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NONLINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATION WITH A DAMPING TERM

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Abstract. The existence of solutions for nonlinear elliptic partial differential equations with general flux and damping terms is investigated.

1. Introduction

In this paper we study the existence theory of nonlinear elliptic equations with a damping term described by

$$-\nabla \cdot \mathbf{J} = f - D(u). \tag{1.1}$$

The *flux* field $\mathbf{J}$ explains the movement of some physical contents such as temperature, chemical potential, electrostatic potential or fluid flows, and so forth [2]. These equations (1.1) with or without a damping term $D(u)$ commonly and naturally have been observed in a lot of physical phenomena, so the problem (1.1) has been one of the most fundamental topics in the theory of partial differential equations. Our research has focused on a mathematical development of an existence theory of the equation (1.1) with very *realistic* flux field $\mathbf{J}$ [3, 4].

For an irrotational flux field (curl $\mathbf{J} = 0$), $\mathbf{J}$ can be represented as $\mathbf{J} = \nabla\phi(u)$ with an appropriate potential function $\phi(u)$. Furthermore, if we suppose ϕ is linear, then one can simply represent $\mathbf{J}$ as $\mathbf{J} = A\nabla u$ with a square matrix A on an-isotropic medium or simply $\mathbf{J} = c\nabla u$ on isotropic medium [3, 4]. The equation in the latter case is what we call the Poisson's equation or Laplace's equation with $f = 0$. Many brilliant mathematicians have studied these linear types during the past

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two centuries, and as a consequence one has gotten very good and enough understandings of these linear equations [2].

However, if ϕ is nonlinear, the situations become much more complicated, and many physical observations tell us these cases are even rather close to realistic. In this nonlinear case, one of the most common assumptions to impose is the p-Laplacian

$$\Delta_p u := \nabla \cdot |\nabla u|^{p-2} \nabla u$$

with flux $\mathbf{J} = |\nabla u|^{p-2}\nabla u$. One of the critical reasons for such assumptions is that the familiar function spaces used to deal with this problem are just the classical Lebesgue space $L^p(X)$. However it is too good to be true in reality [6]! It is definitely required good function spaces that can handle the problems with more natural flux term [3, 4].

The motivation of this research stems from taking a close look at the L^p-norm: $\|f\|_{L^p} = \left(\int_X |f(x)|^p \, d\mu\right)^{1/p}$ of the Lebesgue spaces $L^p(X)$, $1 \le p < \infty$. It can be rewritten as

$$\|f\|_{L^p} := \alpha^{-1}\left(\int_X \alpha(|f(x)|)\, d\mu\right) \tag{1.2}$$

with

$$\alpha(x) := x^p$$

[3, 6]. Even though the positive-real-variable function $\alpha(x) := x^p$ has very beautiful and convenient algebraic and geometric properties, it also has some practical limitations to handle general nonlinear problems [4, 5]. The new space $P_\alpha(X)$ is devised to overcome these limitations without hurting the beauty of L^p-norm too much [3, 6].

We point out that a nonlinear boundary value problem of the type:

$$-\Delta u = f(|u|)\, \mathrm{sgn} u \tag{1.3}$$

has been studied in the space $P_\alpha(\Omega)$ in [6]. In this paper, we present the existence of solutions for the elliptic partial differential equation with a damping term:

$$-\nabla \cdot \mathbf{J} = f - D(u). \tag{1.4}$$

The general flux $\mathbf{J}$ in (1.4) is given by

$$\mathbf{J} := \left(F_1(|u_{x_1}|), F_2(|u_{x_2}|), \cdots, F_n(|u_{x_n}|)\right),$$

and we specify the damping term D and the flux components F_j ($j = 1, 2, \cdots, n$) in Section 3. This problem has more general flux terms than the problem discussed in [3]. Our main observation on this report is that the damping term does not hurt the monotonicity of the diffusion term very much. We employ Galerkin's approximation method and Brouwer's fixed point theorem for the proof.

2. Lebesgue-Pak space $P_\alpha(X)$

We introduce some terminologies to define the Lebesgue-Pak spaces $P_\alpha(X)$ which are defined in [3, 4]. In this section, $(X, \mathcal{M}, \mu)$ represents a given measure space and $\mathbb{R}_+$ denotes the set of all nonnegative real numbers.

