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BOUNDS OF SOLUTIONS OF AN INTEGRO-DIFFERENTIAL EQUATION INVOLVING IMPULSES

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Abstract. In this paper we obtain some integral inequalities involving impulses and apply our results to a certain integro-differential equation with impulses. First, we obtain a bound of the equation, and we use the bound to study some qualitative properties of the equation.

1. Introduction

Differential equations with impulses arise in various real world phenomena in mathematical physics, mechanics, engineering, biology and so on(see, e.g., [6]). And integral inequalities are very useful tools in global existence, uniqueness, stability and other properties of the solutions of various nonlinear differential equations, see, e.g., [4, 5].

In this paper, we discuss some integral inequalities involving impulses and apply the inequalities to the study of some qualitative properties of a certain integro-differential equation involving impulses.

2. Preliminaries

In this section we state some materials that are needed in this paper.

Let $\mathbf{R}, \mathbf{R}^+, \mathbf{N}$ be the set of all real numbers, the set of all nonnegative real numbers, and the set of all positive integers, respectively, and let

$$G(\mathbf{R}^+) = \{f : \mathbf{R}^+ \to \mathbf{R} |\ \forall t \in (0, \infty),\ f(t+), f(t-) \text{ and } f(0+) \text{ exist}\}, \text{ and}$$
$$G([a, b]) = \{f : [a, b] \to \mathbf{R} |\ \forall t \in [a, b],\ f(t+) \text{ and } f(t-) \text{ exist}\},$$

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where for $t \in (0,\infty) \cup (a,b)$, $f(t\pm) = \lim_{s\to t\pm} f(s), f(a+) = \lim_{s\to a+} f(s), f(a-) = f(a)$, and $f(b-) = \lim_{s\to b-} f(s), f(b+) = f(b), f(0+) = \lim_{s\to 0+} f(s)$. If $f \in G(\mathbf{R}^+)$ or $f \in G([a,b])$, then we say that the function f is *regulated* on their domains, respectively. Throughout this paper we define

$$D = \{t_k, k \in \mathbf{N} : 0 < t_1 < t_2 < \cdots < t_n < \cdots\}, D_n = \{t_1, t_2, \cdots, t_n\}.$$

Then we define $PC(\mathbf{R}^+) = \{f : \mathbf{R}^+ \longrightarrow \mathbf{R} : f$ is continuous at every $t \notin D$ and left-continuous at every $t \in D\}$. It is obvious that if $f \in PC(\mathbf{R}^+)$, then f is regulated on $[0,T]$ for every $T > 0$. Throughout this paper we use the Kurzweil-Stieltjes integrals and the Stieltjes derivatives. For the integrals and derivatives, and various properties and notations that are used here, see, e.g., [1, 2, 3, 7, 8, 9] and the references cited there.

A *neighborhood of* t *in* $[a,b]$ is an open interval in $[a,b]$ that contains t. Let a function $\alpha : [a,b] \longrightarrow \mathbf{R}$ be nondecreasing. Then we say that α is *locally constant at* t, if there exists a neighborhood of t in $[a,b]$, where α is constant. Otherwise, we say that the function α is *not locally constant at* t.

Definition 2.1 ([1]). Let $f, g : [a,b] \longrightarrow \mathbf{R}$. If α is not locally constant at $t \in (a,b)$, we define

$$\frac{\mathrm{d}f(t)}{\mathrm{d}\alpha(t)} = \lim_{\eta,\delta\to 0+} \frac{f(t+\eta) - f(t-\delta)}{\alpha(t+\eta) - \alpha(t-\delta)},$$

provided that the limit exists. And for $t = a$ or $t = b$ we define

$$\frac{\mathrm{d}f(a)}{\mathrm{d}\alpha(a)} = \lim_{\eta\to 0+} \frac{f(a+\eta) - f(a)}{\alpha(a+\eta) - \alpha(a)}, \quad \frac{\mathrm{d}f(b)}{\mathrm{d}\alpha(b)} = \lim_{\delta\to 0+} \frac{f(b) - f(b-\delta)}{\alpha(b) - \alpha(b-\delta)},$$

respectively, provided that the limits exist.

If both f and α are constant on some neighborhood of t in $[a,b]$, we define $\frac{\mathrm{d}f(t)}{\mathrm{d}\alpha(t)} = 0$. Frequently we use $f'_\alpha(t)$ instead of $\frac{\mathrm{d}f(t)}{\mathrm{d}\alpha(t)}$.

