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FUZZY TRANSITIVE FILTERS OF BE-ALGEBRAS

M. Sambasiva Rao*

Abstract. The concept of fuzzy transitive filters is introduced in BE-algebras. Some sufficient conditions are established for every fuzzy filter of a BE-algebra to become a fuzzy transitive filter. Some properties of fuzzy transitive filters are studied with respect to fuzzy relations and cartesian products.

1. Introduction

The notion of BE-algebras was introduced and extensively studied by H. S. Kim and Y. H. Kim in [4]. Some properties of filters of BE-algebras were studied by S. S. Ahn and K. S. So in [1] and then by H. S. Kim and Y. H. Kim in [4]. The concepts of a fuzzy set and a fuzzy relation on a set was initially defined by L. A. Zadeh [8]. Fuzzy relations on a group have been studied by Bhattacharya and Mukherjee [2]. In 1996, Y. B. Jun and S. M. Hong [3] discussed the fuzzy deductive systems of Hilbert algebras. In [5], the author introduced the notion of fuzzy filters in BE-algebras and discussed some related properties. Recently, the concept of fuzzy implicative filters [6] is introduced and studied the properties of these filters in BE-algebras. In [7], properties of fuzzy filters and also normal fuzzy filters are studied in BE-algebras.

In this paper, the notion of fuzzy transitive filters is introduced in BE-algebras. Some sufficient condition are derived for every fuzzy filter of a BE-algebra to become a fuzzy transitive filter. An extension property is derived for fuzzy transitive filters of BE-algebras. Some properties of fuzzy transitive filters are studied. Properties of fuzzy transitive filters are studied in terms of fuzzy relations and cartesian

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products. The notion of triangular normed fuzzy transitive filters are introduced and some of the properties of these filters are studied.

2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from [4], [5], [6] and [8] for ready reference of the reader.

Definition 2.1. [4] An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

1) $x * x = 1$,
2) $x * 1 = 1$,
3) $1 * x = x$, and
4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

Theorem 2.2. [4] *Let* $(X, *, 1)$ *be a* BE*-algebra. Then we have*

1) $x * (y * x) = 1$ *and*
2) $x * ((x * y) * y)) = 1$.

We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ implies $x * y = 1$. A BE-algebra X is called self-distributive if $x * (y * z) = (x*y)*(x*z)$ for all $x, y, z \in X$. A BE-algebra X is called commutative if $(x * y) * y = (y * x) * x$ for all $x, y \in X$.

Definition 2.3. [8] For any set X, a fuzzy set in X is a function $\mu : X \longrightarrow [0, 1]$.

Definition 2.4. [5] Let X be a BE-algebra. A fuzzy set μ of X is called a fuzzy filter if it satisfies the following properties, for all $x, y \in X$:

(F_1) $\mu(1) \geq \mu(x)$ and
(F_2) $\mu(y) \geq \min\{\mu(x), \mu(x * y)\}$.

Definition 2.5. [5] Let μ be a fuzzy set in a BE-algebra X. For any $\alpha \in [0, 1]$, the set $\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$ is called a level subset of X.

Definition 2.6. [6] A fuzzy relation on a set S is a fuzzy set $\mu : S \times S \longrightarrow [0, 1]$.

Definition 2.7. [6] Let μ be a fuzzy relation on a set S and ν a fuzzy set in S. Then μ is a fuzzy relation on ν if $\mu(x, y) \leq \min\{\nu(x), \nu(y)\}$ for all $x, y \in S$.

Definition 2.8. [6] Let μ, ν be two fuzzy sets in a BE-algebra X. The cartesian product of μ and ν is defined for all $x, y \in X$ as follows:

$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}.$$

3. Fuzzy transitive filters

In this section, the notion of fuzzy transitive filters is introduced in BE-algebras. Some properties of these fuzzy transitive filters are studied. Some sufficient conditions are derived for any fuzzy filter of a BE-algebra to become a fuzzy transitive filter.

DEFINITION 3.1. A fuzzy set μ of a BE-algebra X is called a fuzzy transitive filter of X if it satisfies the following properties:

(1) $\mu(1) \geq \mu(x)$,
(2) $\mu(x * z) \geq \min\{\mu(x * y), \mu(y * z)\}$ for all $x, y, z \in X$.

