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# CALABI–YAU THREEFOLDS FROM BUILDING BLOCKS OF $G_2$-MANIFOLDS

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Abstract. We construct Calabi–Yau threefolds by smoothing normal crossing varieties, which are made from the building blocks of $G_2$-manifolds. We compute the Hodge numbers of those Calabi–Yau threefolds. Some of those Hodge number pairs $(h^{1,1}, h^{1,2})$ do not overlap with those of Calabi–Yau threefolds constructed in the toric setting.

## 1. Introduction

A *Calabi–Yau manifold* is a compact Kähler manifold with trivial canonical class such that the intermediate cohomologies of its structure sheaf are all trivial ($h^i(M, \mathcal{O}_M) = 0$ for $0 < i < \dim(M)$). A $K3$ surface is a Calabi–Yau twofold in this definition. Calabi–Yau threefolds have attracted much interest from both of mathematics and physics but the classification of them is widely open. Even boundedness of their Hodge numbers is still unknown. Thus developing methods of constructing Calabi–Yau threefolds and finding new examples are of interest. If a normal crossing variety is the central fiber of a semistable degeneration of Calabi–Yau manifolds, it can be regarded as a member in a deformation family of those Calabi–Yau manifolds. A remarkable difference between two-dimensional cases of K3 surfaces and thee-dimensional cases is that there are multiple deformation types for Calabi–Yau threefolds. So building a normal crossing variety smoothable to a Calabi–Yau threefold can be regarded as building a deformation type of Calabi–Yau threefolds. In this note, we consider the construction of Calabi–Yau

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threefolds by smoothing normal crossing varieties which are made from building blocks of $G_2$-manifolds. A $G_2$*-manifold* is a Riemannian manifold of real 7-dimension whose holonomy group is the $G_2$ group. A construction method of compact $G_2$-manifolds has been proposed by A. Kovalev ([4]) — the 'twisted connected sum'. It starts with two smooth projective threefolds $Z_1, Z_2$ which have smooth $K3$ surfaces $S_1, S_2$ in their anticanonical systems respectively. It is also required that the normal bundle in $Z_i$ of $S_i$ is trivial. $G_2$-manifolds are obtained by doing a topological surgery on $(Z_1 - S_1) \times S^1$, $(Z_2 - S_2) \times S^1$ ([4]). The threefolds $Z_i$'s are called 'building blocks' of $G_2$-manifold and several exotic examples of them were introduced in [2]. Some similarity between this construction of $G_2$-manifolds by the twisted connected sum and that of Calabi–Yau threefolds by smoothing normal crossing varieties of two components has been pointed out ([9]). We build normal crossing varieties from those $Z_i$'s and show that they are smoothable to Calabi–Yau threefolds. We compute Hodge numbers of Calabi–Yau threefolds (Table 1). It turns that some of the Hodge number pairs $(h^{1,1}, h^{1,2})$ do not overlap with those of Calabi–Yau threefolds constructed in the toric setting and two of them seems a mirror pair. Note that most of the known Hodge number pairs of Calabi–Yau threefolds come from complete intersections in toric varieties or some crepant resolutions of them.

## 2. Method and examples

We start with defining some terminologies. Let $Y = Y_1 \cup Y_2$ be a variety that is composed of two smooth varieties of dimension three. $Y$ is called a normal crossing if, near any point $p \in Y_1 \cap Y_2$, $Y$ is locally isomorphic to

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 | x_1 x_2 = 0\}$$

with $p$ corresponding to the origin in $\mathbb{C}^4$. Then $D := Y_1 \cap Y_2$ is a smooth surface. Suppose that there is a proper map $\pi : \mathcal{X} \to \Delta$ from a Kähler manifold $\mathcal{X}$ of dimension four onto the unit disk $\Delta = \{t \in \mathbb{C} | \|t\| \leq 1\}$ such that the fiber $\mathcal{X}_t = \pi^{-1}(t)$ is smooth threefold for every $t \neq 0$ and $\mathcal{X}_0 = Y$. We say that $Y$ is a semistable degeneration of a smooth threefold $M = \mathcal{X}_t$ ( $t \neq 0$) and that $M$ is a semistable smoothing (simply smoothing) of $Y$. $Y$ is also said to be smoothable to $M$. Let us give more precise description of the building block $Z$ of $G_2$-manifolds. Let $Z$ be a compact Kähler threefold with a smooth $K3$ surface $S$ in the anticanonical system $| - K_Z|$ such that the normal bundle $N_{S/Z}$ is holomorphically trivial. Suppose further that $Z$ is simply-connected and the fundamental group of $Z - S$ is finite. This is the definition of threefolds which Kovalev used in his twisted connected sum method ([4]). It is not hard to see that $Z$ is actually projective (Proportion 2.2 in [5]). We let

$$k(Z) = \operatorname{rank}(\operatorname{im}(H^2(Z,\mathbb{Z}) \to H^2(S,\mathbb{Z}))).$$

For a given building block $Z$ of $G_2$-manifolds with $K3$ surface $S$ in its anticanonical system, prepare two copies $Y_1, Y_2$ of $Z$. We denote by $S_i$ the copy in $Y_i$ of $S$. We construct a normal crossing variety $Y = Y_1 \cup_{S_1 \sim S_2} Y_2$, where '$\cup_{S_1 \sim S_2}$' means pasting $Y_1, Y_2$ along $S_1$ and $S_2$. We simply denote $Y$ by $Y = Y_1 \cup Y_2$ and let $D = Y_1 \cap Y_2$. Then $D$ isomorphic to $S$.