We use the following results frequently.

Theorem 2.2 ([9, Theorem 2.15]). *Assume that* $f \in G([a,b])$ *and* $\alpha \in BV([a,b])$. *Then both* $f\,\mathrm{d}\alpha$ *and* $\alpha\,\mathrm{d}f$ *are Kurzweil-Stieltjes integrable on* $[a,b]$.

Theorem 2.3 ([1]). *Assume that* $f \in G([a,b])$ *and a function* $\alpha : [a,b] \longrightarrow \mathbf{R}$ *is nondecreasing, and that if* α *is constant on some neighborhood of* t *in* $[a,b]$, *then there exists a neighborhood of* t *in* $[a,b]$ *such that both* f *and* α *are constant there.*

Suppose that $f'_\alpha(t)$ *exists at every* $t \in [a,b] - \{c_1, c_2, ...\}$, *where* f *is continuous at every* $t \in \{c_1, c_2, ...\} \subset [a,b]$. *Then we have*

$$(K^*) \int_a^b f'_\alpha(s)\, \mathrm{d}\alpha(s) = f(b) - f(a).$$

Now we define a function $\phi : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ as

$$\phi(t) = \begin{cases} t, & \text{if } t \in [0, t_1] \\ t + k, & \text{if } t \in (t_k, t_{k+1}],\ t_k \in D, k \in \mathbf{N}. \end{cases} \tag{2.1}$$

For the function ϕ we have the following result.

Lemma 2.4 ([2])**.** *Assume that a function* $f \in G(\mathbf{R}^+)$ *is differentiable at* $t \neq t_k \in D, k \in \mathbf{N}$. *Then we have*

$$f'_\phi(t) = f'(t),\ f'_\phi(t_k) = f(t_k+) - f(t_k-),$$

and

$$\int_0^t f(s)\, \mathrm{d}\phi(s) = \int_0^t f(s)\, \mathrm{d}s + \sum_{0<t_k<t} f(t_k),\ \forall t \in \mathbf{R}^+.$$

3. Some Integral Inequalities involving Impulses

Throughout this section, unless otherwise specified, we always assume the following conditions:

(H1) Every one variable function belongs to $PC(\mathbf{R}^+)$ and is nonnegative.

(H2) A function $w : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ is nondecreasing, continuous on $\mathbf{R}^+$, and positive on $(0, \infty)$. We define

$$E(t) = \int_1^t \frac{\mathrm{d}s}{w(s)},\ \forall t \in \mathbf{R}^+,$$

and E^{-1} represents the inverse of the function E, and $\mathrm{Dom}(E^{-1})$ represents the domain of the function E^{-1}.

(H3) $a_k, b_k \geq 0$, and $0 \leq r_k, s_k \leq t_k, , k \in \mathbf{N}$.

Throughout this paper, for every $n, k \in \mathbf{N}$, we define

$$\tilde{f}_n = \begin{cases} f(t), & t \notin D_n \\ 1, & t \in D_n, \end{cases} \quad A_n(t) = \begin{cases} 1, & t \notin D_n \\ 0, & t \in D_n, \end{cases} \quad B_k(t) = \begin{cases} 1, & t = t_k \\ 0, & t \neq t_k. \end{cases}$$

And $f \circ g$ denotes the composite of f and g.

In order to obtain some integral inequalities, we need the following result.

Lemma 3.1 ([3]). *Let a function $\alpha : \mathbf{R}^+ \longrightarrow \mathbf{R}$ be strictly increasing. Assume that a positive left-continuous function z is nondecreasing on $\mathbf{R}^+$. If z is continuous at t and $z'_\alpha(t)$ exists, then we have*

$$\frac{\mathrm{d}}{\mathrm{d}\alpha(t)}E(z(t)) = \frac{\mathrm{d}}{\mathrm{d}\alpha(t)}\int_1^{z(t)} \frac{\mathrm{d}s}{w(s)} = \frac{z'_\alpha(t)}{w(z(t))}.$$

If α is not continuous at t, then we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha(t)}E(z(t)) = \frac{\mathrm{d}}{\mathrm{d}\alpha(t)}\int_1^{z(t)} \frac{\mathrm{d}s}{w(s)} \le \frac{z'_\alpha(t)}{w(z(t))}.$$

The following result is an Ou-Yang-type integral inequality.

