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FUZZY ALMOST q-CUBIC FUNCTIONAL EQATIONS

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Abstract. In this paper, we approximate a fuzzy almost cubic function by a cubic function in a fuzzy sense. Indeed, we investigate solutions of the following cubic functional equation

$$\begin{aligned}&3f(kx+y)+3f(kx-y)-kf(x+2y)-2kf(x-y)\\&-3k(2k^2-1)f(x)+6kf(y)=0.\end{aligned}$$

and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces.

1. Introduction and preliminaries

In 1940, Ulam proposed the following stability problem (cf. [24]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exists a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [11] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki [1] for additive mappings, and by Rassias [22] for linear mappings, to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians ([5], [6], [7], [10], [20]).

Recently, the stability problems in the fuzzy spaces has been extensively studied ([13], [18], [19]). The concept of fuzzy norm on a linear space was introduced by Katsaras [15] in 1984. Later, Cheng and Mordeson [4] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [14]. In

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2008, for the first time, Mirmostafaee and Moslehian ([18], [19]) used the definition of a fuzzy norm in [2] to obtain a fuzzy version of stability for the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.2}$$

We call a solution of (1.1) *an additive mapping* and a solution of (1.2) is called *a quadratic mapping.* In 2001, Rassias [23] introduced the following cubic functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) = 0 \tag{1.3}$$

and every solution of the cubic functional equation is called *a cubic mapping*

In this paper, we consider the following functional equation

$$\begin{aligned} &3f(kx+y) + 3f(kx-y) - kf(x+2y) - 2kf(x-y) \\ &- 3k(2k^2-1)f(x) + 6kf(y) = 0 \end{aligned} \tag{1.4}$$

for some fixed non-zero rational number k and show the generalized Hyers-Ulam stability of (1.4) in a fuzzy sense.

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is called *a fuzzy norm on* X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x,\ t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,\ t) = 1$ for all $t > 0$;
(N3) $N(cx,\ t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y,\ s+t) \geq \min\{N(x,s),\ N(y,t)\}$;
(N5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N6) for any $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X, N) is called *a fuzzy normed space.*

Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent* in (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* (X, N) and one denotes it by $N - \lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is said to be *Cauchy* if for any $\epsilon > 0$, there is an $m \in N$ such that for any $n \geq m$ and any positive integer p, $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $t > 0$.

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each

Cauchy sequence in it is convergent and the a complete fuzzy normed space is called *a fuzzy Banach space*.

2. Solutions and the stability of (1.4)

In this section, we investigate solutions of (1.4) and prove the generalized Hyers-Ulam stability of (1.4) in fuzzy Banach spaces.

We start with the following theorem.

THEOREM 2.1. *Let X and Y be normed spaces and $f : X \longrightarrow Y$ a mapping with $f(0) = 0$. Suppose that f satisfies (1.4) and $k \neq 0$. Then f is a cubic mapping.*

Proof. Suppose that f satisfies (1.4). If $k = 1$, then f satisfies (1.3) and so f is a cubic mapping. Suppose that $k \neq 1$.

Setting $y = 0$ in (1.4), we have

$$f(kx) = k^3 f(x) \tag{2.1}$$

for all $x \in X$ and setting $x = 0$ and $y = x$ in (1.4), we have

$$3f(x) + 3f(-x) = kf(2x) + 2kf(-x) - 6kf(x) \tag{2.2}$$

for all $x \in X$. Replacing x by $-x$ in (2.2), we have

$$3f(x) + 3f(-x) = kf(-2x) + 2kf(x) - 6kf(-x) \tag{2.3}$$

for all $x \in X$. Since $k \neq 0$, by (2.2) and (2.3), we have

$$f(2x) - f(-2x) = 8[f(x) - f(-x)] \tag{2.4}$$

for all $x \in X$. Relpacing y by ky in (1.4), by (2.1), we have

$$\begin{aligned} &3k^2[f(x+y) + f(x-y)] \\ &\quad - f(x+2ky) - 2f(x-ky) - 3(2k^2-1)f(x) + 6k^3 f(y) = 0 \end{aligned} \tag{2.5}$$