EXAMPLE 3.2. *Let $X = \{1, a, b, c\}$ be a non-empty set. Define a binary operation $*$ on X as follows:*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then $(X, *, 1)$ is a BE-algebra. Define a fuzzy set $\mu : X \longrightarrow [0, 1]$ as

$$\mu(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be easily verified that μ is a fuzzy transitive filter of X.

PROPOSITION 3.3. *Every fuzzy transitive filter of a BE-algebra is a fuzzy filter.*

Proof. Let μ be a fuzzy transitive filter of a BE-algebra X. Hence

$$\begin{aligned} \mu(y) &= \mu(1 * y) \\ &\geq \min\{\mu(1 * x), \mu(x * y)\} \\ &= \min\{\mu(x), \mu(x * y)\}. \end{aligned}$$

Therefore μ is a fuzzy filter of X. □

The converse of the above proposition is not true. That is a fuzzy filter of a BE-algebra is not a fuzzy transitive filter as shown in the following:

EXAMPLE 3.4. *Let $X = \{1, a, b, c, d\}$ be a non-empty set. Define a binary operation $*$ on X as follows:*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	1	1

.

Then it can be easily verified that $(X, *, 1)$ is a BE-algebra. Define a fuzzy set μ on X as follows:

$$\mu(x) = \begin{cases} 0.8, & \text{if } x = 1 \\ 0.3, & \text{otherwise} \end{cases}$$

for all $x \in X$. Then clearly μ is a fuzzy filter of X, but μ is not a fuzzy transitive filter of X since $\mu(b * c) < \min\{\mu(b * d), \mu(d * c)\}$.

We now derive some sufficient conditions for every fuzzy filter of a BE-algebra to become a fuzzy transitive filter.

THEOREM 3.5. *Every fuzzy filter μ of a BE-algebra X is a fuzzy transitive filter if it satisfies the following condition for all $x, y, z \in X$.*
(T_5) $\mu(z) \geq \min\{\mu(x * y), \mu(x * (y * z)\}$.

Proof. Let μ be a fuzzy filter of X such that the condition (T_5) holds for all $x, y, z \in X$. By interchanging y and z in condition (T_5), we get

$$\begin{aligned} \mu(y) &\geq \min\{\mu(x * (z * y)), \mu(x * z)\} \\ &\geq \min\{\mu(z * (x * y)), \mu(x * z)\} \\ &\geq \min\{\mu(x * y), \mu(x * z)\}. \end{aligned}$$

Since μ is a fuzzy filter, we get the following consequence:

$$\begin{aligned} \mu(x * z) &\geq \min\{\mu(y), \mu(y * (x * z))\} \\ &= \min\{\mu(y), \mu(x * (y * z)) \\ &\geq \min\{\min\{\mu(x * y), \mu(x * z)\}, \mu(y * z)\} \\ &= \min\{\mu(x * y), \mu(y * z)\}. \end{aligned}$$

Therefore μ is a fuzzy transitive filter of X. □

THEOREM 3.6. *Let μ be a fuzzy filter of a BE-algebra X which satisfies the following condition, for all $x, y, z \in X$:*

$$\mu(z) \geq \min\{\mu(x), \mu(x * y), \mu(y * z)\}.$$

Then μ is a fuzzy transitive filter of X.

Proof. Assume that the condition holds in X. Clearly $\mu(1) \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$. Clearly $\mu(y * z) \leq \mu((x * y) * (y * z))$. Since μ is a fuzzy filter, by the assumed condition, we get the following:

$$\begin{aligned}
\mu(x * z) &\geq \min\{\mu(x * y), \mu((x * y) * (x * z))\} \\
&\geq \min\{\mu(x * y), \min\{\mu(x * y), \mu((x * y) * (y * z))\}\} \\
&= \min\{\mu(x * y), \mu((x * y) * (y * z))\} \\
&\geq \min\{\mu(x * y), \mu(y * z)\}.
\end{aligned}$$

Therefore μ is a fuzzy transitive filter in X. □

As a converse of the above theorem, it can be easily observed that a fuzzy transitive filter of a BE-algebra can not satisfy the condition of the Theorem 2.5. We now derive a sufficient condition for a transitive filter of a BE-algebra to satisfies the condition of the Theorem 2.5.