Theorem 3.2. *Let $k(t,s) : \mathbf{R}^+ \times \mathbf{R}^+ \longrightarrow \mathbf{R}^+$, and $c, \psi, \varphi : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$, where c is nondecreasing, ψ is strictly increasing, continuous, and φ is nondecreasing, continuous on $\mathbf{R}^+$ and positive on $(0,\infty)$. Assume that $k(\cdot, s)$ is nondecreasing for each fixed $s \in \mathbf{R}^+$. Suppose that $k(t,\cdot) \in PC(\mathbf{R}^+)$ for each fixed $t \in \mathbf{R}^+$ and $\psi(0) = 0, \psi(\infty) = \infty$. If a function u satisfies*

$$\psi(u(t)) \le c(t) + \int_0^t k(t,s)\,\varphi(u(s))\,\mathrm{d}s + \sum_{0<t_k<t} a_k\,\varphi(u(r_k)),\ \forall t \in \mathbf{R}^+,$$

then for $t \in [0, M]$ we have

$$u(t) \le \psi^{-1} \circ E^{-1}[\gamma(t)], \tag{3.1}$$

where $w = \varphi \circ \psi^{-1}$ and

$$\gamma(t) = E(c(t)) + \int_0^t k(t,s)\,\mathrm{d}s + \sum_{0<t_k<t} a_k,$$

and the number M is chosen so that, for all $t \in [0, M], \gamma(t) \in \mathrm{Dom}(E^{-1})$.

Proof. Let $0 \le t \le T \le M$, where the number T is fixed, and let

$$z(t) = c(T) + \int_0^t k(T,s)\,\varphi(u(s))\,\mathrm{d}s + \sum_{0<t_k<t} a_k\,\varphi(u(r_k)).$$

Then since $\psi(u(t)) \le z(t)$ implies $u(t) \le \psi^{-1}(z(t))$, by Lemma 2.4, for every $t \in ([0,T) - D) \cup \{T\}$, we have

$$z'_\phi(t) = k(T,t)\,\varphi(u(t)) \le k(T,t)\,\varphi \circ \psi^{-1}(z(t)) = k(T,t)w(z(t)),$$

and, since $0 \le r_k \le t_k$, by Lemma 2.4, for every $t_k \in D \cap [0, T)$, we get

$$z'_\phi(t_k) = a_k\,\varphi(u(r_k)) \le a_k\,\varphi \circ \psi^{-1}(z(r_k)) = a_k w(z(r_k)) \le a_k w(z(t_k)).$$

Now assume that $D \cap [0, T) = D_n$. Then by Lemma 3.1, for every $t \in [0, T]$, we have

$$(E \circ z)'_\phi(t) \le \frac{z'_\phi(t)}{w(z(t))} \le A_n(t)k(T, t) + \sum_{k=1}^{n} B_k(t)a_k.$$

By Theorem 2.3, Lemma 2.4, and since $E(z(0)) = E(c(T))$, the above inequality implies

$$z(t) \le E^{-1}\left[E(c(T)) + \int_0^t k(T, s)\,\mathrm{d}s + \sum_{0<t_k<t} a_k\right], \forall t \in [0, T].$$

Thus we get, for all $t \in [0, T]$,

$$u(t) \le \psi^{-1} \circ z(t) \le \psi^{-1} \circ E^{-1}\left[E(c(T)) + \int_0^t k(T, s)\,\mathrm{d}s + \sum_{0<t_k<t} a_k\right].$$

So this implies

$$u(T) \le \psi^{-1} \circ E^{-1}\left[E(c(T)) + \int_0^T k(T, s)\,\mathrm{d}s + \sum_{0<t_k<T} a_k\right].$$

Since T was arbitrary the inequalities (3.1) is true for all $t \in [0, M]$. □

Corollary 3.3. *In Theorem 3.2, if a function u satisfies*

$$u(t) \le c(t) + \int_0^t f(s)w(u(s))\,\mathrm{d}s + \sum_{0<t_k<t} a_k w(u(r_k)), \tag{3.2}$$

then for $t \in [0, M]$ we have

$$u(t) \le E^{-1}[\gamma(t)],$$

where

$$\gamma(t) = E(c(t)) + \int_0^t f(s)\,\mathrm{d}s + \sum_{0<t_k<t} a_k,$$

and the number M is chosen so that for all $t \in [0, M]$ $\gamma(t) \in \mathrm{Dom}(E^{-1})$.

The following result is a Pachpatte-type integral inequality.

