A NOTE ON GENERALIZED $\ast$-DERIVATIONS OF PRIME $\ast$-RINGS

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Abstract. The aim of the present paper is to establish some results involving generalized $\ast$-derivations in $\ast$-rings and investigate the commutativity of prime $\ast$-rings admitting generalized $\ast$-derivations of $R$ satisfying certain identities and some related results have also been discussed.

1. Introduction

Over the last few decades, several authors have investigated the relationship between the commutativity of the ring $R$ and certain specific types of derivations of $R$. The first result in this direction is due to E. C. Posner [8] who proved that if a ring $R$ admits a nonzero derivation $d$ such that $[d(x), x] \in Z(R)$ for all $x \in R$, then $R$ is commutative. This result was subsequently, refined and extended by a number of authors. In [7], Bresar and Vukman showed that a prime ring must be commutative if it admits a nonzero left derivation. Recently, many authors have obtained commutativity theorems for prime and semiprime rings admitting derivation. Bresar and Vukman [5] studied the notions of a $\ast$-derivation and a Jordan $\ast$-derivation of $R$. The aim of the present paper is to establish some results involving generalized $\ast$-derivations in $\ast$-rings and investigate the commutativity of prime $\ast$-rings admitting generalized $\ast$-derivations of $R$ satisfying certain identities and some related results have also been discussed.

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2. Preliminaries

Throughout $R$ will represent an associative ring with center $Z(R)$. For all $x, y \in R$, as a usual commutator, we shall write $[x, y] = xy -yx$, and $x \circ y = xy + yx$. Also, we make use of the following two basic identities without any specific mention:

\[
x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z
\]

\[
(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]
\]

\[
[x, y, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z.
\]

Let $R$ be a ring. Then $R$ is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $x \rightarrow x^*$ of $R$ into itself is called an involution if the following conditions are satisfied (i) $(xy)^* = y^*x^*$, (ii) $(x^*)^* = x$ for all $x, y \in R$. A ring equipped with an involution is called an *-ring or ring with involution. Let $R$ be a *-ring. An additive mapping $d : R \rightarrow R$ is called an *-derivation if $d(xy) = d(x)y^* + xd(y)$ holds for all $x, y \in R$. An additive mapping $d : R \rightarrow R$ is called a reverse *-derivation if $d(xy) = d(y)x^* + yd(x)$ holds for all $x, y \in R$. An additive mapping $F : R \rightarrow R$ is called a generalized derivation if there exists a derivation $d$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Let $R$ be an *-ring. An additive mapping $F : R \rightarrow R$ is called a generalized *-derivation if there exists an *-derivation $d$ such that $F(xy) = F(x)y^* + xd(y)$ for all $x, y \in R$. An additive mapping $F : R \rightarrow R$ is called a generalized reverse *-derivation if there exists an *-derivation $d$ such that $F(xy) = F(y)x^* + yd(x)$ for all $x, y \in R$. An additive mapping $F : R \rightarrow R$ is called a right *-multiplier of $R$ if $F(xy) = x^*F(y)$ for all $x, y \in R$. Also, an additive mapping $F : R \rightarrow R$ is called a left *-multiplier of $R$ if $F(xy) = F(x)y^*$ for all $x, y \in R$. An additive mapping $F : R \rightarrow R$ is called a reverse left *-multiplier of $R$ if $F(xy) = F(y)x^*$ for all $x, y \in R$ and an additive mapping $F : R \rightarrow R$ is called a reverse right *-multiplier of $R$ if $F(xy) = y^*F(x)$ for all $x, y \in R$.