A *pre-Hölder's function* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is an *absolutely continuous* bijective function satisfying $\alpha(0) = 0$ [3]. If there exist pre-Hölder's functions β and λ that satisfy

$$\alpha^{-1}(x)\beta^{-1}(x) = \lambda(x), \tag{2.1}$$

then β is called the *conjugate* (pre-Hölder's) *function of* α *linked by* λ [3]. In the relation (2.1), the notations α^{-1}, β^{-1} denote the inverse functions of α, β, respectively [3]. For $\frac{1}{p} + \frac{1}{q} = 1$ $(p,\ q > 1)$, the Lebesgue base functions $(\alpha(x), \beta(x)) = (x^p, x^q)$ is a pre-Hölder's pair linked by the identity function $\lambda(x) = x$. The transcendental pair

$$(\alpha(x), \beta(x)) = (2e^x - 2x - 2, 2(1+x)\log(1+x) - 2x) \quad (x \geq 0)$$

is also a pre-Hölder's pair [3]. In fact, for any Orlicz N-function A together with complementary N-function $\tilde{A}$, $(A, \tilde{A})$ is a pre-Hölder's pair with $\lambda := A^{-1}(x)\tilde{A}^{-1}(x)$ [3, 6].

The followings show some basic identities for a pre-Hölder's pair (α, β) with respect to λ : for $\tilde{\alpha} := \lambda \circ \alpha$, and $\tilde{\beta} := \lambda \circ \beta$,

$$x = \beta\left(\frac{\lambda(x)}{\alpha^{-1}(x)}\right) \quad \text{or} \quad \alpha(x) = \beta\left(\frac{\tilde{\alpha}(x)}{x}\right), \tag{2.2}$$

$$x = \frac{\tilde{\alpha}(x)}{\beta^{-1}(\alpha(x))} \quad \text{or} \quad \frac{\tilde{\alpha}(x)}{x} = (\beta^{-1} \circ \alpha)(x) = (\tilde{\beta}^{-1} \circ \tilde{\alpha})(x), \tag{2.3}$$

$$\tilde{\alpha}^{-1}(x) = \frac{x}{\tilde{\beta}^{-1}(x)}, \tag{2.4}$$

$$\frac{\beta^{-1}(x)}{\alpha'(\alpha^{-1}(x))} + \frac{\alpha^{-1}(x)}{\beta'(\beta^{-1}(x))} = \lambda'(x), \tag{2.5}$$

$$\frac{y}{\alpha'(x)} + \frac{x}{\beta'(y)} = \lambda'(\alpha(x)) \quad \text{for} \quad y := \frac{\tilde{\alpha}(x)}{x}, \tag{2.6}$$

$$\tilde{\alpha}'(x) = \frac{\tilde{\alpha}(x)}{x} + \frac{\tilde{\alpha}(x)}{\tilde{\beta}\left(\frac{\tilde{\alpha}(x)}{x}\right) - x}. \tag{2.7}$$

The notations

$$\tilde{\alpha} := \lambda \circ \alpha \quad \text{and} \quad \tilde{\beta} := \lambda \circ \beta$$

will be used throughout this paper. Also, in the following discussion, a function Φ represents the function of two variables on $\mathbb{R}_+ \times \mathbb{R}_+$ defined by:

$$\Phi(x, y) := \tilde{\alpha}^{-1}(x)\tilde{\beta}^{-1}(y),$$

provided that a pre-Hölder's pair (α, β) exists [3].

Definition 2.1. *A pre-Hölder's function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ together with the conjugate function β for a link-function λ is said to be a Hölder's function if for any positive constants a, b and $\hbar$ >0, there exist constants θ_1, θ_2 (depending on a, b) such that*

$$\theta_1 + \theta_2 \le \hbar$$

and that a comparable condition

$$\Phi(x, y) \le \theta_1 \frac{ab}{\tilde{\alpha}(a)} x + \theta_2 \frac{ab}{\tilde{\beta}(b)} y \tag{2.8}$$

holds for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$.