for all $x, y \in X$ and letting $y = -y$ in (2.5), we have

$$\begin{aligned} &3k^2[f(x+y) + f(x-y)] \\ &\quad - f(x-2ky) - 2f(x+ky) - 3(2k^2-1)f(x) + 6k^3 f(-y) = 0 \end{aligned} \tag{2.6}$$

for all $x, y \in X$. By (2.5) and (2.6), we have

$$[f(x+2ky) - f(x-2ky)] - 2[f(x+ky) - f(x-ky)] - 6k^3[f(y) - f(-y)] = 0 \tag{2.7}$$

for all $x, y \in X$. Letting $y = \frac{1}{k}y$ in (2.7), we have

$$[f(x+2y) - f(x-2y)] - 2[f(x+y) - f(x-y)] - 6[f(y) - f(-y)] = 0 \tag{2.8}$$

for all $x, y \in X$.

Let $f_o(x) = \frac{f(x)-f(-x)}{2}$. Then f_o satisfies (2.8). By (2.4), we have

$$f_o(2y) = 8f_o(y) \tag{2.9}$$

for all $y \in X$. Letting $x = 2x$ in (2.8), by (2.9), we have

$$4[f_o(x+y) - f_o(x-y)] = f_o(2x+y) - f_o(2x-y) + 6f_o(y) \tag{2.10}$$

for all $x, y \in X$. Interchanging x and y in (2.10), we have

$$4[f_o(x+y) + f_o(x-y)] = f_o(x+2y) + f_o(x-2y) + 6f_o(x) \tag{2.11}$$

for all $x, y \in X$. By (2.8) and (2.11), we have

$$f_o(x+2y) - 3f_o(x+y) + 3f_o(x) - f_o(x-y) - 6f_o(y) = 0$$

for all $x, y \in X$ and hence f_0 is a cubic mapping.

Let $f_e(x) = \frac{f(x)+f(-x)}{2}$. Then f_e satisfies (2.8) and so we have

$$f_e(x+2y) - f_e(x-2y) - 2[f_e(x+y) - f_e(x-y)] = 0 \tag{2.12}$$

for all $x, y \in X$. Letting $y = x$ in (2.12), we have

$$f_e(3x) = 2f_e(2x) + f_e(x)$$

for all $x \in X$ and letting $y = 2x$ in (2.12), we have

$$f_e(4x) = 2f_e(3x) - 2f_e(x)$$

for all $x \in X$. Hence we have $f_e(4x) = 4f_e(2x)$ for all $x \in X$ and so

$$f_e(2x) = 4f_e(x), \;\; f_e(3x) = 9f_e(x), \;\; f_e(4x) = 16f_e(x)$$

for all $x \in X$. By induction on n, we have

$$f_e(nx) = n^2 f_e(x)$$

for all $x \in X$ and all $n \in \mathbb{N}$ and hence

$$f_e(rx) = r^2 f_e(x) \tag{2.13}$$

for all $x \in X$ and all rational number r. Since k is a non-zero rational number, by (2.1) and (2.13), we have

$$k^3 f_e(x) = k^2 f_e(x).$$

Since $k \neq 0, 1$, $f_e(x) = 0$ for all $x \in X$. Hence $f = f_o + f_e = f_o$ is a cubic mapping. □

Let (X, N) be a fuzzy normed space and (Y, N') a fuzzy Banach space. For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$\begin{aligned} Df(x,y) = {} & 3f(kx+y) + 3f(kx-y) - kf(x+2y) - 2kf(x-y) \\ & - 3k(2k^2-1)f(x) + 6kf(y) \end{aligned}$$

for all $x, y \in X$ and some non-zero real number k.
For a given real number q with $q > 0$, the mapping f is said to be *a fuzzy q-almost cubic mapping if*

$$N'(Df(x,y),\ t+s) \geq \min\{N(x,\ t^q), N(y,\ s^q)\} \tag{2.14}$$

for all $x, y \in X$ and all positive real numbers t, s.