Theorem 3.4. *Let a function* φ *be as in Theorem 3.2, and* c *is a nonnegative constant, and let*

$$\alpha(t) = 1 + \int_0^t g(s)\,\mathrm{d}s.$$

If for every $t \in \mathbf{R}^+$ *a function* u *satisfies*

$$u(t) \le c + \int_0^t f(s)\left[u(s) + \int_0^s g(\sigma)\big[u(s) + \varphi(u(\sigma))\big]\,\mathrm{d}\sigma\right]\mathrm{d}s + \sum_{0<t_k<t}[a_k u(r_k) + b_k\,\varphi(u(s_k))], \tag{3.3}$$

then we have for every $t \in [0, M]$

$$u(t) \le c + \int_0^t f(s)\lambda(s)\,\mathrm{d}s + \sum_{0<t_k<t}[a_k\lambda(t_k) + b_k\,\varphi(\lambda(t_k))], \tag{3.4}$$

where

$$\lambda(t) = E^{-1}[\gamma(t)], \quad E(t) = \int_1^t \frac{\mathrm{d}s}{w(s)},\; w(s) = \max\{s, \varphi(s)\},\; \text{and}$$

$$\gamma(t) = E(c) + \int_0^t [f(s)\,\alpha(s) + 2g(s)]\,\mathrm{d}s + \sum_{0<t_k<t}\alpha(t_k)(a_k + b_k),$$

and the number M *is chosen so that, for all* $t \in [0, M]$, $\gamma(t) \in \mathrm{Dom}(E^{-1})$.

Proof. Let $t \in [0, M]$. Denote a function $v(t)$ by

$$v(t) = c + \int_0^t f(s)\left[u(s) + \int_0^s g(\sigma)\big[u(s) + \varphi(u(\sigma))\big]\,\mathrm{d}\sigma\right]\mathrm{d}s + \sum_{0<t_k<t}[a_k u(r_k) + b_k\,\varphi(u(s_k))].$$

Then, $v(0) = c, u(t) \le v(t),\ \forall t \in [0, M]$. By Lemma 2.4, for $t \in ([0, M) - D) \cup \{M\}$, we have

$$v'_\phi(t) = f(t)\left[u(t) + \int_0^t g(\sigma)[u(t) + \varphi(u(\sigma))]\,\mathrm{d}\sigma\right]$$
$$\leq f(t)\left[v(t) + \int_0^t g(\sigma)[v(t) + \varphi(v(\sigma))]\,\mathrm{d}\sigma\right],$$

and since $0 \leq s_k, r_k \leq t_k$ and v is nondecreasing, by Lemma 2.4, for every $t_k \in [0, M) \cap D$, we have

$$v'_\phi(t_k) = a_k u(r_k) + b_k\,\varphi(u(s_k)) \leq a_k v(r_k) + b_k\,\varphi(v(s_k)) \leq a_k v(t_k) + b_k\,\varphi(v(t_k)).$$

Now assume that $D \cap [0, M) = D_n$. Then for every $t \in [0, M]$ we have

$$v'_\phi(t) \leq \tilde{f}_n(t)\Bigg[A_n(t)\left(v(t) + \int_0^t g(\sigma)[v(t) + \varphi(v(\sigma))]\,\mathrm{d}\sigma\right) + \sum_{k=1}^{n} B_k(t)[a_k v(t_k) + b_k\,\varphi(v(t_k))]\Bigg]. \tag{3.5}$$

Define a function $m(t)$ by

$$m(t) = A_n(t)\left(v(t) + \int_0^t g(\sigma)[v(t) + \varphi(v(\sigma))]\,\mathrm{d}\sigma\right) + \sum_{k=1}^{n} B_k(t)[a_k v(t_k) + b_k\,\varphi(v(t_k))],$$

then $m(0) = v(0) = c$, and

$$v'_\phi(t) \leq \tilde{f}_n(t) m(t). \tag{3.6}$$

Let

$$z(t) = v(t) + \int_0^t g(\sigma)[v(t) + \varphi(v(\sigma))]\,\mathrm{d}\sigma, \ \forall t \in [0, M]. \tag{3.7}$$