We now define a function space $P_\alpha(X)$ as follows:

$P_\alpha(X) := \{f \mid f$ is a measurable function on X satisfying $\|f\|_{P_\alpha} < \infty\}$,

where we set

$$\|f\|_{P_\alpha} := \tilde{\alpha}^{-1}\left(\int_X \tilde{\alpha}(|f(x)|)d\mu\right). \tag{2.9}$$

In [3], it shows that the metric space $P_\alpha(X)$ is complete with respect to the metric :

$$d(f, g) := \|f - g\|_{P_\alpha} \quad \text{for } f, g \in P_\alpha(X)$$

and says the inhomogeneity of $\|\cdot\|_{P_\alpha}$: for all $k \ge 0$ and $f \in P_\alpha(X)$,

$$\frac{k}{\hbar}\|f\|_{P_\alpha} \le \|kf\|_{P_\alpha} \le k\hbar\|f\|_{P_\alpha}.$$

Unfortunately the new space $P_\alpha(X)$ lacks the homogeneity property: $\|kf\| = |k|\ \|f\|$. However, for nonlinear problems such as the equation (1.1), the homogeneity property may not be an essential factor - we try to explain that the new space $P_\alpha(X)$ accommodates the solutions of nonlinear problems without homogeneity as we pointed out in [3].

We say that a pre-Hölder function β is to satisfy a *slope condition* if there exists a positive constant $c > 1$ for which

$$\tilde{\beta}'(x) \geq c\,\frac{\tilde{\beta}(x)}{x}. \tag{2.10}$$

The slope condition (2.10), in fact, corresponds to the Δ_2-condition for Orlicz spaces [6]. We present a Poincaré-type inequality on $P^1_\alpha(\Omega)$ for the proof of our main theorem. The proof can be found in [5].

THEOREM 2.1 (Poincaré's inequality). *Let (α, β) be a Hölder pair with the slope condition (2.10) and Ω be an open set in $\mathbb{R}^n$ which is bounded in some direction, that is, there is a vector $v \in \mathbb{R}^n$ such that*

$$\sup\{|x \cdot v| : x \in \Omega\} < \infty.$$

Then there is a positive constant $C > 0$ such that for any $f \in P^1_{\alpha,0}(\Omega)$,

$$\|f\|_{P_\alpha} \leq C\|D_v f\|_{P_\alpha},$$

where $D_v f$ represents the directional derivative of f in the direction v; $D_v f = v \cdot \nabla f$.

We refer [3, 6] for a detailed discussion of the space $P_\alpha(X)$.

3. Nonlinear elliptic equation with a damping term

In this section, Ω represents a fixed bounded open set in $\mathbb{R}^n$ with smooth boundary. We are concerned with an elliptic partial differential equation with a damping term:

$$-\nabla \cdot \mathbf{J}(u) = f - D(u), \tag{3.1}$$

where the *flux* vector field is given by

$$\mathbf{J}(u) := (F_1(|u_{x_1}|), F_2(|u_{x_2}|), \cdots, F_n(|u_{x_n}|))$$

with

$$F_j(|\partial_{x_j} u|) := \tilde{\beta}_j^{-1} \circ \tilde{\alpha}_j(|\partial_{x_j} u|)\,\mathrm{sgn}(\partial_{x_j} u) = \frac{\tilde{\alpha}_j(|\partial_{x_j} u|)}{\partial_{x_j} u}$$

$(j = 1, 2, \cdots, n)$ and also the damping term:

$$D(u) := \tilde{\beta}_0^{-1} \circ \tilde{\alpha}_0(|u|)\,\mathrm{sgn}(u) = \frac{\tilde{\alpha}_0(|u|)}{u}.$$

This problem contains more general flux term than ones originally introduced in [3]. We are looking for solutions of the elliptic equation (3.1) with the flux term and the damping term in an appropriate function space. In fact, a natural function space that can permit solutions of the equation (3.1) turns out to be the space

$$V := \{u \,|\, \partial_{x_j} u \in P_{\alpha_j}(\Omega),\ j = 1, 2 \cdots, n \text{ and } u = 0 \text{ on } \partial\Omega\} \cap P_{\alpha_0}(\Omega),$$

equipped with the norm

$$\|u\|_V := \|u\|_{P_{\alpha_0}} + \sum_{j=1}^{n} \|u_{x_j}\|_{P_{\alpha_j}}. \tag{3.2}$$

We now state our main theorem for the problem (3.1).