THEOREM 2.2. *Let q be a positive real number withwith $|k|^{3q-1} > 1$ and $f : X \longrightarrow Y$ a fuzzy q-almost cubic mapping with $f(0) = 0$. Then there exists a unique cubic mapping $F : X \longrightarrow Y$ such that*

$$N'(F(x) - f(x), t) \geq N\Big(x, \frac{3^q(|k|^3 - |k|^p)^q t^q}{2^q |k|^{3q-1}}\Big) \tag{2.15}$$

for all $x \in X$ and all $t > 0$, where $p = \frac{1}{q}$.

Proof. Letting $y = 0$ and $s = t$ in (2.14), we get

$$N'\Big(f(kx) - k^3 f(x), \frac{t}{3}\Big) \geq N(x, t^q) \tag{2.16}$$

for all $x \in X$ and all $t > 0$ and replacing x by $k^n x$ in (2.16), we get

$$N'\Big(f(k^{n+1}x) - k^3 f(k^n x), \frac{t}{3}\Big) \geq N\Big(x, \frac{t^q}{|k|^n}\Big)$$

for all $x \in X$, all $n \in \mathbb{N}$, and all $t > 0$. Hence we have

$$N'\Big(f(k^{n+1}x) - k^3 f(k^n x), \frac{1}{3}|k|^{\frac{n}{q}} t^{\frac{1}{q}}\Big) \geq N(x, t)$$

for all $x \in X$, all $n \in \mathbb{N}$, and all $t > 0$ and so we get

$$N'\Big(\frac{f(k^{n+1}x)}{k^{3(n+1)}} - \frac{f(k^n x)}{k^{3n}}, \frac{1}{3}|k|^{n(p-3)-3} t^p\Big) \geq N(x, t)$$

for all $x \in X$, all $n \in \mathbb{N}$, and all $t > 0$. For $n > m \geq 0$, we have

(2.17)

$$\begin{aligned}
&N'\Big(\frac{f(k^n x)}{k^{3n}} - \frac{f(k^m x)}{k^{3m}}, \sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(p-3)-3}\ t^p\Big)\\
&= N'\Big(\sum_{i=m+1}^{n} \Big[\frac{f(k^i x)}{k^{3i}} - \frac{f(k^{i-1}x)}{k^{3(i-1)}}\Big], \sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(p-3)-3}\ t^p\Big)\\
&\geq \min\Big\{N'\Big(\frac{f(k^i x)}{k^{3i}} - \frac{f(k^{i-1}x)}{k^{3(i-1)}}, \frac{1}{3}|k|^{i(p-3)-3}\ t^p\Big)\ \Big|\ m+1 \leq i \leq n\Big\}\\
&\geq N(x, t)
\end{aligned}$$

for all $x \in X$ and all positive integer n.

Let $x \in X$, $c > 0$ and $\epsilon > 0$. By (N5), there is a t_0 such that

$$N(x, t_0) \geq 1 - \epsilon$$

for all $x \in X$. Since $|k|^{3q-1} > 1$, $|k|^{p-3} < 1$ and so $\sum_{n=1}^{\infty} |k|^{n(p-3)-3}\ t_0^p$ is convergent. Hence there is a positive integer n_0 such that for any $n \in \mathbb{N}$ with $n \geq n_0$, $\sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(p-3)-3}\ t_0^p < c$ and so for $n > m \geq 0$, we have

$$\begin{aligned}
&N'\Big(\frac{f(k^n x)}{k^{3n}} - \frac{f(k^m x)}{k^{3m}}, c\Big) \\
&\geq N'\Big(\frac{f(k^n x)}{k^{3n}} - \frac{f(k^m x)}{k^{3m}}, \sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(p-3)-3}\ t_0^p\Big) \\
&\geq N(x, t_0) \geq 1 - \epsilon
\end{aligned}$$

and thus $\Big\{\frac{f(k^n x)}{k^{3n}}\Big\}$ is a Cauchy sequence in (Y, N'). Since (Y, N') is a fuzzy Banach space, there is an $F(x)$ in Y such that