Then, by (3.6) and (3.7), we have

$$\forall t \in [0, M], v(t) \leq z(t), \ \text{and } \forall t \in [0, M] - D_n, v'_\phi(t) \leq \tilde{f}_n(t) m(t) = f(t) z(t), \tag{3.8}$$

and so for every $t \in [0, M] - D_n$, by Lemma 2.4 we get

$$\begin{aligned}
z'_\phi(t) &= v'_\phi(t) + g(t)\,\varphi(v(t)) + v'_\phi(t) \int_0^t g(\sigma)\,\mathrm{d}\sigma + v(t)g(t) \\
&\le f(t)z(t) + g(t)\,\varphi(z(t)) + f(t)z(t) \int_0^t g(\sigma)\,\mathrm{d}\sigma + z(t)g(t) \\
&\le \left[f(t) + g(t) + f(t) \int_0^t g(\sigma)\,\mathrm{d}\sigma + g(t) \right] w(z(t)) \\
&= [f(t)\,\alpha(t) + 2g(t)]w(z(t)).
\end{aligned} \tag{3.9}$$

And for every $t_k \in D_n$ by (3.5), (3.7), (3.8), and by Lemma 2.4, we have

$$\begin{aligned}
z'_\phi(t_k) &= z(t_k+) - z(t_k-) = z(t_k+) - z(t_k) \\
&= \left(1 + \int_0^{t_k} g(\sigma)\,\mathrm{d}\sigma\right) v'_\phi(t_k) \le \alpha(t_k)[a_k v(t_k) + b_k\,\varphi(v(t_k))] \\
&\le \alpha(t_k)[a_k z(t_k) + b_k\,\varphi(z(t_k))] \le \alpha(t_k)(a_k + b_k)w(z(t_k)).
\end{aligned} \tag{3.10}$$

Thus by (3.9) and (3.10), we get

$$\begin{aligned}
z'_\phi(t) \le & A_n(t)[f(t)\,\alpha(t) + 2g(t)]w(z(t)) \\
& + \sum_{k=1}^{n} B_k(t)\,\alpha(t_k)(a_k + b_k)w(z(t_k)).
\end{aligned}$$

So by Theorem 2.3, Lemma 2.4, and $z(0) = v(0) = c$, this implies that

$$\begin{aligned}
z(t) \le c + & \int_0^t [f(s)\,\alpha(s) + 2g(s)]w(z(s))\,\mathrm{d}s \\
& + \sum_{0<t_k<t} \alpha(t_k)(a_k + b_k)w(z(t_k)).
\end{aligned}$$

Thus by Corollary 3.3, for all $t \in [0, M]$, we get

$$\begin{aligned}
z(t) &\le E^{-1}\left[E(c) + \int_0^t [f(s)\,\alpha(s) + 2g(s)]\,\mathrm{d}s + \sum_{0<t_k<t} \alpha(t_k)(a_k + b_k) \right] \\
&= E^{-1}[\gamma(t)] = \lambda(t).
\end{aligned}$$

So we have

$$\forall t \in [0, M] - D_n,\ m(t) = z(t) \leq \lambda(t), \tag{3.11}$$

and by (3.8) we get

$$\begin{aligned} m(t_k) = a_k v(t_k) + b_k\, \varphi(v(t_k)) &\leq a_k z(t_k) + b_k\, \varphi(z(t_k)) \\ &\leq a_k \lambda(t_k) + b_k\, \varphi(\lambda(t_k)). \end{aligned} \tag{3.12}$$

Thus by (3.6), (3.11) and (3.12) we have

$$\begin{aligned} v'_\phi(t) &\leq \tilde{f}_n(t) m(t) \\ &\leq \tilde{f}_n(t) \Big[A_n(t)\lambda(t) + \sum_{k=1}^{n} B_k(t)[a_k \lambda(t_k) + b_k\, \varphi(\lambda(t_k))] \Big] \\ &\leq A_n(t) f(t) \lambda(t) + \sum_{k=1}^{n} B_k(t)[a_k \lambda(t_k) + b_k\, \varphi(\lambda(t_k))]. \end{aligned}$$

Since $v(0) = c$, using Theorem 2.3 and Lemma 2.4, for all $t \in [0, M]$, we get

$$u(t) \leq v(t) \leq c + \int_0^t f(s)\lambda(s)\, \mathrm{d}s + \sum_{0 < t_k < t} [a_k \lambda(t_k) + b_k\, \varphi(\lambda(t_k))].$$

This completes the proof. □

4. Some Applications

There are many applications of the inequalities obtained in the previous section.