THEOREM 3.1 (Main theorem). *Let* (α_i, β_i) $(i = 0, 1, 2, \cdots, n)$ *be Hölder's pairs satisfying the slope condition* (2.10). *Then for any functional* $f \in V'$ *there exists a solution* $u \in V$ *satisfying the equation* (3.1).

4. Argument of the main theorem

The main idea of the proof is borrowed from [3]. We divide the proof into several steps.

1. We note that V is a separable reflexive complete metric space. Hence we can choose an independent (complete) set $\{w_1, w_2, \cdots\}$ whose linear spans are dense in V. For each $m \geq 1$, let V_m be the subspace of V spanned by the set $\{w_1, w_2, \cdots, w_m\}$, and the (natural) isomorphism $j_m : V_m \to \mathbb{R}^m$ is defined by

$$\sum_{i=1}^{m} a_i w_i \mapsto (a_1, a_2, \cdots, a_m).$$

We note that $j_m^{-1} : \mathbb{R}^m \to V_m$ is continuous, since it is a linear combination of continuous functions. Then

$$\pi_m := i_m \circ j_m^{-1} : \mathbb{R}^m \to V$$

is continuous wherein $i_m : V_m \to V$ means the inclusion map.

2. (Functional formulation) We set up the functional formulation associated with the equation (3.1). For any $\phi \in C_c^\infty(\Omega)$, we have

$$-\int_\Omega \nabla \cdot \mathbf{J}(u)\ \phi\, d\mu = \int_\Omega f\phi\, d\mu - \int_\Omega D(u)\phi\, d\mu.$$

Then by virtue of Green theorem, the left-hand side is

$$-\int_\Omega \nabla \cdot (\mathbf{J}(u))\phi\, d\mu = \int_\Omega \mathbf{J}(u) \cdot \nabla\phi\, d\mu = \int_\Omega \sum_{j=1}^{n} \frac{\tilde{\alpha}_j(|\partial_{x_j} u|)}{\partial_{x_j} u} \partial_{x_j}\phi\ d\mu.$$

We also have

$$\int_\Omega D(u)\phi\,d\mu = \int_\Omega \frac{\tilde{\alpha}_0(|u|)}{u}\,\phi\;d\mu.$$

Now we define the operators $\mathcal{A}$ and $\mathcal{B}$ by

$$\mathcal{A}u(v) := \sum_{j=1}^{n}\int_\Omega \frac{\tilde{\alpha}_j(|u_{x_j}|)}{u_{x_j}}\,v_{x_j}\,d\mu = \sum_{j=1}^{n}\int_\Omega \tilde{\beta}_j^{-1}\circ\tilde{\alpha}_j(|u_{x_j}|)\,v_{x_j}d\mu, \tag{4.1}$$

$$\mathcal{B}u(v) := \int_\Omega \frac{\tilde{\alpha}_0(|u|)}{u}\,v\;d\mu = \int_\Omega \tilde{\beta}_0^{-1}\circ\tilde{\alpha}_0(|u|)\,v\,d\mu \tag{4.2}$$

for $u, v \in V$. In the following discussion, we are going to find a solution $u \in V$ of

$$\mathcal{A}(u) + \mathcal{B}(u) = f.$$

3. (Continuities of operators) First, for each $m \in \mathbb{N}$, we plan to find a solution $u_m \in V_m$ for the system:

$$\mathcal{A}(u_m)(w_j) + \mathcal{B}(u_m)(w_j) = f(w_j), \quad 1 \le j \le m.$$

To accomplish it, set

$$\pi_m^*(\phi)(x) := \phi(\pi_m(x)), \qquad \text{for } \phi \in V',\ x \in \mathbb{R}^m,$$

and we notice that $\pi_m^* \circ (\mathcal{A}+\mathcal{B}) \circ \pi_m : \mathbb{R}^m \to (\mathbb{R}^m)'$ is continuous because it is well-known that the dual linear operator $\pi_m^* : V' \to (\mathbb{R}^m)'$ of continuous operator π_m is continuous and from the facts that