$$F(x) = N - \lim_{n \to \infty} \frac{f(k^n x)}{k^{3n}}.$$

Clearly, $F : X \longrightarrow Y$ is a mapping. Letting $m = 0$ in (2.17), we get

$$N'\Big(\frac{f(k^n x)}{k^{3n}} - f(x), \sum_{i=1}^{n} \frac{1}{3}|k|^{i(p-3)-3}\ t^p\Big) \geq N(x, t)$$

for all $x \in X$ and all positive integer n and so we have

$$N'\Big(\frac{f(k^n x)}{k^{3n}} - f(x), t\Big) \geq N\Big(x, \frac{t^q}{\Big(\sum_{i=1}^{n} \frac{1}{3}|k|^{i(p-3)-3}\Big)^q}\Big) \tag{2.18}$$

for all $x \in X$, $n \in \mathbb{N}$, and $t > 0$.

Now, we will show that F satisfies (1.4). Let $x, y \in X$. By (N4), we have

$$\begin{aligned}
&N'(DF(x,y),t)\\
&\geq \min\Big\{N'\Big(3F(kx+y)-\frac{3f(k^n(kx+y))}{k^{3n}},\frac{t}{7}\Big),\\
&\qquad N'\Big(3F(kx-y)-\frac{3f(k^n(kx-y))}{k^{3n}},\frac{t}{7}\Big),\\
&\qquad N'\Big(kF(x+2y)-\frac{kf(k^n(x+2y))}{k^{3n}},\frac{t}{7}\Big),\\
&\qquad N'\Big(2kF(x-y)-\frac{2kf(k^n(x-y))}{k^{3n}},\frac{t}{7}\Big),\\
&\qquad N'\Big(3k(2k^2-1)F(x)-\frac{3k(2k^2-1)f(k^nx)}{k^{3n}},\frac{t}{7}\Big),\\
&\qquad N'\Big(6kF(y)-\frac{6kf(k^ny)}{k^{3n}},\frac{t}{7}\Big),N'\Big(\frac{Df(k^nx,k^ny)}{k^{3n}},\frac{t}{7}\Big)\Big\}
\end{aligned}$$

for all $x,y\in X$ and all positive integer n. The first six terms on the right-hand of the above inequality tend to 1 as $n\to\infty$ and by (2.14), we have

$$N'\Big(\frac{Df(k^nx,k^ny)}{k^{3n}},\frac{t}{7}\Big)\geq N\Big(x,|k|^{(3q-1)n}\Big(\frac{t}{14}\Big)^q\Big)$$

for all $x,y\in X$, all positive integer n and all $t>0$. Since $|k|^{3q-1}>1$, Hence $N'(DF(x,y),t)=0$ for all $x,y\in X$ and all $t>0$. By (N2), $DF(x,y)=0$ for all $x,y\in X$ and by Theorem 2.1, F is a cubic mapping.

Now, we will show that (2.15) holds. Let $x\in X$ and $t>0$. By (2.18), for large enough n, we get

$$\begin{aligned}
&N'(F(x)-f(x),t)\\
&\geq \min\Big\{N'\Big(F(x)-\frac{f(k^nx)}{k^{3n}},\frac{t}{2}\Big),N'\Big(f(x)-\frac{f(k^nx)}{k^{3n}},\frac{t}{2}\Big)\Big\}\\
&\geq N\Big(x,\frac{t^q}{\Big(\sum_{i=1}^n\frac{2}{3}|k|^{i(p-3)-3}\Big)^q}\Big)\\
&\geq N\Big(x,\frac{3^q(|k|^6-|k|^{p+3})^qt^q}{2^q|k|}\Big)
\end{aligned}$$

for all $x\in X$ and all $n\in\mathbb{N}$ and so we have (2.15).

