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UNIFORMLY LIPSCHITZ STABILITY OF PERTURBED NONLINEAR DIFFERENTIAL SYSTEMS

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Abstract. In this paper, we study that the solutions to perturbed differential system

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s))ds + h(t, y(t), T_2 y(t))$$

have uniformly Lipschitz stability by imposing conditions on the perturbed part $\int_{t_0}^{t} g(s, y(s), T_1 y(s))ds, h(t, y(t), T_2 y(t))$, and on the fundamental matrix of the unperturbed system $y' = f(t, y)$ using integral inequalities.

1. Introduction and Preliminaries

It is well known that one of the important techniques for investigating the stability properties of solutions of nonlinear differential systems[7-9, 13-15] is through the use of the corresponding linear variational systems. Dannan and Elaydi introduced a notion of uniformly Lipschitz stability (ULS) [9]. This notion of ULS lies somewhere between uniformly stability on one side and the notions of asymptotic stability in variation of Brauer [4] and uniformly stability in variation of Brauer and Strauss [3] on the other side. An important feature of ULS is that for linear systems, the notion of uniformly Lipschitz stability and that of uniformly stability are equivalent, but for nonlinear systems, the two notions are quite distinct. Pachpatte [14, 15] studied the stability and asymptotic behavior of the solutions of perturbed nonlinear systems under some suitable conditions on the perturbation term. Choi *et al.* [7, 8] examined Lipschitz and exponential asymptotic stability for nonlinear functional systems.

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Also, Goo [10, 11] and Goo *et al.* [5, 6, 12] investigated Lipschitz and asymptotic stability for perturbed differential systems.

In the current paper, we study ULS for solutions of perturbed nonlinear systems using integral inequalities.

We consider the unperturbed nonlinear system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We suppose that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed functional differential system of (1.1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s), T_1y(s))ds + h(t, y(t), T_2y(t)),\ y(t_0) = y_0, \tag{1.2}$$

where $g, h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0, 0) = h(t, 0, 0) = 0$, and $T_1, T_2 : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ are continuous operators. The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^n$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x| \le 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t),\ v(t_0) = v_0 \tag{1.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t),\ z(t_0) = z_0. \tag{1.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.3).

This connection between the stability of the zoro solution of (1.1) and the zero solutions of (1.3) and (1.4) has been extensively studied in [2-4, 6, 7, 14, 15].

The following definition is due to Dannan and Elaydi [9].

Definition 1.1. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called

(S) *stable* if for any $\epsilon > 0$ and $t_0 \ge 0$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \epsilon$ for all $t \ge t_0 \ge 0$,

(US) *uniformly stable* if the δ in (S) is independent of the time t_0,

(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \le M|x_0|$ whenever $|x_0| \le \delta$ and $t \ge t_0 \ge 0$,
(ULSV) *uniformly Lipschitz stable in variation* if there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, x_0)| \le M$ for $|x_0| \le \delta$ and $t \ge t_0 \ge 0$.

Let us recall some results that will be used throughout this work.

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ y(t_0) = y_0, \tag{1.5}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.5) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 1.2. [2] *Let x and y be solutions of (1.1) and (1.5), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \ge t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Lemma 1.3. *(Bihari-type inequality) Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \le c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds,\ t \ge t_0 \ge 0.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s) ds\Big],$$

where $t_0 \le t < b_1$, $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$, *and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 1.4. [6] *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10} \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$ and $0 \le t_0 \le t$,*

$$\begin{aligned} u(t) \le & c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds + \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \Big(\lambda_4(\tau)u(\tau) \\ & + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)u(r)dr + \lambda_7(\tau) \int_{t_0}^{\tau} \lambda_8(r)w(u(r))dr\Big) d\tau ds \\ & + \int_{t_0}^{t} \lambda_9(s) \int_{t_0}^{s} \lambda_{10}(\tau)w(u(\tau))d\tau ds. \end{aligned}$$

Then

$$\begin{aligned} u(t) \le & W^{-1}\Big[W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} (\lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)dr \\ & + \lambda_7(\tau) \int_{t_0}^{\tau} \lambda_8(r)dr)d\tau + \lambda_9(s) \int_{t_0}^{s} \lambda_{10}(\tau)d\tau\Big) ds\Big], \end{aligned}$$

where $t_0 \le t < b_1$, W, W^{-1} are the same functions as in Lemma 1.3, and

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : & W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} (\lambda_4(\tau) + \\ & \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)dr + \lambda_7(\tau) \int_{t_0}^{\tau} \lambda_8(r)dr)d\tau + \\ & \lambda_9(s) \int_{t_0}^{s} \lambda_{10}(\tau)d\tau\Big) ds \in \mathrm{domW}^{-1}\Big\}. \end{aligned}$$

For the proof we need the following two corollaries.