THEOREM 3.7. *A fuzzy transitive filter μ of a commutative BE-algebra X satisfies the following condition for all $x, y, z \in X$.*

$$\mu((x * y) * (x * z)) \geq \min\{\mu(x * y), \mu((x * y) * (y * z))\}.$$

Proof. Let μ be a fuzzy transitive filter of X. Hence μ is a fuzzy filter of X. Let $x, y, z \in X$. Consider $(x * y) * ((x * y) * (x * z)) = t$ for brevity. Then we get

$$\begin{aligned}
\mu(t) &\geq \min\{\mu((x * y) * (y * z)), \mu((y * z) * ((x * y) * (x * z)))\} \\
&= \min\{\mu((x * y) * (y * z)), \mu((x * y) * ((y * z) * (x * z)))\} \\
&= \min\{\mu((x * y) * (y * z)), \mu((x * y) * (x * ((y * z) * z)))\} \\
&= \min\{\mu((x * y) * (y * z)), \mu((x * y) * (x * ((z * y) * y)))\} \\
&= \min\{\mu((x * y) * (y * z)), \mu((x * y) * ((z * y) * (x * y)))\} \\
&= \min\{\mu((x * y) * (y * z)), \mu((z * y) * ((x * y) * (x * y)))\} \\
&= \min\{\mu((x * y) * (y * z)), \mu((z * y) * 1))\} \\
&= \min\{\mu((x * y) * (y * z)), \mu(1)\} \\
&= \mu((x * y) * (y * z)).
\end{aligned}$$

Since μ is a fuzzy filter and from the above observation, we get

$$\begin{aligned}
\mu((x * y) * (x * z)) &\geq \min\{\mu(x * y), \mu((x * y) * ((x * y) * (x * z)))\} \\
&\geq \min\{\mu(x * y), \mu((x * y) * (y * z))\}.
\end{aligned}$$

This is the complete of the proof. □

In the following theorem, an extension property for fuzzy transitive filters of BE-algebras is obtained.

Theorem 3.8. *(Extension property for fuzzy transitive filters). Let μ and ν be two fuzzy filters of a commutative BE-algebra X such that $\mu(1) = \nu(1)$ and $\mu \subseteq \nu$ (i.e. $\mu(x) \leq \nu(x)$ for all $x \in X$). If μ is a fuzzy transitive filter, then so is ν.*

Proof. Assume that μ is a fuzzy transitive filter of X. Let $x, y, z \in X$. Then

$$\begin{aligned}
\nu((y*z)*((x*y)*(x*z))) &\geq \mu((y*z)*((x*y)*(x*z))) \\
&= \mu((x*y)*((y*z)*(x*z))) \\
&= \mu((x*y)*(x*((y*z)*z))) \\
&= \mu((x*y)*(x*((z*y)*y))) \\
&= \mu((x*y)*((z*y)*(x*y))) \\
&= \mu((z*y)*((x*y)*(x*y))) \\
&= \mu((z*y)*1) \\
&= \mu(1) \\
&= \nu(1).
\end{aligned}$$

Hence $\nu((y*z)*((x*y)*(x*z))) = \nu(1)$. Since ν is a fuzzy filter, we get

$$\begin{aligned}
\nu((x*y)*(x*z)) &\geq \min\{\nu(y*z), \nu((y*z)*((x*y)*(x*z)))\} \\
&= \min\{\nu(y*z), \nu(1)\} \\
&= \nu(y*z) \\
&\geq \min\{\nu(x*y), \nu((x*y)*(y*z))\}.
\end{aligned}$$

Hence by the above theorem, ν is a fuzzy transitive filter of X. □

4. Cartesian products

In this section, the homomorphic images of fuzzy transitive filters are studied in BE-algebras. Some properties of fuzzy transitive filters of BE-algebras are also studied with respect to cartesian products and fuzzy relations.