3. Generalized *-derivations of prime *-rings

**Lemma 3.1.** Let $R$ be a semiprime *-ring and $a \in R$. If $R$ admits a generalized *-derivation $F$ associated with an *-derivation $d$ of $R$ and $F(x) = [x, a]$ for all $x \in R$, then $F = 0$. 
Proof. By hypothesis, we have
(3.1) \[ F(xy) = F(x)y^* + xd(y), \forall x, y \in R. \]
Replacing \(y\) by \(yz\) in (3.1), we have
(3.2) \[ F(x(yz)) = F(x)z^*y^* + xd(y)z^* + xyd(z), \forall x, y, z \in R \]
and
(3.3) \[ F((xy)z) = F(x)y^*z^* + xd(y)z^* + xyd(z), \forall x, y, z \in R. \]
Combining (3.2) and (3.3), we have
(3.4) \[ F(x)[y^*, z^*] = 0, \forall x, y, z \in R. \]
Substituting \(y^*\) for \(y\) and \(z^*\) for \(z\) in (3.4), we have \(F(x)[y, z] = 0\) for all \(x, y, z \in R\). Again, replacing \(y\) by \(yx\) in the last relation, we have \(F(x)y[x, z] = 0\) for all \(x, y, z \in R\). This implies that \(F(x)R[x, z] = \{0\}\) for all \(x, z \in R\). Taking \(a\) in stead of \(z\) in this relation, we get \(F(x)R[x, a] = \{0\}\) for all \(x \in R\). If \(F(x) = [x, a]\), then, we have \([a, x]R[a, x] = \{0\}\) for all \(x \in R\). Since \(R\) is semiprime, we have \([x, a] = 0\), that is, \(F(x) = [x, a] = 0\) for all \(x \in R\). This implies that \(F = 0\).

Theorem 3.2. Let \(R\) be a semiprime \(*\)-ring. If \(R\) admits a generalized \(*\)-derivation \(F\) associated with \(*\)-derivation \(d\) of \(R\), then \(F\) maps from \(R\) to \(Z(R)\).

Proof. By hypothesis, we have
(3.5) \[ F(xy) = F(x)y^* + xd(y), \forall x, y \in R. \]
Replacing \(y\) by \(yz\) in (3.5), we have
(3.6) \[ F(x(yz)) = F(x)z^*y^* + xd(y)z^* + xyd(z), \forall x, y, z \in R. \]
On the other hand,
(3.7) \[ F(xyz) = F((xy)z) = F(x)y^*z^* + xd(y)z^* + xyd(z), \forall x, y, z \in R. \]
Combining (3.6) with (3.7), we have \(F(x)[y^*, z^*] = 0\) for all \(x, y, z \in R\). Substituting \(y^*\) for \(y\) and \(z^*\) for \(z\) in this relation, we have \(F(x)[y, z] = 0\) for all \(x, y, z \in R\). Taking \(yF(x)\) instead of \(y\) in the last relation, we have
(3.8) \[ F(x)y[F(x), z] = 0, \forall x, y \in R. \]
Multiplying the left side of (3.8) by \(zF(x)\), we have
(3.9) \[ zF(x)F(xy)[F(x), z] = 0, \forall x, y, z \in R. \]
Again, multiplying the left side of (3.8) by \(F(x)z\), we have
(3.10) \[ F(x)zF(x)F(x)[F(x), z] = 0, \forall x, y, z \in R. \]
Subtracting (3.9) from (3.10), we have \([F(x), z]F(x)(F(x), z) = 0\) for all \(x, y, z \in R\). Hence we have \([F(x), z]R[F(x), z] = \{0\}\) for all \(x, z \in R\). Since \(R\) is semiprime, we have \([F(x), z] = 0\) for all \(x, z \in R\). Therefore \(F\) is a mapping form \(R\) into \(Z(R)\).

**Theorem 3.3.** Let \(R\) be a prime \(*\)-ring. If \(R\) admits a generalized \(*\)-derivation \(F\) associated with an \(*\)-derivation \(d\) such that \(F(x) \neq x\) and \(F(xy) = F(x)F(y)\) for all \(x, y \in R\), then \(d = 0\).

**Proof.** By hypothesis, we have

\[(3.11) \quad F(xy) = F(x)y^* + xd(y) = F(x)F(y), \quad \forall \, x, y \in R.\]

Replacing \(x\) by \(xz\) in (3.11), we have

\[F(x)F(z)y^* + zxd(y) = F(x)F(z)F(y) = F(x)(F(z)y^* + zd(y)),\]

which implies that \((x - F(x))zd(y) = 0\) for all \(x, y, z \in R\). Hence we have \((x - F(x))Rd(y) = \{0\}\) for all \(x, y \in R\). Since \(R\) is prime, we have \(x - F(x) = 0\) or \(d(y) = 0\) for all \(x, y \in R\). But \(F(x) \neq x\), and so \(d(y) = 0\) for all \(y \in R\), that is, \(d = 0\).

