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CHAIN TRANSITIVE SETS AND DOMINATED SPLITTING FOR GENERIC DIFFEOMORPHISMS

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Abstract. Let $f : M \to M$ be a diffeomorphism of a compact smooth manifold M. In this paper, we show that C^1 generically, if a chain transitive set Λ is locally maximal then it admits a dominated splitting. Moreover, C^1 generically if a chain transitive set Λ of f is locally maximal then it has zero entropy.

1. Introduction

Let M be a closed C^∞ Riemannian manifold with $\dim M \geq 2$, and let Diff(M) be the space of diffeomorphisms of M endowed with the C^1 topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM.

Let Λ be a closed f invariant set. We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. In differentiable dynamical systems, the notion is an important concept. For that, many results published in [3, 6, 7, 8, 9, 10, 11, 12, 14]. In fact, they were used to various dynamical properties (expansive, continuum-wise expansive, continuum-wise fully expansive, shadowing, inverse shadowing, average shadowing, asymptotic average shadowing, etc). In the paper, we consider that if a closed invariant set which is locally maximal then it admits a dominated splitting for C^1 generic sense. An invariant closed set Λ is called *chain transitive* if for

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any $\delta > 0$ and $x, y \in \Lambda$, there is δ-pseudo orbit $\{x_i\}_{i=0}^{n} (n \geq 1) \subset \Lambda$ such that $x_0 = x$ and $x_n = y$.

We say that Λ is *locally maximal* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$. Here the neighborhood U is called *locally maximal neighborhood* of Λ. We say that a subset $\mathcal{G} \subset \mathrm{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\mathrm{Diff}(M)$; in this case $\mathcal{G}$ is dense in $\mathrm{Diff}(M)$. A property "P" is said to be *(C^1)-generic* if "P" holds for all diffeomorphisms which belong to some residual subset of $\mathrm{Diff}(M)$. We use the terminology "for C^1 generic f" to express "there is a residual subset $\mathcal{G} \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G} \ldots$". In the paper, we show the following which is a main theorem.

Theorem A *For C^1 generic $f \in \mathrm{Diff}(M)$, if a chain transitive set Λ of f is locally maximal then it admits a dominated splitting.*

2. Proof of Theorem A

Let M be as before, and let $f \in \mathrm{Diff}(M)$. We say that $p \in P(f)$ with period $\pi(p)$ is a *sink* if all the eigenvalues of $D_p f^{\pi(p)}$ are less than 1, and $p \in P(f)$ with period $\pi(p)$ is a *source* if all eigenvalues of $D_p f^{\pi(p)}$ is greater than 1.

THEOREM 2.1. [1, Theorem 2.1] *There is a residual set $\mathcal{G} \subset \mathrm{Diff}(M)$ such that given any chain transitive set Λ of $f \in \mathcal{G}$ then either there is a dominated splitting over Λ or the set Λ is contained in the Hausdorff limit of a sequence of periodic sinks or sources of f.*

We also recall that the Hausdoroff distance between two compact subsets A and B of M is given by:

$$d_H(A, B) = \max\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\}.$$

LEMMA 2.2. *There is a residual set $\mathcal{G} \subset \mathrm{Diff}(M)$ such that for any chain transitive set Λ of $f \in \mathcal{G}$, if Λ is locally maximal and it does not admits a dominated splitting then Λ contains a sink or a source.*

Proof. Let $f \in \mathcal{G}$ and let U be a locally maximal neighborhood of Λ. Suppose that Λ does not admits a dominated splitting. Since Λ is compact, there is $\eta > 0$ such that $\Lambda \subset B_\eta(\Lambda) \subset U$. Since Λ does not admits a dominated splitting, by Theorem 2.1, there is a sequence of periodic sinks $Orb(s_n)$ such that $Orb(s_n)$ is the Hausdorff limit to Λ.

For sufficiently large n, we have $d_H(Orb(s_n), \Lambda) < \eta/2$. Then there is a periodic sink $s \in Orb(s_n)$ such that $s \in B_{\eta/2}(\Lambda) \subset U$. Since Λ is a locally maximal in U, we know that $s \in \Lambda$. The case of a sequence of periodic source is similar. □

Proof of Theorem A. Let $f \in \mathcal{G} \cap \mathcal{D}$. Assume that a locally maximal chain transitive set Λ does not admit a dominated splitting. Since Λ does not admits a dominated splitting, by Lemma 2.2 we know that Λ contains a sink or a source. Since Λ is a chain transitive set of f by [13, Lemma 2.1], Λ has neither sinks nor sources. This is a contradiction by Theorem 2.1. Thus C^1 generically, a chain transitive set Λ admits a dominated splitting if Λ is locally maximal. □

A compact f invariant set Δ is said to be *transitive* if there is a point $x \in \Delta$ such that $\omega(x) = \Delta$, where $\omega(x)$ is the omega limit set of x. In general, a chain transitive set is not a transitive set (see [4, Example 1.5]).

LEMMA 2.3. [4, Corollary 2] *There is a residual set $\mathcal{C} \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{C}$, a chain transitive set Λ of f is a transitive set Δ of f.*

We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \le C\lambda^n \quad \text{and} \quad \|D_x f^{-n}|_{E^u_x}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. Let p be a hyperbolic periodic point of f. Then the homoclinic class of $Orb(p)$ is the set $H_f(p) = H_f(Orb(p)) = W^s(p) \pitchfork W^u(p)$, and a neighborhood V of $Orb(p)$, then the homoclinic class of p relative to V is the set

$$H_V(Orb(p)) = H_V(p) = \overline{\{x \in W^s(p) \pitchfork W^u(p) : Orb(x) \subset V\}}.$$

It is clear that if the homoclinic class $H_f(p)$ is locally maximal then it is relative to V, that is, $H_f(p) = H_V(p)$.

LEMMA 2.4. [2, Thereom 4.10] *There is a residual set $\mathcal{T} \subset \mathrm{Diff}(M)$ such that for any transitive set Λ of f if the transitive set Λ is locally maximal then $\Lambda = H_f(p)$ for some periodic point p of f.*

It is well known that if a diffeomorphism is More-Smale then it has zero entropy. The set of diffomorphisms having zero entropy is contained in the closure of the Morse-Smale diffeomorphism. Denote by $\mathcal{MS}$ the set of all Mores-Smale diffeomorphisms. $U = \mathrm{Diff}(M^2) \setminus \overline{\mathcal{MS}}$. Then

Pujals and Smbarino [15] proved that there exists an open and dense set $\mathcal{R} \subset U$ such that every $f \in \mathcal{R}$ has a transversal homoclinic orbit. In particular, the closure of the interior of the set formed by the diffeomorphisms having zero entropy, is equal to $\overline{\mathcal{MS}}$. In the paper, we have zero entropy if for a C^1 generic diffeomorphism f, a chain transitive set is locally maximal.

THEOREM 2.5. *For C^1 generic $f \in \mathrm{Diff}(M)$, if any chain transitive set Λ of f is locally maximal then it has zero entropy.*

Proof. Let $f \in \mathcal{C} \cap \mathcal{T}$. Since $f \in \mathcal{C} \cap \mathcal{T}$, by Lemmas 2.3 and 2.4 a locally maximal chain transitive set $\Lambda = H_f(p)$. Since $H_f(p)$ a locally maximal homclinic class, the homoclinic class $H_f(p)$ is a relative homoclinic class. Since the homoclinic class $H_f(p)$ is relative homoclinic class, by [2, Theorem 3.1] there is a measure $\mu \in \mathcal{M}_f(H_f(p))$ such that $h_\mu(f) = 0$. $\square$

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