Lemma 4.1. *The operators $\mathcal{A}, \mathcal{B} : V \to V'$ are continuous.*

Proof. For each $i = 1, 2, \cdots, n$, we note that $\|u_{x_i}\|_{P_{\alpha_i}} \le \|u\|_V$ (from the definition of the norm of V, see (3.2)). Hence the operators $\frac{\partial}{\partial x_i}$: $V \to P_{\alpha_i}(\Omega)$ are continuous. Also, for each $i = 0, 1, 2, \cdots, n$, we define an operator $T_i : P_{\alpha_i}(\Omega) \to P_{\beta_i}(\Omega)$ by

$$T_i(u) = \frac{\tilde{\alpha_i}(|u|)}{u}$$

for $u \neq 0$ and $T_i(0) = 0$. Then the operator T_i is well-defined because

$$\int_\Omega \tilde{\beta}_i(|T_i(u)|)d\mu = \int_\Omega \tilde{\beta}_i\left(\frac{\tilde{\alpha_i}(|u|)}{|u|}\right)d\mu = \int_\Omega \tilde{\alpha_i}(|u|)d\mu < \infty$$

for any $0 \neq u \in P_{\alpha_i}$. This implies that $\|T_i(u)\|_{P_{\beta_i}} = \tilde{\beta}_i^{-1}\circ\tilde{\alpha}_i(\|u\|_{P_{\alpha_i}})$, which yields the continuity of T_i.

Define $\left(\frac{\partial}{\partial x_i}\right)^* : P_{\beta_i} \to V'$ by

$$\left(\frac{\partial}{\partial x_i}\right)^* v(\phi) := \int_\Omega v(x)\phi_{x_i}(x)dx,$$

for $v \in P_{\beta_i}$ and $\phi \in V$. Then the operator $\left(\frac{\partial}{\partial x_i}\right)^*$ is continuous, since

$$\begin{aligned}\left\|\left(\frac{\partial}{\partial x_i}\right)^* v\right\|_{V'} &:= \sup_{\|\phi\|_V \neq 0} \frac{\left|\int_\Omega v(x)\phi_{x_i}(x)dx\right|}{\|\phi\|_V} \\ &\leq \hbar \sup_{\|\phi\|_V \neq 0} \frac{\|v\|_{P_{\beta_i}}\|\phi_{x_i}\|_{P_{\alpha_i}}}{\|\phi\|_V} \leq \hbar \|v\|_{P_{\beta_i}}.\end{aligned}$$

Therefore the composition maps $S_i := \left(\frac{\partial}{\partial x_i}\right)^* \circ T_i \circ \frac{\partial}{\partial x_i} : V \to V'$, $i = 1, 2, \cdots, n$ are continuous. We obtain the continuity of the operator $\mathcal{A}$ since it is a linear combination of the continuous operators S_i.

By virtue of Poincaré's inequality(Theorem 2.1), we have $\|u\|_{P_{\alpha_0}} \leq \|u\|_V$, so the inclusion $\iota_0 : V \hookrightarrow P_{\alpha_0}(\Omega)$ and its dual operator $\iota_0^* : P_{\beta_0}(\Omega) \to V'$ are continuous. Hence $\mathcal{B} = \iota_0^* \circ T_0 \circ \iota_0 : V \to V'$ is also continuous. □

4. Now we plan to show that the operator $\mathcal{F}_m : \mathbb{R}^m \to (\mathbb{R}^m)'$ defined by

$$\mathcal{F}_m(v) := \pi_m^* \circ (\mathcal{A} + \mathcal{B}) \circ \pi_m(v) - \pi_m^*(f), \quad v \in \mathbb{R}^m$$

has a zero.

5. (Coercivity of $\mathcal{A}+\mathcal{B}$) For it, we observe that the operator $\mathcal{A}+\mathcal{B} : V \to V'$ is coercive. That is to say, we observe that

$$\lim_{\|u\|_V \to \infty} \frac{(\mathcal{A}+\mathcal{B})u(u)}{\|u\|_V} = \infty. \tag{4.3}$$