**Theorem 3.4.** Let \(R\) be a prime \(*\)-ring. If \(R\) admits a generalized \(*\)-derivation \(F\) associated with an \(*\)-derivation \(d\) of \(R\) and \(F(xy) = F(y)F(x)\) for all \(x, y \in R\), then \(d = 0\).

**Proof.** By hypothesis, we have

\[(3.12) \quad F(xy) = F(x)y^* + xd(y) = F(x)F(x), \quad \forall \, x, y \in R.\]

Replacing \(x\) by \(xy\) in (3.12), we have

\[F(xy)y^* + xyd(y) = F(y)F(xy) = F(y)(F(x)y^* + xd(y)),\]

which implies that \(F(y)F(xy)y^* + xyd(y) = F(y)F(x)y^* + F(y)xd(y)\) for all \(x, y \in R\). Hence we have

\[(3.13) \quad xyd(y) = F(y)xd(y), \quad \forall \, x, y \in R.\]

Taking \(wx\) instead of \(x\) in (3.13) and using (3.13), we have \(wF(y)xd(y) = F(y)wxd(y)\) for all \(x, y, w \in R\). This implies that \([w, F(y)]xd(y) = 0\), and so \([w, F(y)]Rd(y) = \{0\}\) for all \(w, y \in R\). Since \(R\) is prime, we have \(d(y) = 0\) or \([w, F(y)] = 0\) for all \(y \in R\). Let \(K = \{y \in R : d(y) = 0\}\) and \(L = \{y \in R : [w, F(y)] = 0, \forall \, w \in R\}\). Then \(K\) and \(L\) are both additive subgroups and \(K \cup L = R\), but \((R, +)\) is not union of two its proper subgroups, which implies that either \(K = R\) or \(L = R\). In the former case, we have \(d = 0\). If \(L = R\), then \([w, F(y)] = 0\) for all \(w, y \in R\). Hence we have \(F(y) \in Z(R)\), and so \(F(xy) = F(y)F(x) = F(x)F(y)\) for
all \(x, y \in R\). Since \(F\) acts an endomorphism of \(R\), it follows that \(d = 0\) via Theorem 3.3.

**Theorem 3.5.** Let \(R\) be a noncommutative prime \(*\)-ring. If \(R\) admits a generalized \(*\)-derivation \(F\) associated with an \(*\)-derivation \(d\) and \(F(x) \in Z(R)\) for all \(x, y \in R\), then \(d = 0\).

**Proof.** By hypothesis, we have

\[
[F(xy), z] = 0, \quad \forall \ x, y, z \in R,
\]

which implies that \([F(x)y^* + xd(y), z] = 0\), and so

\[
F(x)[y^*, z] + x[d(y), z] + [x, z]d(y) = 0
\]

for all \(x, y, z \in R\). Replacing \(z\) by \(y^*\) in the above relation, we have \(x[d(y), y^*] + [x, y^*]d(y) = 0\) for all \(x, y \in R\), that is,

\[
(3.15)
\]

\[
xd(y)y^* = y^*xd(y), \quad \forall \ x, y \in R.
\]

Substituting \(xz\) for \(x\) in (3.15), we have \(xzd(y)y^* = y^*xzd(y)\) for all \(x, y, z \in R\). Using the relation (15), we have \(xy^*zd(y) = y^*xzd(y)\) for all \(x, y, z \in R\), that is, \([x, y^*]zd(y) = 0\) for all \(x, y, z \in R\). Hence \([x, y^*]Rd(y) = \{0\}\) for all \(x, y \in R\). Since \(R\) is prime, we have \([x, y^*] = 0\) or \(d(y) = 0\) for all \(x, y \in R\). But \(R\) is noncommutative, and so \(d(y) = 0\) for all \(y \in R\), which means that \(d = 0\).