To prove the uniqueness of F, let $F_1:X\longrightarrow Y$ be another cubic mapping satisfying (2.15). Then we have

$$N'(F(x)-F_1(x),t)=N'(F(k^nx)-F_1(k^nx),|k|^{3n}t)$$
$$\geq \min\Big\{N'\Big(F(k^nx)-f(k^nx),|k|^{3n}\frac{t}{2}\Big),N'\Big(f(k^nx)-F_1(k^nx),|k|^{3n}\frac{t}{2}\Big)\Big\}$$
$$\geq N\Big(x,\frac{3^q(|k|^6-|k|^{p+3})^q|k|^{n(3q-1)-1}t^q}{4^q}\Big)$$

for all $x,y\in X$, all positive integer n and all $0<s<t$. Since $|k|^{3q-1}>1$,

$$\lim_{n\to\infty}N\Big(x,\frac{3^q(|k|^6-|k|^{p+3})^q|k|^{n(3q-1)-1}t^q}{4^q}\Big)=1$$

and so $N'(F(x)-F_1(x),t)=1$ for all $t>0$. Hence $F=F_1$. □

Theorem 2.3. *Let q be a positive real number withwith $|k|^{3q-1}<1$ and $f:X\longrightarrow Y$ a fuzzy q-almost cubic mapping with $f(0)=0$. Then there exists a unique cubic mapping $F:X\longrightarrow Y$ such that*

$$N'(F(x)-f(x),t)\geq N\Big(x,\frac{3^q(|k|^p-|k|^3)^qt^q}{2^q|k|^{3q-1}}\Big) \tag{2.19}$$

for all $x\in X$ and all $t>0$.

Proof. Letting $y=\frac{x}{k}$ in (2.16), we get

$$N'\Big(f(x)-k^3f\Big(\frac{x}{k}\Big),\frac{t}{3}\Big)\geq N(x,|k|t^q) \tag{2.20}$$

for all $x\in X$ and all $t>0$ and replacing x by $\frac{x}{k^n}$ in (2.20), we get

$$N'\Big(f\Big(\frac{x}{k^n}\Big)-k^3f\Big(\frac{x}{k^{n+1}}\Big),\frac{t}{3}\Big)\geq N\Big(x,|k|^{n+1}t^q\Big)$$

for all $x\in X$, all $n\in\mathbb{N}$, and all $t>0$. Hence we have

$$N'\Big(k^{3n}f\Big(\frac{x}{k^n}\Big)-k^{3(n+1)}f\Big(\frac{x}{k^{n+1}}\Big),\frac{1}{3}|k|^{n(3-p)-p}t^p\Big)\geq N(x,t)$$

for all $x\in X$, all $n\in\mathbb{N}$, and all $t>0$. For $n>m\geq 0$, we have

(2.21)
$$N'\Big(k^{3n}f\Big(\frac{x}{k^n}\Big)-k^{3m}f\Big(\frac{x}{k^m}\Big)\sum_{i=m+1}^{n}\frac{1}{3}|k|^{i(3-p)-p}\ t^p\Big)$$
$$\geq \min\Big\{N'\Big(k^{3i}f\Big(\frac{x}{k^i}\Big)-k^{3(i-1)}f\Big(\frac{x}{k^{i-1}}\Big),\frac{1}{3}|k|^{i(3-p)-p}\ t^p\Big)\ \Big|\ m+1\leq i\leq n\Big\}$$
$$\geq N(x,t)$$

for all $x\in X$ and all positive integer n.

Let $x \in X$, $c > 0$ and $\epsilon > 0$. By (N5), there is a t_0 such that

$$N(x, t_0) \geq 1 - \epsilon$$

for all $x \in X$. Since $|k|^{3q-1} < 1$, $|k|^{3-p} < 1$ and so $\sum_{n=1}^{\infty} |k|^{n(3-p)-p}\ t_0^p$ is convergent. Hence there is a positive integer n_0 such that for any $n \in \mathbb{N}$ with $n \geq n_0$, $\sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(3-p)-p}\ t_0^p < c$ and so for $n > m \geq 0$, we have

$$\begin{aligned}
&N'\Big(k^{3n} f\Big(\frac{x}{k^n}\Big) - k^{3m} f\Big(\frac{x}{k^m}\Big), c\Big) \\
&\geq N'\Big(k^{3n} f\Big(\frac{x}{k^n}\Big) - k^{3m} f\Big(\frac{x}{k^m}\Big), \sum_{i=m+1}^{n} \frac{1}{3}|k|^{i(3-p)-p}\ t_0^p\Big) \geq N(x, t_0) \\
&\geq 1 - \epsilon
\end{aligned}$$