In fact, for each $u \in V$, we have

$$\begin{aligned}\|u\|_V &:= \|u\|_{P_{\alpha_0}} + \sum_{j=1}^n \|u_{x_j}\|_{P_{\alpha_j}} \\ &\leq (n+1) \max\left\{\|u\|_{P_{\alpha_0}}, \|u_{x_1}\|_{P_{\alpha_1}}, \|u_{x_2}\|_{P_{\alpha_2}}, \cdots, \|u_{x_n}\|_{P_{\alpha_n}}\right\}.\end{aligned}$$

There is an index j_0 $(0 \leq j_0 \leq n)$ such that

$$\max\left\{\|u\|_{P_{\alpha_0}}, \|u_{x_1}\|_{P_{\alpha_1}}, \|u_{x_2}\|_{P_{\alpha_2}}, \cdots, \|u_{x_n}\|_{P_{\alpha_n}}\right\} = \|u_{x_{j_0}}\|_{P_{\alpha_{j_0}}},$$

wherein $\|u_{x_{j_0}}\|_{P_{\alpha_{j_0}}}$ may be replaced by $\|u\|_{P_{\alpha_0}}$ if $j_0 = 0$. Then we obtain

$$\begin{aligned}\frac{(\mathcal{A}+\mathcal{B})u(u)}{\|u\|_V} &= \frac{\tilde{\alpha_0}(\|u\|_{P_{\alpha_0}}) + \sum_{j=1}^n \tilde{\alpha_j}(\|u_{x_j}\|_{P_{\alpha_j}})}{\|u\|_{P_{\alpha_0}} + \sum_{j=1}^n \|u_{x_j}\|_{P_{\alpha_j}}} \\ &\geq \frac{\tilde{\alpha_{j_0}}(\|u_{x_{j_0}}\|_{P_{\alpha_{j_0}}})}{(n+1)\|u_{x_{j_0}}\|_{P_{\alpha_{j_0}}}} \\ &= \frac{1}{n+1}\,\tilde{\beta_{j_0}}^{-1} \circ \tilde{\alpha_{j_0}} \left(\|u_{x_{j_0}}\|_{P_{\alpha_{j_0}}}\right) \\ &\geq \frac{1}{n+1}\,\tilde{\beta_{j_0}}^{-1} \circ \tilde{\alpha_{j_0}} \left(\frac{1}{n+1}\|u\|_V\right).\end{aligned}$$

For any positive real number $L > 0$, there is $N > 0$ such that, for all $0 \leq j \leq n$,

$$\left(\tilde{\beta_j}^{-1} \circ \tilde{\alpha_j}\right)(t) \geq (n+1)L$$

whenever $t \geq N$. Hence for any $u \in V$ with $\|u\|_V \geq (n+1)N$, we have

$$\frac{1}{n+1}\,\tilde{\beta_{j_0}}^{-1} \circ \tilde{\alpha_{j_0}} \left(\frac{1}{n+1}\|u\|_V\right) \geq L.$$

Therefore $\frac{(\mathcal{A}+\mathcal{B})u(u)}{\|u\|_V}$ goes to infinity as $\|u\|_V \to \infty$, which explains the coercivity of the operator $\mathcal{A}+\mathcal{B} : V \to V'$.

6. (Solution of $\mathcal{F}_m(u) = 0$) We revisit the meaning of the limit (4.3) to find that for any $M > 0$, there exists a number $N > 0$ such that $(\mathcal{A}+\mathcal{B})u(u) > M\|u\|_V$ for any $\|u\|_V \geq N$. By taking $M := \|f\|_{V'}$, we have

$$f(u) \leq |f(u)| \leq \|f\|_{V'}\|u\|_V < (\mathcal{A}+\mathcal{B})u(u)$$

if $\|u\|_V \geq N$. Therefore for any $\|\pi_m(v)\|_V \geq N$, we obtain

$$\begin{aligned}\mathcal{F}_m(v)(v) &= \pi_m^* \circ (\mathcal{A}+\mathcal{B}) \circ \pi_m(v)(v) - \pi_m^* f(v) \\ &= (\mathcal{A}+\mathcal{B})(\pi_m(v))(\pi_m(v)) - f(\pi_m(v)) > 0.\end{aligned} \tag{4.4}$$