Corollary 1.5. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$ and $0 \le t_0 \le t$,*

$$\begin{aligned} u(t) \le c & + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \Big(\lambda_3(\tau)u(\tau) \\ & + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)u(r)dr + \lambda_6(\tau) \int_{t_0}^{\tau} \lambda_7(r)w(u(r))dr\Big) d\tau ds \\ & + \int_{t_0}^{t} \lambda_8(s) \int_{t_0}^{s} \lambda_9(\tau)w(u(\tau))d\tau ds. \end{aligned}$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr + \lambda_6(\tau) \int_{t_0}^{\tau} \lambda_7(r)dr)d\tau + \lambda_8(s) \int_{t_0}^{s} \lambda_9(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.3, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr + \lambda_6(\tau) \int_{t_0}^{\tau} \lambda_7(r)dr)d\tau + \lambda_8(s) \int_{t_0}^{s} \lambda_9(\tau)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.$$

Corollary 1.6. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds + \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)u(\tau)d\tau ds + \int_{t_0}^{t} \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)w(u(\tau))d\tau ds, \quad 0 \le t_0 \le t.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.3, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.$$

2. Main Results

In this section, we investigate ULS for solutions of the perturbed functional differential systems.

To obtain ULS, the following assumptions are needed:

(H1) The solution $x = 0$ of (1.1) is ULSV.

(H2) $w(u)$ be nondecreasing in u such that $u \leq w(u)$ and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$.

THEOREM 2.1. *Suppose that (H1), (H2), and that the perturbing term g in (1.2) satisfies*

$$|g(t, y, T_1 y)| \leq a(t)|y(t)| + b(t)w(|y(t)|) + |T_1 y(t)|, \tag{2.1}$$

$$|T_1 y(t)| \leq b(t) \int_{t_0}^{t} k(s)|y(s)|ds + m(t) \int_{t_0}^{t} p(s)w(|y(s)|)ds, \tag{2.2}$$

$$|h(t, y(t), T_2 y(t))| \leq \int_{t_0}^{t} c(s)|y(s)|ds + |T_2 y(t)|, \tag{2.3}$$

and

$$|T_2 y(t)| \leq d(t)|y(t)| + n(t)w(|y(t)|), \tag{2.4}$$

where $a, b, c, d, k, m, n, p \in C(\mathbb{R}^+)$, $a, b, c, d, k, m, n, p \in L^1(\mathbb{R}^+)$, $w \in C((0, \infty))$, T_1, T_2 *are continuous operators, and*

$$\begin{aligned} M(t_0) =& W^{-1}\Big[W(M) + M \int_{t_0}^{\infty} \Big(d(s) + n(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) + c(\tau) \\ &+ b(\tau) \int_{t_0}^{\tau} k(r)dr + m(\tau) \int_{t_0}^{\tau} p(r)dr)d\tau\Big)ds\Big], \end{aligned} \tag{2.5}$$

where $M(t_0) < \infty$, $b_1 = \infty$, $t_0 \leq t < b_1$, *and* W, W^{-1} *are the same functions as in Lemma 1.3. Then the zero solution of (1.2) is ULS.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the assumption (H1), it is ULS([9], Theorem 3.3). Applying the nonlinear variation of constants formula due to Lemma 1.2, together with (H2), (2.1), (2.2), (2.3), and (2.4), we obtain

$$
\begin{aligned}
|y(t)| &\le |x(t)| \\
&\quad + \int_{t_0}^{t} |\Phi(t,s,y(s))| \Big(\int_{t_0}^{s} |g(\tau, y(\tau), T_1 y(s))| d\tau + |h(s, y(s), T_2 y(s))| \Big) ds \\
&\le M|y_0| + \int_{t_0}^{t} M|y_0| \Big(\int_{t_0}^{s} ((a(\tau)+c(\tau)) \frac{|y(\tau)|}{|y_0|} + b(\tau) w(\frac{|y(\tau)|}{|y_0|}) \\
&\quad + b(\tau) \int_{t_0}^{\tau} k(r) \frac{|y(r)|}{|y_0|} dr + m(\tau) \int_{t_0}^{\tau} p(r) w(\frac{|y(r)|}{|y_0|}) dr) d\tau \\
&\quad + d(s) \frac{|y(s)|}{|y_0|} + n(s) w(\frac{|y(s)|}{|y_0|}) \Big) ds.
\end{aligned}
$$

Set $u(t) = |y(t)||y_0|^{-1}$. Then, an application of Lemma 1.4 yields

$$
\begin{aligned}
|y(t)| \le |y_0| W^{-1} \Big[W(M) + M \int_{t_0}^{t} \Big(d(s) + n(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) \\
+ c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r) dr + m(\tau) \int_{t_0}^{\tau} p(r) dr) d\tau \Big) ds \Big],
\end{aligned}
$$

Thus, by (2.5), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. Hence the proof is complete. □

REMARK 2.2. Letting $c(t) = d(t) = k(t) = n(t) = 0$ in Theorem 2.1, we obtain the same result as that of Theorem 3.1 in [5].