Definition 4.1. Let $f : X \longrightarrow Y$ be a homomorphism of BE-algebras and μ is a fuzzy set in Y. Then define a mapping $\mu^f : X \longrightarrow [0, 1]$ such that $\mu^f(x) = \mu(f(x))$ for all $x \in X$.

Clearly the above mapping μ^f is well-defined and a fuzzy set in X.

THEOREM 4.2. *Let $f : X \longrightarrow Y$ be an onto homomorphism of BE-algebras and μ is a fuzzy set in Y. Then μ is a fuzzy transitive filter in Y if and only if μ^f is a fuzzy transitive filter in X.*

Proof. Assume that μ is a fuzzy transitive of Y. For any $x \in X$, we have $\mu^f(1) = \mu(f(1)) = \mu(1') \geq \mu(f(x)) = \mu^f(x)$. Let $x, y, z \in X$. Then

$$\begin{aligned} \mu^f(x * z) &= \mu(f(x * z)) \\ &= \mu(f(x) * f(z)) \\ &\geq \min\{\mu(f(x) * f(y)), \mu(f(y) * f(z))\} \\ &= \min\{\mu(f(x * y)), \mu(f(y * z))\} \\ &= \min\{\mu^f(x * y), \mu^f(y * z)\}. \end{aligned}$$

Hence μ^f is a fuzzy transitive filter of X. Conversely, assume that μ^f is a fuzzy transitive filter of X. Let $x \in Y$. Since f is onto, there exists $y \in X$ such that $f(y) = x$. Then $\mu(1') = \mu(f(1)) = \mu^f(1) \geq \mu^f(y) = \mu(f(y)) = \mu(x)$. Let $x, y, z \in Y$. Then there exist $a, b, c \in X$ such that $f(a) = x, f(b) = y$ and $f(c) = z$. Hence we get

$$\begin{aligned} \mu(x * z) &= \mu(f(a) * f(c)) \\ &= \mu(f(a * c)) \\ &= \mu^f(a * c) \\ &\geq \min\{\mu^f(a * b), \mu^f(b * c)\} \\ &= \min\{\mu(f(a) * f(b)), \mu(f(b) * f(c))\} \\ &= \min\{\mu(x * y), \mu(y * z)\}. \end{aligned}$$

Therefore μ is a fuzzy transitive filter in Y. □

Let μ and ν be two fuzzy sets in a BE-algebra X. Then obviously $\mu \times \nu$ is a fuzzy relation on X and hence a fuzzy set in $X \times X$. For any two BE-algebras X and Y, define an operation $*$ on $X \times Y$ as follows:

$$(x, y) * (x', y') = (x * x', y * y') \text{ for all } x, x' \in X \text{ and } y, y' \in Y.$$

Then it can be easily observed that $(X \times Y, *, (1, 1))$ is a BE-algebra.

THEOREM 4.3. *Let μ and ν be two fuzzy transitive filters of a BE-algebra X. Then $\mu \times \nu$ is a fuzzy transitive filter in $X \times X$.*

Proof. Let $(x, y) \in X \times X$. Since μ, ν are fuzzy transitive filters in X, we get

$$\begin{aligned} (\mu \times \nu)(1, 1) &= \min\{\mu(1), \nu(1)\} \\ &\geq \min\{\mu(x), \nu(y)\} \qquad \text{for all } x, y \in X \\ &= (\mu \times \nu)(x, y). \end{aligned}$$

Let $(x,x'),(y,y'),(z,z') \in X \times X$. Since μ and ν are fuzzy transitive filters in X, we can obtain the following consequence.

$$\begin{aligned}(\mu\times\nu)((x,x')*(z,z')) &= (\mu\times\nu)(x*z,x'*z')\\ &= \min\{\mu(x*z),\nu(x'*z')\}\\ &\geq \min\{\min\{\mu(x*y),\mu(y*z)\},\min\{\nu(x'*y'),\nu(y'*z')\}\}\\ &= \min\{\min\{\mu(x*y),\nu(x'*y')\},\min\{\mu(y*z),\nu(y'*z')\}\}\\ &= \min\{(\mu\times\nu)(x*y,x'*y'),(\mu\times\nu)(y*z,y'*z')\}\\ &= \min\{(\mu\times\nu)((x,x')*(y,y')),(\mu\times\nu)((y,y')*(z*z'))\}.\end{aligned}$$