Since as $|v|$ goes to infinity, $\pi_m(v)$ also goes to infinity, and there is $r > 0$ such that $|v| \geq r$ implies $\|\pi_m(v)\|_V \geq N$. Therefore if we denote the Riesz map as $\mathcal{R} : \mathbb{R}^m \to (\mathbb{R}^m)'$ and put $\tilde{\mathcal{F}}_m := \mathcal{R}^{-1} \circ \mathcal{F}_m$, we get

$$\tilde{\mathcal{F}}_m(v) \cdot v = \mathcal{F}_m(v)(v) > 0, \qquad \text{if } |v| \geq r.$$

The Brouwer's fixed point theorem together with the continuity of $\tilde{\mathcal{F}}_m$ implies that $\tilde{\mathcal{F}}_m$ has a zero inside the ball $\{x \in \mathbb{R}^m : |x| \leq r\}$. Hence

$\mathcal{F}_m$ has a zero: there is $|\bar{u}_m| \le r$ satisfying $\mathcal{F}_m(\bar{u}_m) = 0$. Therefore, for each $m \ge 1$, we obtain

$$(\mathcal{A}+\mathcal{B}) \circ \pi_m(\bar{u}_m)(\pi_m(v)) - f(\pi_m(v)) = 0 \quad \text{for all } v \in \mathbb{R}^m,$$

or equivalently

$$(\mathcal{A}+\mathcal{B}) \circ \pi_m(\bar{u}_m)(w) - f(w) = 0 \quad \text{for all } w \in V_m.$$

Denote $\pi_m(\bar{u}_m) := u_m$, and we get

$$(\mathcal{A}+\mathcal{B})u_m = f \ \ in \ \ V_m'. \tag{4.5}$$

7. (Weak-convergence of solutions) Hölder's inequality together with the identity (2.2) implies

$$\begin{aligned}
|\mathcal{A}u(\phi)| &\le \sum_{j=1}^{n} \left| \int_\Omega \frac{\tilde{\alpha_j}(|\partial_{x_j} u|)}{\partial_{x_j} u} \, \partial_{x_j}\phi \, d\mu \right| \\
&\le \hbar \sum_{j=1}^{n} \tilde{\beta_j}^{-1}\!\left(\int_\Omega \tilde{\beta_j}\left(\frac{\tilde{\alpha_j}(|\partial_{x_j} u|)}{|\partial_{x_j} u|} \right) d\mu \right) \tilde{\alpha_j}^{-1}\!\left(\int_\Omega \tilde{\alpha_j}(|\partial_{x_j}\phi|) \, d\mu \right) \\
&= \hbar \sum_{j=1}^{n} \ \tilde{\beta_j}^{-1} \left(\int_\Omega \tilde{\alpha_j}(|\partial_{x_j} u|) \, d\mu \right) \|\partial_{x_j}\phi\|_{P_{\alpha_j}} \\
&= \hbar \sum_{j=1}^{n} \ \tilde{\beta_j}^{-1} \circ \tilde{\alpha_j}(\|\partial_{x_j} u\|_{P_{\alpha_j}}) \|\partial_{x_j}\phi\|_{P_{\alpha_j}} \\
&\le \hbar \sum_{j=1}^{n} \ \tilde{\beta_j}^{-1} \circ \tilde{\alpha_j}(\|u\|_V) \cdot \|\phi\|_V.
\end{aligned}$$

Similarly, we can have an estimate: for $u \in V$,

$$|\mathcal{B}u(\phi)| \le \tilde{\beta_0}^{-1} \circ \tilde{\alpha_0}(\|u\|_V) \cdot \|\phi\|_V.$$

Since $\tilde{\alpha}_j$ and $\tilde{\beta}_j^{-1}$ are continuous on $\mathbb{R}_+ (j = 0, 1, 2, \cdots, n)$, the image $(\mathcal{A}+\mathcal{B})(S)$ of a bounded set S in V is bounded in V'. It follows from (4.4) that $\|u_m\|_V \le N$ and so the sequence $\{\|(\mathcal{A}+\mathcal{B})u_m\|_{V'}\}$ is bounded. So the sequence $\{|f(u_m)|\}$ is bounded in $\mathbb{R}$. Then we can extract a subsequence $\{u_{m_k}\}$ of $\{u_m\}$ and take an element u in V satisfying the following properties

(1) u_{m_k} converges weakly to u in V

(2) $(\mathcal{A}+\mathcal{B})u_{m_k}$ converges weakly to f in V'

(3) $(\mathcal{A}+\mathcal{B})u_{m_k}(u_{m_k}) = f(u_{m_k})$ converges to $f(u)$.