THEOREM 2.3. *Suppose that (H1), (H2), and that the perturbing term g in (1.2) satisfies*

$$
\int_{t_0}^{t} |g(s, y(s), T_1 y(s))| ds \le a(t)|y(t)| + b(t) w(|y(t)|) + |T_1 y(t)|, \tag{2.6}
$$

$$
|T_1 y(t)| \le b(t) \int_{t_0}^{t} k(s)|y(s)| ds + m(t) \int_{t_0}^{t} p(s) w(|y(s)|) ds, \tag{2.7}
$$

$$
|h(t, y(t), T_2 y(t))| \le m(t) \int_{t_0}^{t} c(s) w(|y(s)|) ds + |T_2 y(t)|, \tag{2.8}
$$

and

$$
|T_2 y(t)| \le b(t) \int_{t_0}^{t} q(s)|y(s)| ds + d(t)|y(t)|, \tag{2.9}
$$

where $a, b, c, d, k, m, p, q \in C(\mathbb{R}^+)$, $a, b, c, d, k, m, p, q \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 are continuous operators, and

(2.10)

$$\begin{aligned} M(t_0) =& W^{-1}\Big[W(M) + M\int_{t_0}^{\infty}\Big(a(s)+b(s)+d(s) \\ &+ b(s)\int_{t_0}^{s}(k(\tau)+q(\tau))d\tau + m(s)\int_{t_0}^{s}(c(\tau)+p(\tau))d\tau\Big)ds\Big], \end{aligned}$$

where $M(t_0) < \infty$ and $b_1 = \infty$, $t_0 \le t < b_1$, and W, W^{-1} are the same functions as in Lemma 1.3. Then the zero solution of (1.2) is ULS.

Proof. Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (1.1) and (1.2), respectively. By the assumption (H1), it is ULS. Using Lemma 1.2, together with (H2), (2.6), (2.7), (2.8), and (2.9), we have

$$\begin{aligned} |y(t)| \le M|y_0| + \int_{t_0}^{t} M|y_0|\Big((a(s)+d(s))\frac{|y(s)|}{|y_0|} + b(s)w(\frac{|y(s)|}{|y_0|}) \\ + b(s)\int_{t_0}^{s}(k(\tau)+q(\tau))\frac{|y(\tau)|}{|y_0|}d\tau \\ + m(s)\int_{t_0}^{s}(c(\tau)+p(\tau))w(\frac{|y(\tau)|}{|y_0|})d\tau\Big)ds. \end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Then, an application of Corollary 1.6 yields

$$\begin{aligned} |y(t)| \le |y_0|W^{-1}\Big[W(M) + M\int_{t_0}^{t}\Big(a(s)+b(s)+d(s) \\ +b(s)\int_{t_0}^{s}(k(\tau)+q(\tau))d\tau + m(s)\int_{t_0}^{s}(c(\tau)+p(\tau))d\tau\Big)ds\Big]. \end{aligned}$$

Hence, by (2.10), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$, and so the proof is complete. □

Remark 2.4. Letting $c(t) = d(t) = k(t) = q(t) = 0$ in Theorem 2.3, we obtain the same result as that of Theorem 3.3 in [5].

Theorem 2.5. *Suppose that (H1), (H2), and that the perturbing term g in (1.2) satisfies*

$$|g(t,y,T_1y)| \le a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|, \tag{2.11}$$

$$|T_1y(t)| \le b(t)\int_{t_0}^{t} k(s)|y(s)|ds + m(t)\int_{t_0}^{t} p(s)w(|y(s)|)ds, \tag{2.12}$$

$$|h(t,y(t),T_2y(t))| \le \int_{t_0}^{t} c(s)w(|y(s)|)ds + |T_2y(t)|, \tag{2.13}$$

and

$$|T_2y(t)| \le \int_{t_0}^{t} d(s)|y(s)|ds + n(t)|y(t)|, \tag{2.14}$$

where $a, b, c, d, k, m, n, p \in C(\mathbb{R}^+)$, $a, b, c, d, k, m, n, p \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 *are continuous operators, and*
(2.15)

$$\begin{aligned} M(t_0) =& W^{-1}\Big[W(M) + M\int_{t_0}^{\infty}\Big(n(s) + \int_{t_0}^{s}(a(\tau) + b(\tau) + c(\tau) + d(\tau) \\ &+ b(\tau)\int_{t_0}^{\tau} k(r)dr + m(\tau)\int_{t_0}^{\tau} p(r)dr)d\tau\Big)ds\Big], \end{aligned}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$, $t_0 \le t < b_1$, *and* W, W^{-1} *are the same functions as in Lemma 1.3. Then the zero solution of (1.2) is ULS.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the assumption (H1), it is ULS. Applying the nonlinear variation of constants formula due to Lemma 1.2, together with (H2), (2.11), (2.12), (2.13), and (2.14), we have

