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BEHAVIOR OF HOLOMORPHIC FUNCTIONS ON THE BOUNDARY OF THE UNIT DISC

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Abstract. In this paper, we establish lower estimates for the modulus of the non-tangential derivative of the holomorphic function $f(z)$ at the boundary of the unit disc. Also, we shall give an estimate below $|f''(b)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z = 0$ and $z_0 \neq 0$.

1. Introduction

Let us consider a function $f(z)$ holomorphic in the unit disc $E = \{z : |z| < 1\}$ with $f(E) \subset E$ and $f(0) = 0$. The Schwarz lemma asserts that

$$|f(z)| \leq |z|,$$

for every $z \in E$ and

$$\left|f'(0)\right| \leq 1.$$

Moreover, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| = 1$ then f is a rotation, that is, $f(z) = ze^{i\theta}$, θ real [6]. More generally, the Schwarz lemma can be applied to a function with the information $f(z_0) = \eta$ for some z_0, η with $|z_0| < 1$, $|\eta| < 1$ instead of $f(0) = 0$ and it is called the Schwarz-Pick lemma [6].

Let f be a holomorphic function of E into E with $f(z_0) = \eta$ for some z_0, η with $|z_0| < 1$, $|\eta| < 1$. Then

$$\left|\frac{f(z) - f(z_0)}{1 - \overline{f(z_0)}f(z)}\right| \leq \left|\frac{z - z_0}{1 - \overline{z_0}z}\right|, \ z \neq z_0,$$

and

$$\left|f'(z)\right| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

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Equality holds at some point z if and only if $f(z)$ is a Möbius transformation.

For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [2], [19]).

The basic tool in proving our results is the following lemma due to Jack [2].

Lemma 1.1 (Jack's lemma)**.** *Let the function $f(z)$ defined by*

$$f(z) = c_p z^p + c_{p+1} z^{p+1} + ...$$

be holomorphic in E with $f(0) = 0$. If $|f(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in E$, then there exists a real number $k \geq p$ such that

$$\frac{z_0 f'(z_0)}{f(z_0)} = k.$$

Let $\mathcal{A}_p$ denote the class of functions

$$f(z) = z + c_{p+1} z^{p+1} + c_{p+2} z^{p+2} + ...$$

that are holomorphic in the unit disc E. Also, $\mathcal{M}(\alpha)$ be the subclass of $\mathcal{A}_p$ consisting of all functions $f(z)$ which satisfy

$$\left| z f''(z) - \gamma (f'(z) - 1) \right| < (1 - \alpha) |p - \gamma| \tag{1.1}$$

for some real $0 \leq \alpha < 1$ and some complex γ with $\Re(\gamma) < p$.

Let $f(z) \in \mathcal{M}(\alpha)$ and define $\varphi(z)$ in E by

$$\varphi(z) = \frac{f'(z) - 1}{1 - \alpha}. \tag{1.2}$$

Obviously, $\varphi(z)$ is holomorphic function in the unit disc E and $\varphi(0) = 0$. That is;

$$\varphi(z) = \frac{(p+1)\, c_{p+1}}{1 - \alpha} z^p + \frac{(p+2)\, c_{p+2}}{1 - \alpha} z^{p+1} + ...$$

We want to prove $|\varphi(z)| < 1$ for $|z| < 1$. Differentiating (1.2) and simplifiying, we obtain

$$z f''(z) = (1 - \alpha)\, z \varphi'(z)$$

and, therefore

$$\begin{aligned} \left| z f''(z) - \gamma \left(f'(z) - 1 \right) \right| &= \left| (1 - \alpha)\, z \varphi'(z) - \gamma (1 - \alpha)\, \varphi(z) \right| \\ &= (1 - \alpha) |\varphi(z)| \left| \frac{z \varphi'(z)}{\varphi(z)} - \gamma \right| \\ &< (1 - \alpha) |p - \gamma| . \end{aligned}$$

If there exists a point $z_0 \in E$ such that

$$\max_{|z| \leq |z_0|} |\varphi(z)| = |\varphi(z_0)| = 1,$$

then Jack's lemma gives us that $\varphi(z_0) = e^{i\theta}$ and $z_0\varphi'(z_0) = k\varphi(z_0)$ $(k \geq p)$.