Here we shall apply an integral inequality that was obtained in the previous section to obtain a bound of solutions of the following integro-differential equation with impulses:

$$\begin{cases} x'(t) = F(t, x(t), \Lambda x(t)),\ t \notin D, \\ \Lambda x(t) = \displaystyle\int_0^t K(t,\, \sigma, x(t), x(\,\sigma))\, \mathrm{d}\sigma, \\ \Delta x(t_k) = I_k(x(r_k)) + J_k(x(s_k)),\ k \in \mathbf{N}, \end{cases} \tag{4.1}$$

where $t \in \mathbf{R}^+$ and $x(0) = x_0$.

In this section we assume the following conditions.

(C1) Functions $x, f, g \in PC(\mathbf{R}^+)$, where f and g are all nonnegative.

(C2) A continuous function $I_k, J_k : \mathbf{R} \longrightarrow \mathbf{R}$ satisfies

$$|I_k(x)| + |J_k(x)| \leq a_k|x| + b_k\, \varphi(|x|),\ a_k, b_k \geq 0,\ k \in \mathbf{N},$$

where a function $\varphi : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ is continuous, nondecreasing on $\mathbf{R}^+$, and positive on $(0, \infty)$.

(C3) Continuous functions $F : \mathbf{R}^+ \times \mathbf{R}^2 \longrightarrow \mathbf{R}$, $K : (\mathbf{R}^+)^2 \times \mathbf{R}^2$ satisfy

$$|F(t, u, v)| \le f(t)|u| + |v|,$$
$$|K(t, s, u, v)| \le f(t)g(s)[|u| + \varphi(|v|)].$$

Now we obtain a bounded for the equation (4.1).

Theorem 4.1. *Let* $|x_0| = c, |x(t)| = u(t), M^* \ge 0$. *Assume that, in Theorem* 3.4,

$$\gamma(t) \le M^* \in \mathrm{Dom}(E^{-1}), \ \forall t \in \mathbf{R}^+.$$

Then, there exists a nonnegative number M such that for any solution x of the equation (4.1),

$$\sup_{t \in \mathbf{R}^+} |x(t)| \le M. \tag{4.2}$$

Proof. If x is a solution of the equation (4.1), then we have

$$x(t) = x_0 + \int_0^t F(s, x(s), \Lambda x(s))\,\mathrm{d}s + \sum_{0<t_k<t} [I_k(x(r_k)) + J_k(x(s_k))].$$

So we get

$$\begin{aligned}
|x(t)| &\le |x_0| + \int_0^t |F(s, x(s), \Lambda x(s))|\,\mathrm{d}s + \sum_{0<t_k<t} [|I_k(x(r_k))| + |J_k(x(s_k))|] \\
&\le |x_0| + \int_0^t [f(s)|x(s)| + |\Lambda x(s)|]\,\mathrm{d}s + \sum_{0<t_k<t} [|I_k(x(r_k))| + |J_k(x(s_k))|] \\
&\le |x_0| + \int_0^t \left[f(s)|x(s)| + \int_0^s |K(s, \sigma, x(s).x(\sigma))|\,\mathrm{d}\sigma \right] \mathrm{d}s \\
&\qquad + \sum_{0<t_k<t} [a_k|x(r_k)| + b_k\,\varphi(|x(s_k)|)] \\
&\le |x_0| + \int_0^t \left[f(s)|x(s)| + \int_0^s f(s)g(\sigma)[|x(s)| + \varphi(|x(\sigma)|)]\,\mathrm{d}\sigma \right] \mathrm{d}s \\
&\qquad + \sum_{0<t_k<t} [a_k|x(r_k)| + b_k\,\varphi(|x(s_k)|)]
\end{aligned}$$

$$\leq |x_0| + \int_0^t f(s)\left[|x(s)| + \int_0^s g(\sigma)[|x(s)| + \varphi(|x(\sigma)|)]\,\mathrm{d}\sigma\right]\mathrm{d}s$$

$$+ \sum_{0<t_k<t} [a_k|x(r_k)| + b_k\,\varphi(|x(s_k)|)].$$

Since $\forall t \in \mathbf{R}^+, \gamma(t) \leq M^* < \infty$, $\alpha(t) \geq 1$ implies that

$$\int_0^\infty f(s)\,\mathrm{d}s + \sum_{0<t_k<\infty} (a_k + b_k) < \infty,$$

by Theorem 3.4, we have

$$|x(t)| \leq |x_0| + \int_0^t f(s)\lambda(s)\,\mathrm{d}s + + \sum_{0<t_k<t} [a_k\lambda(t_k) + b_k\,\varphi(\lambda(t_k))]$$

$$\leq |x_0| + E^{-1}[M^*] \int_0^t f(s)\,\mathrm{d}s + + \sum_{0<t_k<t} [a_k E^{-1}[M^*] + b_k\,\varphi(E^{-1}[M^*])]$$

$$\leq |x_0| + E^{-1}[M^*] \int_0^\infty f(s)\,\mathrm{d}s + w(E^{-1}[M^*]) \sum_{0<t_k<\infty} (a_k + b_k) \equiv M < \infty.$$