Therefore $\mu\times\nu$ is a fuzzy transitive filter in $X \times X$. □

Definition 4.4. Let ν be a fuzzy set in a BE-algebra X. Then the strongest fuzzy relation μ_ν is a fuzzy relation on X defined for all $x, y \in X$ as follows:

$$\mu_\nu(x,y) = \min\{\nu(x),\nu(y)\}.$$

Theorem 4.5. *Let ν be a fuzzy set in a BE-algebra X and μ_ν the strongest fuzzy relation on X. Then ν is a fuzzy transitive filter of X if and only if μ_ν is a fuzzy transitive filter of $X \times X$.*

Proof. Assume that ν is a fuzzy transitive filter of X. Then for any $(x,y) \in X \times X$,

$$\mu_\nu(x,y) = \min\{\nu(x),\nu(y)\} \leq \min\{\nu(1),\nu(1)\} = \mu_\nu(1,1).$$

Let $(x,x'),(y,y'),(z,z') \in X \times X$. Then we get the following:

$$\begin{aligned}&\mu_\nu((x,x')*(z,z'))\\ &\quad= \mu_\nu(x*z,x'*z')\\ &\quad= \min\{\nu(x*z),\nu(x'*z')\}\\ &\quad\geq \min\{\min\{\nu(x*y),\nu(y*z)\},\min\{\nu(x'*y'),\nu(y'*z')\}\}\\ &\quad= \min\{\min\{\nu(x*y),\nu(x'*y')\},\min\{\nu(y*z),\nu(y'*z')\}\}\\ &\quad= \min\{\mu_\nu(x*y,x'*y'),\mu_\nu(y*z,y'*z')\}\\ &\quad= \min\{\mu_\nu((x,x')*(y,y')),\mu_\nu((y,y')*(z,z'))\}.\end{aligned}$$

Therefore μ_ν is a fuzzy transitive filter of $X \times X$. Conversely, assume that μ_ν is a fuzzy transitive filter of $X \times X$. Then

$$\nu(1) = \min\{\nu(1),\nu(1)\} = \mu_\nu(1,1) \geq \mu_\nu(x,x) = \min\{\nu(x),\nu(x)\} = \nu(x)$$

for all $x \in X$. Hence it yields that $\nu(x) \leq \nu(1)$ for all $x \in X$. Let $x, y, z \in X$. Then we have the following consequence.

$$\begin{aligned}
\nu(x * z) &= \min\{\nu(x * z), \nu(1)\} \\
&= \mu_\nu(x * z, 1) \\
&= \mu_\nu((x, 1) * (z * 1)) \\
&\geq \min\{\mu_\nu((x, 1) * (y, 1)), \mu_\nu((y, 1) * (z, 1))\} \\
&= \min\{\mu_\nu(x * y, 1), \mu_\nu(y * z, 1)\} \\
&= \min\{\min\{\nu(x * y), \nu(1)\}, \min\{\nu(y * z), \nu(1)\}\} \\
&= \min\{\nu(x * y), \nu(y * z)\}.
\end{aligned}$$

Therefore ν is a fuzzy transitive filter of X. □

5. Triangular normed fuzzification

It this section, the notion of triangular normed fuzzy transitive filters in BE-algebras. Some sufficient conditions are derived for every triangular normed fuzzy filter of a BE-algebra to become a triangular normed fuzzy transitive filter.

Definition 5.1. Let $I = [0, 1]$. Then by a t-norm T, we mean a function $T : I \times I \longrightarrow I$ satisfying the following:

(1) $T(x, x) = 1$,
(2) $y \leq z$ implies $T(x, y) \leq T(x, z)$,
(3) $T(x, y) = T(y, x)$, and
(4) $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in I$.

Let $I = [0, 1]$ and $T : I \times I \longrightarrow I$ a function defined as follows:

$$T_m(x) = min\{x, y\} = \begin{cases} x & \text{if } x \leq y \\ y & y < x. \end{cases}$$

Then clearly T_m is a t-norm on I. For any t-norm T on I, it can be easily observed that $T(\alpha, \beta) \leq min\{\alpha, \beta\}$ for all $\alpha, \beta \in I$. For any t-norm T on I, define $\Delta_T = \{\alpha \in I \mid T(\alpha, \alpha) = \alpha\}$. A t-norm T is continuous if T is a continuous function.