**Theorem 3.6.** Let \(R\) be a semiprime \(*\)-ring. If \(R\) admits a generalized reverse \(*\)-derivation \(F\) associate with a nonzero reverse \(*\)-derivation \(d\), then \([d(x), z] = 0\) for all \(x, z \in R\).

**Proof.** By hypothesis, we have

\[
(3.16)
\]

\[
F(xy) = F(y)x^* + yd(x), \quad \forall \ x, y \in R.
\]

Replacing \(x\) by \(xz\) in (3.16), we have

\[
F((xz)y) = F(y)(xz)^* + yd(xz)
\]

\[
= F(y)z^*x^* + y(d(z)x^* + zd(x))
\]

\[
= F(y)z^*x^* + yd(z)x^* + yzd(x)
\]

(3.17)

for every \(x, y, z \in R\). On the other hand, we have

\[
F(x(zy)) = F(zy)x^* + zyd(x)
\]

\[
= (F(y)z^* + yd(z))x^* + zyd(x)
\]

\[
= F(y)z^*x^* + yd(z)x^* + zyd(x)
\]

(3.18)
for every $x, y, z \in R$. Comparing (3.17) and (3.18), we get $[y, z]d(x) = 0$ for all $x, y, z \in R$. Substituting $d(x)y$ for $y$ in this relation, we obtain

\begin{equation}
[d(x), z]yd(x) = 0, \forall x, y, z \in R.
\end{equation}

Multiplying the right side of (3.19) by $zd(x)$, we have

\begin{equation}
[d(x), z]yd(x)zd(x) = 0, \forall x, y, z \in R.
\end{equation}

Multiplying the right side of (3.19) by $d(x)z$, we have

\begin{equation}
[d(x), z]yd(x)d(x)z = 0, \forall x, y, z \in R.
\end{equation}

Subtracting (3.20) from (3.21), we have $[d(x), z]yd(x)zd(x) = 0$ for all $x, y, z \in R$. This implies that $[d(x), z]R[d(x), z] = \{0\}$ for all $x, z \in R$. Since $R$ is semiprime, we have $[d(x), z] = 0$ for all $x, z \in R$.

**Theorem 3.7.** Let $R$ be a noncommutative prime $*$-ring. If $R$ admits a generalized reverse $*$-derivation $F$ associated with a nonzero reverse $*$-derivation $d$ of $R$, then $F$ is a reverse left $*$-multiplier of $R$.

**Proof.** By Theorem 3.6, we have

\begin{equation}
[y, z]d(x) = 0, \forall x, y, z \in R.
\end{equation}

Replacing $y$ by $xy$ in (3.22), we have $[xy, z]d(x) = 0$, and so $[x, z]yd(x) = 0$ for all $x, y, z \in R$. This implies that $[x, z]Rd(x) = 0$ for all $x, z \in R$. Since $R$ is prime, we have $[x, z] = 0$ or $d(x) = 0$ for all $x, z \in R$. Let $K = \{x \in R \mid d(x) = 0\}$ and $L = \{x \in R \mid [x, z] = 0, \forall z \in R\}$. Then $K$ and $L$ are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $L = R$ or $K = R$. In the former case, $R$ is commutative, contradiction. On the other hand, if $K = R$, then $d(x) = 0$ for all $x \in R$, that is, $d = 0$. Hence $F(xy) = F(y)x^*$ for all $x, y \in R$. This implies that $F$ is a reverse left $*$-multiplier of $R$.

**Theorem 3.8.** Let $R$ be a prime $*$-ring. If $R$ admits a generalized $*$-derivation $F$ associated with an $*$-derivation $d$ such that $F([x, y]) = 0$ for all $x, y \in R$, then $d = 0$ or $R$ is commutative.

**Proof.** By hypothesis, we have

\begin{equation}
F([x, y]) = 0, \forall x, y \in R.
\end{equation}

Replacing $x$ by $xy$ in (3.23), we have

\begin{equation}
F([x, y]) = F([x, y])y^* + [x, y]d(y) = 0
\end{equation}

for all $x, y \in R$. By the relation (3.23), we have $[x, y]d(y) = 0$ for all $x, y \in R$. Substituting $sx$ for $x$ in this relation, we have $[s, y]xd(y)$ for
Let $x, y, s \in R$. This implies that $[s, y]Rd(y) = \{0\}$ for all $s, y \in R$. Since $R$ is prime, we have $[s, y] = 0$ or $d(y) = 0$ for all $s, y \in R$. Let $K = \{y \in R \mid d(y) = 0\}$ and $L = \{y \in R \mid [s, y] = 0, \forall \ s \in R\}$. Then $K$ and $L$ are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$, and in second case, $R$ is commutative.