THEOREM 2.1. *The normal crossing variety $Y$ is smoothable to a Calabi–Yau threefold $M_Y$ and the Hodge numbers of $M_Y$ are given by*

1. $h^{1,1}(M_Y) = 2h^2(Z) - 1 - k(Z)$,
2. $h^{1,2}(M_Y) = h^{2,1}(M_Y) = 21 + h^3(Z) - k(Z)$.

*Proof.* We will use Theorem 4.2 in [3] (see p. 716 of [8] for detailed conditions). Note that $Z$ is projective. So there is an ample divisor $H$ on $Z$. Denote the copy in $Y_i$ of $H$ by $H_i$. Clearly $H_1|_D = H_2|_D$. So $Y$ is projective. Both of the normal bundles $N_{D/Y_1}, N_{D/Y_2}$ are trivial. Since $D = Y_1 \cap Y_2$ is an anticanonical divisor of $Y_i$ for $i = 1, 2$, $Y$ has trivial dualizing sheaf. From the exact sequence of structure sheaves

$$0 \to \mathcal{O}_Y \to \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} \to \mathcal{O}_D \to 0,$$

we have an exact sequence

$$H^1(D, \mathcal{O}_D) \to H^2(Y, \mathcal{O}_Y) \ \to H^2(Y, \mathcal{O}_{Y_1}) \oplus H^2(Y, \mathcal{O}_{Y_1}).$$

Using the fact that $H^1(D, \mathcal{O}_D) = H^2(Y_1, \mathcal{O}_{Y_1}) = H^2(Y_2, \mathcal{O}_{Y_2}) = 0$, we have

$$H^2(Y, \mathcal{O}_Y) = 0.$$

Since $Y_i$ is simply-connected, $H^1(Y_i, \mathcal{O}_{Y_i}) = 0$ for $i = 1, 2$. We showed that all the conditions in Theorem 4.2 in [3] are satisfied. So $Y$ is smoothable to Calabi–Yau threefold $M_Y$. By Corollary 8.2 in [8],

$$\begin{aligned} h^{1,1}(M_Y) &= h^2(Y_1) + h^2(Y_2) - 1 \\ &\qquad - \operatorname{rank}(\operatorname{im}(H^2(Y_1,\mathbb{Z}) \oplus H^2(Y_2,\mathbb{Z}) \to H^2(D,\mathbb{Z})) \\ &= 2h^2(Z) - 1 - k(Z) \end{aligned}$$

Table 1. Calabi–Yau threefolds $M_Y$'s from the building blocks $Z$'s

| Example No. | $h^2(Z)$ | $h^3(Z)$ | $k(Z)$ | $h^{1,1}(M_Y)$ | $h^{1,2}(M_Y)$ |
|---|---|---|---|---|---|
| 7.3 | 3 | 50 | 2 | 3 | 69 |
| 7.4 | 3 | 44 | 2 | 3 | 63 |
| 7.5 | 3 | 34 | 2 | $3^*$ | $53^*$ |
| 7.6 | 3 | 36 | 2 | $3^*$ | $55^*$ |
| 7.7 | 17 | 6 | 16 | $17^*$ | $11^*$ |
| 7.8 | 5 | 24 | 1 | 8 | 44 |
| 7.9 | 4 | 30 | 3 | 4 | 48 |
| 7.10 | 11 | 24 | 10 | 11 | 35 |
| 7.11 | 23 | 0 | 10 | 35 | 11 |
| 7.12 | 3 | 46 | 2 | 3 | 65 |

and

$$\begin{aligned} h^{1,2}(M_Y) &= 21 + h^{1,2}(Y_1) + h^{1,2}(Y_2) - k(Z) \\ &= 21 + 2h^{1,2}(Z) - k(Z) = 21 + h^3(Z) - k(Z). \end{aligned}$$

This is the complete of the proof. □

Let us take a simple example.

Example 2.2. Consider three-dimensional projective space $\mathbb{P}^3$ and a smooth quartic hypersuface $\Gamma$ in $\mathbb{P}^3$. Let $Z$ be the blow-up of $\mathbb{P}^3$ along a smooth curve $c \in |\mathcal{O}_\Gamma(4)|$ and $S$ be the proper transform of $\Gamma$. Then it is easy to check that $Z$ is a building block of $G_2$-manifolds with $k(Z) = 1$. As in Theorem 2.1, we build a normal crossing variety $Y$ and smooth it to a Calabi–Yau threefold $M_Y$. Its Hodge numbers are

$$h^{1,1}(M_Y) = 2h^2(Z) - 1 - k(Z) = 4 - 1 - 1 = 2,$$
$$h^{1,2}(M_Y) = h^{2,1}(M_Y) = 21 + h^3(Z) - k(Z) = 21 + 66 - 1 = 86.$$

In §7 of [2], the authors constructed various examples of building blocks of $G_2$ manifolds and whose invariants are summarized in Table 7.2 in [2]. We apply the above procedure to all those examples of building blocks to construct Calabi–Yau manifolds $M_Y$'s. We give the Hodge numbers of $M_Y$'s in Table 1, where '*' means that the Hodge pair $(h^{1,1}(M_Y), h^{1,2}(M_Y))$ does not overlap with those of any Calabi–Yau threefolds that are desingularizations of anticanonical sections of Gorenstein toric Fano fourfolds ([1, 6, 7]). In Table 1, 'Example No.' refers to examples of building blocks in [2]. See §7 in [2] for the detailed description of them. Although some pairs of Hodge numbers appear in the toric construction, probably those Calabi–Yau threefolds may be different from ones of same Hodge numbers in the toric construction – there seems no reason that they are the same ones. One possible way of distinguishing them is comparing the cubic forms on their second integral cohomology classes.

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