Definition 5.2. A fuzzy set μ in a BE-algebra X is said to satisfy *imaginable property* if $T(\mu(x), \mu(x)) = \mu(x)$ for all $x \in X$.

Definition 5.3. A fuzzy set μ of a BE-algebra X is called a fuzzy filter of X with respect to a t-norm T (simply called T-fuzzy filter) if it satisfies:

(1) $\mu(1) \geq \mu(x)$ for all $x \in X$,
(2) $\mu(y) \geq T(\mu(x), \mu(x * y)\}$ for all $x, y \in X$.

Definition 5.4. A fuzzy set μ of a BE-algebra X is called a fuzzy transitive filter X with respect to a t-norm T (simply called T-fuzzy transitive filter) if it satisfies:

(1) $\mu(1) \geq \mu(x)$ for all $x \in X$,
(2) $\mu(x * z) \geq T(\mu(x * y), \mu(y * z)\}$ for all $x, y, z \in X$.

Proposition 5.5. *Every T-fuzzy transitive filter of a BE-algebra is a T-fuzzy filter.*

Proof. Let μ be a T-fuzzy transitive filter of F. Let $x, y \in X$. Then $\mu(y) = \mu(1 * y) \geq T(\mu(1 * x), \mu(x * y)) = T(\mu(x), \mu(x * y))$. Hence μ is a T-fuzzy filter of X. □

In general, the converse of the above proposition is not true. However, in the following, we derive some sufficient conditions for every T-fuzzy filter of a BE-algebra to become a T-fuzzy transitive filter.

Theorem 5.6. *Every T-fuzzy filter μ of a BE-algebra X is a T-fuzzy transitive filter if it satisfies the following condition for all $x, y \in X$.*

(TF_1) $\mu(y) \geq \mu(x * (x * y))$

Proof. Let μ be a T-fuzzy filter of X such that the condition (TF_1) holds for all $x, y \in X$. Let $x, y, z \in X$. Then we get the following:

$$\begin{aligned}
\mu(x * z) &\geq T(\mu(y), \mu(y * (x * z))) \\
&= T(\mu(y), \mu(x * (y * z))) \\
&\geq T(\mu(y), \mu(y * z)) \\
&\geq T(\mu(x * (x * y)), \mu(y * z)) \\
&\geq T(\mu(x * y), \mu(y * z)).
\end{aligned}$$

Therefore μ is a T-fuzzy transitive filter of X. □

Theorem 5.7. *Every T-fuzzy filter μ of a BE-algebra X is a T-fuzzy transitive filter if it satisfies the following condition for all $x, y, z \in X$.*

(TF_2) $\mu((x * y) * z) \geq \mu(x * (y * z))$.

Proof. Let μ be a T-fuzzy filter of X such that the condition (TF_2) holds for all $x, y, z \in X$. Let $x, y, z \in X$. Then we get the following:

$$\begin{aligned}
\mu(x * z) &\geq \mu(z) \\
&= T(\mu(x * y), \mu((x * y) * z)) \\
&\geq T(\mu(x * y), \mu(x * (y * z))) \\
&\geq T(\mu(x * y), \mu(y * z)).
\end{aligned}$$

Therefore μ is a T-fuzzy transitive filter of X. □

Lemma 5.8. *Every imaginable T-fuzzy transitive filter of a BE-algebra X is order preserving.*

Proof. Let μ be a T-fuzzy transitive filter of X. Let $x, y \in L$ be such that $x \leq y$. Then $x * y = 1$. Hence we get the following:

$$\begin{aligned} \mu(y) &= \mu(1 * y) \\ &\geq T(\mu(1 * x), \mu(x * y)) \\ &= T(\mu(x), \mu(1)) \geq T(\mu(x), \mu(x)) \\ &= \mu(x). \end{aligned}$$

Therefore μ is order preserving. □

Proposition 5.9. *Every fuzzy transitive filter is a T-fuzzy transitive.*

Proof. Let μ be a fuzzy transitive filter of a BE-algebra X. For $x, y, z \in X$, we have

$$\mu(x * z) \geq \min\{\mu(x * y), \mu(y * z)\} \geq T(\mu(x * y), \mu(y * z)).$$

Therefore μ is a T-fuzzy transitive filter in X. □

The converse of the above proposition is not true. However, we derive a sufficient condition for every T-fuzzy transitive filter to become a fuzzy transitive filter.

