f_i(x^*, u_i^*),\ i = 1, \cdots, p$ and $\zeta_j^* \in \partial_x g_j(x^*, v_j^*),\ j = 1, \cdots, m$ such that

$$\sum_{i=1}^{p} \mu_i \xi_i^*(\eta(x, x^*)) + \sum_{j=1}^{m} \lambda_j \zeta_j^*(\eta(x, x^*)) = 0. \tag{2.1}$$

Suppose that x^* is not a weakly robust efficient solution of (RMP). Then there exist a feasible solution x of (RMP) such that

$$\max_{u_i \in U_i} f_i(x, u_i) < \max_{u_i \in U_i} f_i(x^*, u_i),\ i = 1, \cdots, p.$$

Then there exist $u_i^* \in U_i(x^*),\ i = 1, \cdots, p$, such that

$$f_i(x, u_i^*) < f_i(x^*, u_i^*).$$

By the invexity of $f_i(\cdot, u_i^*)$ at x^*, $\xi_i^*(\eta(x, x^*)) < 0$. Since $\mu_i \geqq 0, i = 1, \cdots, p$, not all zero, $\sum_{i=1}^{p} \mu_i \xi_i^*(\eta(x, x^*)) < 0$.
From (2.1), $\sum_{j=1}^{m} \lambda_j \zeta_j^*(\eta(x, x^*)) > 0$. By the invexity of $g_j(\cdot, v_j^*)$ at x^*,

$$\lambda_j g_j(x, v_j^*) - \lambda_j g_j(x^*, v_j^*) > 0,\ j = 1, \cdots, m.$$

Hence $\lambda_j g_j(x, v_j^*) > \lambda_j g_j(x^*, v_j^*),\ j = 1, \cdots, m.$ Since $\lambda_j g_j(x^*, v_j^*) = 0$, $\lambda_j g_j(x, v_j^*) > 0,\ j = 1, \cdots, m$, which contradicts the fact that $\lambda_j g_j(x, v_j^*) \leqq 0,\ j = 1, \cdots, m$. Therefore, x^* is a weakly robust efficient solution of (RMP). $\square$

3. Duality theorems

In this section, we establish Wolfe type robust duality theorems which hold between (RMP) and the following Wolfe type dual problem (WD) for (RMP).

(WD) Maximize

$$\left(f_1(y, u_1) + \sum_{j=1}^{m} \lambda_j g_j(y, v_j), \cdots, f_p(y, u_p) + \sum_{j=1}^{m} \lambda_j g_j(y, v_j)\right)$$

subject to

$$0 \in \sum_{i=1}^{p} \mu_i \partial_x f_i(y, u_i) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(y, v_j),$$

$$\mu_i \geqq 0,\ \sum_{i=1}^{p} \mu_i = 1,\ \lambda_j \geqq 0,$$

$$u_i \in U_i,\ v_j \in V_j,\ i = 1, \cdots, p, j = 1, \cdots, m.$$

THEOREM 3.1. *(Weak Duality) Let x be robust feasible solution for (RMP) and (y, μ, λ) be feasible for (WD). If each $f_i(\cdot, u_i), i = 1, \cdots, p$*

and $g_j(\cdot, v_j), j = 1, \cdots, m$ *are invex at* y *with respect to the same* η*, then the following does not hold: for all* $i \in \{1, \cdots, p\}$,

$$\max_{u_i \in U_i} f_i(x, u_i) < f_i(y, u_1) + \sum_{j=1}^{m} \lambda_j g_j(y, v_j).$$

Proof. Let x be any feasible for (RMP) and (y, μ, λ) be any feasible for (WD). Then there exist $\mu_i \geqq 0, i = 1, \cdots, p$, not all zero, and $u_i \in U_i, i = 1, \cdots, p$, $\lambda_j \geqq 0$ and $v_j \in V_j, j = 1, \cdots, m$ such that

$$0 \in \sum_{i=1}^{p} \mu_i \partial_x f_i(y, u_i) + \sum_{j=1}^{m} \lambda_j \partial_x g_j(y, v_j).$$

Then there exists $\xi_i^* \in \partial_x f_i(y, u_i),\ i = 1, \cdots, p$ and $\zeta_j^* \in \partial_x g_j(y, v_j),\ j = 1, \cdots, m$ such that

$$\sum_{i=1}^{p} \mu_i \xi_i^*(\eta(x, y)) + \sum_{j=1}^{m} \lambda_j \zeta_j^*(\eta(x, y)) = 0. \tag{3.1}$$