$$\begin{aligned} |y(t)| \le M|y_0| + \int_{t_0}^{t} M|y_0|\Big(&\int_{t_0}^{s}((a(\tau) + d(\tau))\frac{|y(\tau)|}{|y_0|} + (b(\tau) + \\ &c(\tau)w(\frac{|y(\tau)|}{|y_0|}) + b(\tau)\int_{t_0}^{\tau} k(r)\frac{|y(r)|}{|y_0|}dr + \\ &m(\tau)\int_{t_0}^{\tau} p(r)w(\frac{|y(r)|}{|y_0|})dr)d\tau + n(s)\frac{|y(s)|}{|y_0|}\Big)ds. \end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Then, an application of Corollary 1.5 yields

$$\begin{aligned} |y(t)| \le |y_0|W^{-1}\Big[&W(M) + M\int_{t_0}^{t}\Big(n(s) + \int_{t_0}^{s}(a(\tau) + b(\tau) + c(\tau) \\ &+ d(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr + m(\tau)\int_{t_0}^{\tau} p(r)dr)d\tau\Big)ds\Big], \end{aligned}$$

Thus, by (2.15), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. This completes the proof. □

Remark 2.6. Letting $c(t) = d(t) = k(t) = n(t) = 0$ in Theorem 2.5, we obtain the same result as that of Theorem 3.1 in [5].

Theorem 2.7. *Suppose that (H1), (H2), and that the perturbing term* g *in (1.2) satisfies*

$$\int_{t_0}^{t} |g(s, y(s), T_1y(s))|ds \le a(t)|y(t)| + b(t)w(|y(t)|) + |T_1y(t)|, \tag{2.16}$$

$$|T_1y(t)| \le b(t)\int_{t_0}^t k(s)|y(s)|ds + m(t)\int_{t_0}^t p(s)w(|y(s)|)ds, \tag{2.17}$$

$$|h(t,y(t),T_2y(t))| \le m(t)\int_{t_0}^t c(s)w(|y(s)|)ds + |T_2y(t)|, \tag{2.18}$$

and

$$|T_2y(t)| \le b(t)\int_{t_0}^t q(s)|y(s)|ds + d(t)w(|y(t)|), \tag{2.19}$$

where $a, b, c, d, k, m, p, q \in C(\mathbb{R}^+)$, $a, b, c, d, k, m, p, q \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T_1, T_2 *are continuous operators, and*
(2.20)

$$\begin{aligned} M(t_0) =& W^{-1}\Big[W(M) + M\int_{t_0}^\infty \Big(a(s)+b(s)+d(s) \\ &+ b(s)\int_{t_0}^s (k(\tau)+q(\tau))d\tau + m(s)\int_{t_0}^s (c(\tau)+p(\tau))d\tau\Big)ds\Big], \end{aligned}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$, $t_0 \le t < b_1$, *and* W, W^{-1} *are the same functions as in Lemma 1.3. Then the zero solution of (1.2) is ULS.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the assumption (H1), it is ULS. Using Lemma 1.2, together with (H2), (2.16), (2.17), (2.18), and (2.19), we have

$$\begin{aligned} |y(t)| \le M|y_0| + \int_{t_0}^t & M|y_0|\Big(a(s)\frac{|y(s)|}{|y_0|} + (b(s)+d(s))w(\frac{|y(s)|}{|y_0|}) \\ &+ b(s)\int_{t_0}^s (k(\tau)+q(\tau))\frac{|y(\tau)|}{|y_0|}d\tau \\ &+ m(s)\int_{t_0}^s (c(\tau)+p(\tau))w(\frac{|y(\tau)|}{|y_0|})d\tau\Big)ds. \end{aligned}$$

Set $u(t) = |y(t)||y_0|^{-1}$. Then, an application of Corollary 1.6 yields

$$\begin{aligned} |y(t)| \le |y_0| W^{-1}\Big[& W(M) + M\int_{t_0}^t \Big(a(s)+b(s)+d(s) \\ &+ b(s)\int_{t_0}^s (k(\tau)+q(\tau))d\tau + m(s)\int_{t_0}^s (c(\tau)+p(\tau))d\tau\Big)ds\Big]. \end{aligned}$$

Hence, by (2.20), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. Thus the theorem is proved. □

REMARK 2.8. Letting $c(t) = d(t) = k(t) = q(t) = 0$ in Theorem 2.7, we obtain the same result as that of Theorem 3.3 in [5].

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