Theorem 5.10. *Every imaginable T-fuzzy transitive filter of a BE-algebra X is a fuzzy transitive filter.*

Proof. Let μ be an imaginable T-fuzzy filter in X. Let $x, y, z \in X$. Then clearly $\mu(x * z) \geq T(\mu(x * y), \mu(y * z))$. Since μ is imaginable and min $\{\mu(x * y), \mu(y * z)\} \leq \mu(x * y), \mu(y * z)$, we get the following:

$$\begin{aligned} &\min\{\mu(x * y), \mu(y * z)\} \\ &= T(\min\{\mu(x * y), \mu(y * z)\}, \min\{\mu(x * y), \mu(y * z)\}) \\ &\leq T(\min\{\mu(x * y), \mu(y * z)\}, \mu(y * z)\}) \\ &\leq T(\mu(x * y), \mu(y * z)\}) \\ &\leq \min\{\mu(x * y), \mu(y * z)\}. \end{aligned}$$

Hence $T(\mu(x * y), \mu(y * z)) = \min\{\mu(x * y), \mu(y * z)\}$. Thus

$$\mu(x * z) \geq T(\mu(x * y), \mu(y * z)) = \min\{\mu(x * y), \mu(y * z)\}.$$

Therefore μ is a fuzzy transitive filter of X. □

Definition 5.11. Let X and X' be any two set and $f : X \longrightarrow X'$ be a function. If ν is a fuzzy set in $f(X)$, then the fuzzy set μ in X defined for all $x \in X$ by

$$\mu(x) = \nu(f(x))$$

is called the pre-image of ν under f and is denoted by $f^{-1}(\nu)$. Clearly $f^{-1}(\nu) = \nu \circ f$.

Theorem 5.12. *Let $f : X \longrightarrow Y$ be an onto homomorphism of BE-algebras. If ν is a T-fuzzy transitive filter of Y, then $f^{-1}(\nu)$ is a T-fuzzy transitive filter of X. Moreover, if ν satisfies the imaginable property then so does $f^{-1}(\nu)$.*

Proof. For any $x \in X$, $f^{-1}(\nu)(1) = \nu(f(1)) = \nu(1) \geq \nu(f(x)) = f^{-1}(\nu)(x)$. Let $x, y, z \in X$. Then

$$f^{-1}(\nu)(x*z) = \nu(f(x*z)) = \nu(f(x)*f(z)) \geq T(\nu(f(x)*a), \nu(a*f(z)))$$

for some $a \in Y$. Since f is onto, there exists $y_a \in X$ such that $f(y_a) = a$. Now

$$\begin{aligned} f^{-1}(\nu)(x*z) &\geq T(\nu(f(x)*a), \nu(a*f(z))) \\ &= T(\nu(f(x)*f(y_a)), \nu(f(y_a)*f(z))) \\ &= T(\nu(f(x*y_a)), \nu(f(y_a*z))) \\ &= T(f^{-1}(v)(x*y_a), f^{-1}(\nu)(y_a*z)). \end{aligned}$$

Since a is arbitrary, this inequality holds for all $y \in X$. Hence it yields

$$f^{-1}(\nu)(x*z) \geq T(f^{-1}(v)(x*y), f^{-1}(\nu)(y*z)).$$

Therefore $f^{-1}(\nu)$ is a T-fuzzy transitive filter of X. Suppose ν satisfies the imaginable property. Then we get

$$T(f^{-1}(v)(x), f^{-1}(v)(x)) = T(\nu(f(x)), \nu(f(x))) = \nu(f(x)) = f^{-1}(\nu)(x).$$

Therefore $f^{-1}(\nu)$ satisfies the imaginable property. □

Definition 5.13. Let X and X' be any two sets and $f : X \longrightarrow X'$ be any function. If μ is a fuzzy set in X, then the fuzzy set ν in X' defined for all $x \in X'$ by

$$\nu(x) = \sup_{t \in f^{-1}(x)} \mu(t)$$

is called the image of μ under f and is denoted by $f(\mu)$.