8. (Monotonicity) For any real numbers a, b, we notice that

$$\left(\frac{\tilde{\alpha}_j(|a|)}{a} - \frac{\tilde{\alpha}_j(|b|)}{b}\right)(a-b)$$
$$= \left\{\tilde{\beta}_j^{-1} \circ \tilde{\alpha}_j(|a|)\mathrm{sgn}(a) - \tilde{\beta}_j^{-1} \circ \tilde{\alpha}_j(|b|)\mathrm{sgn}(b)\right\}(a-b) > 0,$$

because $\tilde{\beta}_j^{-1} \circ \tilde{\alpha}_j$ is monotone increasing. Hence we get

$$((\mathcal{A}+\mathcal{B})u - (\mathcal{A}+\mathcal{B})v)(u-v)$$
$$= \sum_{j=1}^{n} \int_\Omega \left(\frac{\tilde{\alpha}_j(|u_{x_j}|)}{u_{x_j}} - \frac{\tilde{\alpha}_j(|v_{x_j}|)}{v_{x_j}}\right)(u_{x_j} - v_{x_j})dx$$
$$+ \int_\Omega \left(\frac{\tilde{\alpha}_0(|u|)}{u} - \frac{\tilde{\alpha}_0(|v|)}{v}\right)(u-v)dx > 0,$$

for $u, v \in V$. Hence we have that for all $v \in V$,

$$(\mathcal{A}+\mathcal{B})u_{m_k}(u_{m_k}) - (\mathcal{A}+\mathcal{B})u_{m_k}(v)$$
$$- (\mathcal{A}+\mathcal{B})v(u_{m_k}) + (\mathcal{A}+\mathcal{B})v(v) \geq 0.$$

Then as k goes to infinity, we get

$$f(u) - f(v) - (\mathcal{A}+\mathcal{B})v(u) + (\mathcal{A}+\mathcal{B})v(v) \geq 0.$$

This is equivalent to

$$(f - \mathcal{A}v - \mathcal{B}v)(u-v) \geq 0, \tag{4.6}$$

and this holds for all $v \in V$.

9. Now for any $w \in V$ and any $t > 0$, we put $v := u - tw$, and then plug v into (4.6) to have $t(f - \mathcal{A}(u-tw) - \mathcal{B}(u-tw))w \geq 0$. Or

$$f(w) - \mathcal{A}(u-tw)w - \mathcal{B}(u-tw)w \geq 0.$$

By the continuity of $\mathcal{A}+\mathcal{B}$ (Lemma 4.1), we let $t \to 0$ to obtain

$$f(w) - (\mathcal{A}+\mathcal{B})u(w) \geq 0, \tag{4.7}$$

for all $w \in V$. We replace w in (4.7) with $-w$ to find

$$f(w) - (\mathcal{A}+\mathcal{B})u(w) \leq 0. \tag{4.8}$$

Combine (4.7) with (4.8), and we finally obtain

$$f = (\mathcal{A}+\mathcal{B})u.$$

The proof is now completed. □

References

[1] R. Adams, *Sobolev spaces*, 2^{nd} edition, Academic press, Amsterdam, 2003.
[2] L. C. Evans, *Partial differential equations*, American Math. Soc. 1998.
[3] H-C. Pak, *Existence of solutions for a nonlinear elliptic equation with general flux term*, Fixed Point Theory and Appl. **2011** (2011), Article ID 496417.
[4] H-C. Pak and S-H. Chang, *A new function space $L_\alpha(X)$- Version 1.1*, J. Chung-cheong Math. Soc. **21** (2008), 471-481.
[5] H-C. Pak and S-H. Chang, *Poincaré's inequality on a new function space $L_\alpha(X)$*, J. Chung-cheong Math. Soc. **22** (2009), 309-318.
[6] H-C. Pak and Y-J. Park, *Trace operator and a nonlinear boundary value problem in a new space*, Boundary Value Problems, **2014** (2014):153.
[7] H. Triebel, *Theory of function spaces II*, Birkhäuser, 1992.

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