This implies (4.2). The proof is complete. □

Theorem 4.2. *Assume the same conditions as in Theorem* 4.1*. If x is a solution of the equation* (4.1)*, then there is a constant $c(x)$ which satisfies that*

$$\lim_{t\to\infty} x(t) = c(x).$$

And

$$|x(t) - c(x)| \leq \int_t^\infty f(s)\left[M + \int_0^s g(\sigma)[M + \varphi(M)]\,\mathrm{d}\sigma\right]\mathrm{d}s$$

$$+ \sum_{t\leq t_k<\infty} [a_k M + b_k\,\varphi(M)].$$

Proof. Since $\forall t \in \mathbf{R}^+,\ \gamma(t) \leq M^*,\ 1 \leq \alpha(t)$ implies $\int_0^\infty [f(s) + g(s)]\,\mathrm{d}s < \infty$, by the conditions that we assumed, we have

$$\begin{aligned}
&\int_0^t |F(s,x(s),\Lambda x(s))|\,\mathrm{d}s \\
&\le \int_0^t f(s)\left[|x(s)| + \int_0^s g(\sigma)[|x(s)| + \varphi(|x(\sigma)|)]\,\mathrm{d}\sigma\right]\mathrm{d}s \\
&\le \int_0^t f(s)\left[M + \int_0^s g(\sigma)[M + \varphi(M)]\,\mathrm{d}\sigma\right]\mathrm{d}s \\
&\le \int_0^\infty f(s)\left[M + \int_0^\infty g(\sigma)[M + \varphi(M)]\,\mathrm{d}\sigma\right]\mathrm{d}s < \infty.
\end{aligned}$$

This implies that $\int_0^\infty F(s,x(s),\Lambda x(s))\,\mathrm{d}s$ exists. And since $\sum_{0<t_k<\infty}(a_k+b_k)<\infty$, we get

$$\begin{aligned}
&\sum_{0<t_k<t}[|I_k(x(r_k))| + |J_k(x(s_k))|] \le \sum_{0<t_k<t}[a_k|x(r_k)| + b_k\,\varphi(|x(s_k)|)] \\
&\le \sum_{0<t_k<t}[a_k M + b_k\,\varphi(M)] \le w(M)\sum_{0<t_k<\infty}(a_k+b_k) < \infty.
\end{aligned}$$

So there is a constant $c(x)$ such that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left[x_0 + \int_0^t F(s,x(s),\Lambda x(s))\,\mathrm{d}s + \sum_{0<t_k<t}[I_k(x(r_k)) + J_k(x(s_k))]\right] = c(x).$$

And

$$\begin{aligned}
|x(t)-c(x)| &= \left|x(t) - \left[x_0 + \int_0^\infty F(s,x(s),\Lambda x(s))\,\mathrm{d}s + \sum_{0<t_k<\infty}[I_k(x(r_k)) + J_k(x(s_k))]\right]\right| \\
&= \left|\int_t^\infty F(s,x(s),\Lambda x(s))\,\mathrm{d}s + \sum_{t\le t_k<\infty}[I_k(x(r_k)) + J_k(x(s_k))]\right| \\
&\le \int_t^\infty f(s)\left[|x(s)| + \int_0^s g(\sigma)[|x(s)| + \varphi(|x(\sigma)|)\,\mathrm{d}\sigma\right]\mathrm{d}s \\
&\qquad + \sum_{t\le t_k<\infty}[a_k|x(r_k)| + b_k\,\varphi(|x(s_k)|)]
\end{aligned}$$

$$\leq \int_t^\infty f(s)\left[M + \int_0^s g(\sigma)[M + \varphi(M)]\,\mathrm{d}\sigma\right]\mathrm{d}s + \sum_{t\leq t_k<\infty}[a_k M + b_k\,\varphi(M)].$$