**Theorem 3.9.** Let $R$ be a prime $*$-ring. If $R$ admits a generalized $*$-derivation $F$ associated with an $*$-derivation $d$ such that $F(x \circ y) = 0$ for all $x, y \in R$, then $d = 0$ or $R$ is commutative.

**Proof.** By hypothesis, we have

\[(3.24) \quad F(x \circ y) = 0, \quad \forall \ x, y \in R.\]

Replacing $x$ by $xy$ in (3.24), we have

$$F((x \circ y)y) = F(x \circ y)y^* + (x \circ y)d(y) = 0$$

for all $x, y \in R$. By the relation (3.24), we have $(x \circ y)d(y) = 0$ for all $x, y \in R$. Substituting $sy$ for $x$ in this relation, we have $(s \circ y)y'd(y) = 0$ for all $s, y \in R$. This implies that $(s \circ y)Rd(y) = \{0\}$ for all $s, y \in R$. Since $R$ is prime, we have $(s \circ y) = 0$ or $d(y) = 0$ for all $s, y \in R$. Let $K = \{y \in R \mid d(y) = 0\}$ and $L = \{y \in R \mid s \circ y = 0, \forall \ s \in R\}$. Then $K$ and $L$ are both additive subgroups and $K \cup L = R$, but $(R, +)$ is not union of two its proper subgroups, which implies that either $K = R$ or $L = R$. In the former case, we have $d = 0$. On the other hand, if $L = R$, then we have $s \circ y = 0$ for all $s, y \in R$. Replacing $s$ by $sz$ in the last relation and using the fact that $ys = -sy$, we obtain $s[z, y] = 0$ for all $s, y, z \in R$. That is, $R[z, y] = \{0\}$. This implies that $xR[z, y] = \{0\}$ for $0 \neq x \in R$. Since $R$ is prime, we have $[z, y] = 0$ for all $y, z \in R$, which means that $R$ is commutative.\[\]

**Theorem 3.10.** Let $R$ be a 2-torsion free prime $*$-ring. If $R$ admits a generalized $*$-derivation $F$ associated with an $*$-derivation $d$ such that $F(x \circ y) = [x, y]$ for all $x, y \in R$, then $d = 0$.

**Proof.** By hypothesis, we have

\[(3.25) \quad F(x \circ y) = [x, y], \quad \forall \ x, y \in R.\]

Replacing $y$ by $yx$ in (3.25), we have

$$F((x \circ y)x) = F(x \circ y)x^* + (x \circ y)d(x) = [x, y]x$$

for all $x, y \in R$. By the relation (3.25), we get

\[(3.26) \quad [x, y]x^* + (x \circ y)d(x) = [x, y]x, \quad \forall \ x, y \in R.\]
Substituting $y$ for $x$ in this relation, we have $(y \circ y)d(y) = 0$ for all $x, y \in R$. This implies that $2y^2d(y) = \{0\}$ for all $y \in R$. Since $R$ is 2-torsion free, we have $y^2d(y) = 0$ for all $y \in R$. This implies that $yRd(y) = \{0\}$ for all $y \in R$. Since $R$ is prime, we obtain $y = 0$ or $d(y) = 0$ for all $y \in R$. In the former case, $y = 0$ for all $y \in R$, a contradiction. Hence $d(y) = 0$ for all $y \in R$, that is, $d = 0$.

**Theorem 3.11.** Let $R$ be a 2-torsion free prime $*$-ring. If $R$ admits a generalized $*$-derivation $F$ associated with an $*$-derivation $d$ such that $F(x \circ y) = -[x, y]$ for all $x, y \in R$, then $d = 0$.

**Proof.** Using the similar technique with necessary variations in the above theorem, we get the required result.

**References**


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