and thus $\Big\{k^{3n} f\Big(\frac{x}{k^n}\Big)\Big\}$ is a Cauchy sequence in (Y, N'). Since (Y, N') is a fuzzy Banach space, there is an $F(x)$ in Y such that

$$F(x) = N - \lim_{n \to \infty} \frac{f(k^n x)}{k^{3n}}.$$

The rest of the proof is similar to Theorem 2.2. □

Using Theorem 2.2- 2.3, we have the following corollary :

COROLLARY 2.4. *Let q be a positive real number with $q \neq \frac{1}{3}$ and $f : X \longrightarrow Y$ a fuzzy q-almost cubic mapping. Then there exists a unique cubic mapping $F : X \longrightarrow Y$ such that*

$$N'(F(x) - f(x), t) \geq \begin{cases} N\Big(x, \frac{3^q(|k|^3 - |k|^p)^q}{2^q |k|^{1-3q}} t^q\Big), & \text{if } |k|^{3q-1} > 1 \\ N\Big(x, \frac{3^q(|k|^p - |k|^3)^q}{2^q |k|^{3q-1}} t^q\Big), & \text{if } |k|^{3q-1} < 1 \end{cases}$$

for all $x \in X$ and all $t > 0$.

We can use Theorem 2.2-2.3 to get a classical result in the framework of normed spaces. For example, it is well known that for any normed space $(X, ||\cdot||)$, the mapping $N_X : X \times \mathbb{R} \longrightarrow [0, 1]$, defined by

$$N_X(x, t) = \begin{cases} 0, & \text{if } t < \|x\| \\ 1, & \text{if } t \geq \|x\| \end{cases}$$

a fuzzy norm on X. In [17], [18] *and* [19], *some examples are provided for the fuzzy norm N_X. Here using the fuzzy norm N_X, we have the following corollary.*

Corollary 2.5. *Let X and Y be normed spaces. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\|Df(x,y)\| \leq \|x\|^p + \|y\|^p \tag{2.22}$$

for a fixed positive number p such that $p \neq 3$. Then there exists a unique cubic mapping $F : X \longrightarrow Y$ such that the inequality

$$\|F(x) - f(x)\| \leq \begin{cases} \frac{2\|k\|^p}{3(\|k\|^6 - \|k\|^{p+3})}\|x\|^p, & \text{if } |k|^{3q-1} > 1 \\ \frac{2\|k\|^3}{3(\|k\|^{2p} - \|k\|^{p+3})}\|x\|^p, & \text{if } |k|^{3q-1} < 1 \end{cases}$$

holds for all $x \in X$.

Proof. By the definition of N_Y, we have

$$N_Y(Df(x,y), s+t) = \begin{cases} 0, & \text{if } s+t < \|Df(x,y)\| \\ 1, & \text{if } s+t \geq \|Df(x,y)\|. \end{cases}$$

for all $x, y \in X$ and all $s, t \in \mathbb{R}$. Now, we claim that

$$N_Y(Df(x,y), s+t) \geq \min\{N_X(x, s^q), N_X(y, t^q)\}$$

for all $x, y \in X$ and $s, t > 0$, where $q = \frac{1}{p}$. If $N_Y(Df(x,y), s+t) = 1$, then it is trivial. Suppose that $N_Y(Df(x,y), s+t) = 0$. Then $s + t < \|Df(x,y)\|$. If $s \geq ||x||^p$ and $t \geq \|y\|^p$, then, by (2.22),

$$\|Df(x,y)\| \leq \|x\|^p + \|y\|^p < s + t,$$

which is a contradiction. Hence either $s < \|x\|^p$ or $t < \|y\|^p$, that is, either $N_X(x, s^q) = 0$ or $N_X(y, t^q) = 0$ and thus f is a fuzzy q-almost cubic mapping. By Theorem 2.2 and Theorem 2.3, we have the results. □

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