We say that a fuzzy set μ in X has the sup property if, for any subset A of X, there exists $a_0 \in A$ such that $\mu(a_0) = \sup_{a \in A} \mu(a)$.

THEOREM 5.14. *Let $f : X \longrightarrow Y$ be a homomorphism of a BE-algebra X onto a BE-algebra Y. Let μ be a T-fuzzy transitive filter of X which has the sup property. Then the image of μ under f is a T-fuzzy transitive filter of Y.*

Proof. Since $1 \in f^{-1}(1)$, we get $f(\mu)(1) = \sup_{t \in f^{-1}(1)} \mu(t) = \mu(1) \geq \mu(x)$ for all $x \in X$. Hence $f(\mu)(1) \geq \sup_{t \in f^{-1}(a)} \mu(t) = f(\mu)(a)$ for all $a \in Y$. For any $a, b, c \in Y$, let $x_a \in f^{-1}(a), x_b \in f^{-1}(b)$ and $x_c \in f^{-1}(c)$ be such that $\mu(x_a * x_c) = \sup_{t \in f^{-1}(a*c)} \mu(t), \mu(x_b * x_c) = \sup_{t \in f^{-1}(b*c)} \mu(t)$ and $\mu(x_a * x_b) = \sup_{t \in f^{-1}(a*b)} \mu(t)$. Then we get the following consequence:

$$\begin{aligned} f(\mu)(a * c) &= \sup_{t \in f^{-1}(a*c)} \mu(t) \\ &= \mu(x_a * x_c) \\ &\geq T(\mu(x_a * x_b), \mu(x_b * x_c)) \\ &= T(\sup_{t \in f^{-1}(a*b)} \mu(t), \sup_{t \in f^{-1}(b*c)} \mu(t)) \\ &= T(f(\mu)(a * b), f(\mu)(b * c)). \end{aligned}$$

Therefore $f(\mu)$ is a T-fuzzy transitive filter of Y. □

DEFINITION 5.15. Let μ and ν be two fuzzy sets in a BE-algebra X. Then the T-product of μ and ν is defined by $(\mu \times \nu)_T(x) = T(\mu(x), \nu(x))$ for all $x \in X$

DEFINITION 5.16. Let T and S be two t-norms on $I = [0, 1]$. Then the t-norm S is said to dominate the t-norm T if for all $\alpha, \beta, \gamma, \delta \in [0, 1]$, the following satisfies:

$$S(T(\alpha, \gamma), T(\beta, \delta)) \geq T(S(\alpha, \beta), S(\gamma, \delta))$$

THEOREM 5.17. *Let μ and ν be T-fuzzy transitive filters of a BE-algebra X. If a t-norm S dominates T, then the produce $(\mu \times \nu)_S$ is a T-fuzzy transitive filter of X.*

Proof. For any $x \in X$, we can get that $(\mu \times \nu)_S(1) = S(\mu(1), \nu(1)) \geq S(\mu(x), \nu(x)) = (\mu \times \nu)_S(x)$. Let $x, y, z \in X$. Then

$$\begin{aligned} (\mu \times \nu)_S(x * z) &= S(\mu(x * z), \nu(x * z)) \\ &\geq S(T(\mu(x * y), \mu(y * z)), T(\nu(x * y), \nu(y * z))) \\ &\geq T(S(\mu(x * y), \nu(x * y)); S(\mu(y * z), \nu(y * z))) \\ &= T((\mu \times \nu)_S(x * y), (\mu \times \nu)_S(x * y)). \end{aligned}$$

Therefore $(\mu \times \nu)_S$ is a T-fuzzy transitive filter of X. □

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Department of Mathematics
MVGR College of Engineering
Vizianagaram, Andhra Pradesh 535 005, India
E-mail: mssraomaths35@rediffmail.com