This completes the proof. □

Theorem 4.3. *Assume the same conditions as in Theorem* 4.1. *Suppose that*

$$|x_0| > \int_0^\infty f(s)\left[M + \int_0^s g(\sigma)[M + \varphi(M)]\,\mathrm{d}\sigma\right]\mathrm{d}s + \sum_{0<t_k<\infty}[a_k M + b_k\,\varphi(M)].$$

If x is a solution of the equation (4.1), *then we have*

$$x(t) \neq 0,\ \forall t \in \mathbf{R}^+.$$

Proof. By the triangular inequality we get

$$\begin{aligned}|x(t)| &\geq |x_0| - \left|\int_0^t F(s, x(s), \Lambda x(s))\,\mathrm{d}s + \sum_{0<t_k<t}[I_k(x(r_k)) + J_k(x(s_k))]\right|\\ &\geq |x_0| - \left[\int_0^\infty f(s)\left[M + \int_0^s g(\sigma)[M + \varphi(M)]\,\mathrm{d}\sigma\right]\mathrm{d}s + \sum_{k=1}^\infty[a_k M + b_k\,\varphi(M)]\right]\\ &> 0.\end{aligned}$$

This completes the proof. □

Theorem 4.4. *Assume the same conditions as in Theorem* 4.1. *Suppose that for some positive number T*

$$\forall t \in [0, T], w(t) \equiv \max\{t, \varphi(t)\} = t.$$

Then, for all solutions x of the equation (4.1),

$$\sup_{t\in\mathbf{R}^+}|x(t)| \longrightarrow 0+ \ \text{ as } |x_0| \longrightarrow 0+\,. \tag{4.3}$$

Proof. Let $L = \ln(T) + \int_T^1 \frac{\mathrm{d}s}{w(s)}$. Then, since for all $t \in (0, T]$, $w(t) = t$, we have

$$\begin{aligned}E(t) &= \int_1^t \frac{\mathrm{d}s}{w(s)} = -\left[\int_t^T \frac{\mathrm{d}s}{s} + \int_T^1 \frac{\mathrm{d}s}{w(s)}\right] = -\left[\ln(T) - \ln(t) + \int_T^1 \frac{\mathrm{d}s}{w(s)}\right]\\ &= \ln(t) - L,\ \forall t \in (0, T].\end{aligned}$$

This implies that

$$E^{-1}(t) = \exp(t+L),\ \forall t \in (-\infty, \ln(T)-L].$$

So, if $|x_0|$ is sufficiently small, then we have $E(|x_0|) = \ln(|x_0|) - L$, and for some nonnegative number C, and for all $t \in \mathbf{R}^+$,

$$\begin{aligned}
\lambda(t) &= E^{-1}[\gamma(t)] \\
&= E^{-1}\left[E(|x_0|) + \int_0^t [f(s)\,\alpha(s) + 2g(s)]\,\mathrm{d}s + \sum_{0<t_k<t} \alpha(t_k)(a_k+b_k)\right] \\
&= \exp\left[\ln|x_0| - L + \int_0^t [f(s)\,\alpha(s) + 2g(s)]\,\mathrm{d}s + \sum_{0<t_k<t} \alpha(t_k)(a_k+b_k) + L\right] \\
&\le \exp\left[\ln|x_0| + \int_0^\infty [f(s)\,\alpha(s) + 2g(s)]\,\mathrm{d}s + \sum_{0<t_k<\infty} \alpha(t_k)(a_k+b_k)\right] \\
&\le C\cdot|x_0| < T.
\end{aligned}$$

Hence, for sufficiently small $|x_0|$, since $C|x_0| < T$ implies $\varphi(C|x_0|) \le C|x_0|$, for all $t \in \mathbf{R}^+$, we get

$$\begin{aligned}
|x(t)| &\le |x_0| + \int_0^\infty f(s)\lambda(s)\,\mathrm{d}s + \sum_{0<t_k<\infty} [a_k\lambda(t_k) + b_k\,\varphi(\lambda(t_k))] \\
&\le |x_0| + C|x_0| \int_0^\infty f(s)\,\mathrm{d}s + \sum_{0<t_k<\infty} [a_k C|x_0| + b_k\,\varphi(C|x_0|)] \\
&\le |x_0| + C|x_0| \int_0^\infty f(s)\,\mathrm{d}s + C|x_0| \sum_{0<t_k<\infty} (a_k+b_k).
\end{aligned}$$

This completes the proof. □

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