Now suppose that

$$(\max_{u_1 \in U_1} f_1(x, u_1), \cdots, \max_{u_p \in U_p} f_p(x, u_p))$$
$$< (f_1(y, u_1) + \sum_{j=1}^{m} \lambda_j g_j(y, v_j), \cdots, f_p(y, u_p) + \sum_{j=1}^{m} \lambda_j g_j(y, v_j)).$$

Then

$$\max_{u_i \in U_i} f_i(x, u_i) < f_i(y, u_i) + \sum_{j=1}^{m} \lambda_j g_j(y, v_j),\ i = 1, \cdots, p.$$

Since

$$f_i(x, u_i) < f_i(y, u_i) + \sum_{j=1}^{m} \lambda_j g_j(y, v_j),\ i = 1, \cdots, p.$$

Since $g_j(x, v_j) \leqq 0,\ \lambda_j \geqq 0,\ j = 1, \cdots, m,\ \lambda_j g_j(x, v_j) \leqq 0$,

$$f_i(x, u_i) + \sum_{j=1}^{m} \lambda_j g_j(x, v_j) < f_i(y, u_i) + \sum_{j=1}^{m} \lambda_j g_j(y, v_j),\ i = 1, \cdots, p.$$

Since $f_i(\cdot, u_i),\ i = 1, \cdots, p$ and $g_j(\cdot, v_j).\ j = 1, \cdots, m$ are invex,

$$\xi_i^*(\eta(x, y)) + \sum_{j=1}^{m} \lambda_j \zeta_j^*(\eta(x, y)) < 0.$$

Since $\mu_i \geqq 0$ and $\sum_{i=1}^{p} \mu_i = 1$,

$$\Big[\sum_{i=1}^{p} \mu_i \xi_i^* + \sum_{j=1}^{m} \lambda_j \zeta_j^*\Big](\eta(x,y)) < 0.$$

This contradicts the inclusion (3.1). $\square$

THEOREM 3.2. *(Strong Duality) Let $\bar{x}$ be a solution of (RMP). Assume that the Extended Mangasarian-Fromovitz constraint qualification holds. Then, there exist $(\bar{u}, \bar{v}, \bar{\mu}, \bar{\lambda})$ such that $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu}, \bar{\lambda})$ is feasible for (WD) and the objective values of (RMP) and (WD) are equal. Moreover, if the weak duality holds, then $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu}, \bar{\lambda})$ is a solution of (WD).*

Proof. Since $\bar{x}$ is a robust solution of (RMP) at which the Extended Mangasarian-Fromovitz constraint qualification holds, then by Theorem 2.1, there exist $\bar{\mu}_i \geqq 0, i = 1, \cdots, p, \bar{\lambda}_j \geqq 0, j = 1, \cdots, m$, not all zero, and $\bar{u}_i \in U_i(\bar{x}), i = 1, \cdots, p, \bar{v}_j \in V_j(\bar{x}), j = 1, \cdots, m$ such that

$$0 \in \sum_{i=1}^{p} \bar{\mu}_i \partial_x f_i(\bar{x}, \bar{u}_i) + \sum_{j=1}^{m} \bar{\lambda}_j \partial_x g_j(\bar{x}, \bar{v}_j),$$

$$\bar{\lambda}_j g_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, \cdots, m.$$

Thus $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu}, \bar{\lambda})$ is feasible for (WD) and the objective values of (RMP) and (WD) are equal. Moreover, $\max_{\bar{u}_i \in U_i} f_i(\bar{x}, u_i) = f_i(\bar{x}, \bar{u}_i) = f_i(\bar{x}, \bar{u}_i) + \sum_{j=1}^{j} \bar{\lambda}_j g_j(\bar{x}, \bar{v}_j)$. It follows from weak duality (Theorem 3.1) that for any feasible solution $(\widetilde{x}, \widetilde{u}, \widetilde{v}, \widetilde{\mu}, \widetilde{\lambda})$ for (WD), the following does not hold: for all $i \in \{1, \cdots, p\}$,

$$f_i(\bar{x}, \bar{u}_i) + \sum_{j=1}^{j} \bar{\lambda}_j g_j(\bar{x}, \bar{v}_j) < f_i(\widetilde{x}, \widetilde{u}_i) + \sum_{j=1}^{j} \widetilde{\lambda}_j g_j(\widetilde{x}, \widetilde{v}_j).$$

Hence $(\bar{x}, \bar{u}, \bar{v}, \bar{\mu}, \bar{\lambda})$ is a solution of (WD). $\square$

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