Thus we have

$$\begin{aligned} \left|z_0 f''(z_0) - \gamma\left(f'(z_0) - 1\right)\right| &= (1-\alpha)\,|\varphi(z_0)| \left|\frac{z_0\varphi'(z_0)}{\varphi(z_0)} - \gamma\right| \\ &= (1-\alpha)\,|k-\gamma| \\ &\geq (1-\alpha)\,|p-\gamma|\,. \end{aligned}$$

This contradict (1.1). So, there is no point $z_0 \in E$ such that $|\varphi(z_0)| = 1$. This means that $|\varphi(z)| < 1$ for $|z| < 1$. Thus, from the Schwarz lemma, we obtain

$$|c_{p+1}| \leq \frac{1-\alpha}{1+p}.$$

Moreover, the equality $|c_{p+1}| = \frac{1-\alpha}{1+p}$ holds if and only if

$$f(z) = z + \frac{1-\alpha}{1+p} z^{p+1} e^{i\theta},$$

where θ is a real number.

The proof has been completed. Let us now give the statement of the lemma.

Lemma 1.2. *If $f(z) \in \mathcal{M}(\alpha)$, then we have*

$$|c_{p+1}| \leq \frac{1-\alpha}{1+p}. \tag{1.3}$$

The equality in (1.3) *holds if and only if*

$$f(z) = z + \frac{1-\alpha}{1+p} z^{p+1} e^{i\theta},$$

where θ is a real number.

The following boundary version of the Schwarz lemma was proved in 1938 by Unkelbach in [20] and then rediscovered and partially improved by Osserman in 2000 [16].

Lemma 1.3. *Let $f(z) = c_p z^p + c_{p+1} z^{p+1} + c_{p+2} z^{p+2} + \ldots$ $p \geq 1$, $p \in \mathbb{N}$ be a holomorphic function self-mapping of $E = \{z : |z| < 1\}$, that is $|f(z)| < 1$ for all $z \in E$. Assume that there is a $b \in \partial E$ so that f extend continuously to b, $|f(b)| = 1$ and $f'(b)$ exists. Then*

$$\left|f'(b)\right| \geq p + \frac{1-|c_p|}{1+|c_p|}. \tag{1.4}$$

The equality in (1.4) *holds if and only if* f *is of the form*

$$f(z) = -z^p \frac{a-z}{1-az}, \ \forall z \in E,$$

for some constant $a \in (-1, 0]$.

Corollary 1.4. *Under the hypotheses of Lemma 1.3, we have*

$$\left|f'(b)\right| \geq p, \tag{1.5}$$

with equality only if f *is of the form*

$$f(z) = z^p e^{i\theta},$$

where θ *is a real number.*

The following Lemma 1.5 and Corollary 1.6, known as the Julia-Wolff lemma, is needed in the sequel [18].

Lemma 1.5 (Julia-Wolff lemma)**.** *Let* f *be a holomorphic function in* E, $f(0) = 0$ *and* $f(E) \subset E$. *If, in addition, the function* f *has an angular limit* $f(b)$ *at* $b \in \partial E$, $|f(b)| = 1$, *then the angular derivative* $f'(b)$ *exists and* $1 \leq |f'(b)| \leq \infty$.

Corollary 1.6. *The holomorphic function* f *has a finite angular derivative* $f'(b)$ *if and only if* f' *has the finite angular limit* $f'(b)$ *at* $b \in \partial E$.

Inequality (1.4) and its generalizations have important applications in geometric theory of functions (see, e.g., [6], [18]). Therefore, the interest to such type results is not vanished recently (see, e.g., [1], [2], [4], [5], [10], [11], [16], [17], [19] and references therein).

Vladimir N. Dubinin has continued this line and has made a refinement on the boundary Schwar lemma under the assumption that $f(z) = c_p z^p + c_{p+1} z^{p+1} + ...$, with a zero set $\{z_k\}$ (see [4]).

S. G. Krantz and D. M. Burns [9] and D. Chelst [3] studied the uniqueness part of the Schwarz lemma. In M. Mateljević's papers, for more general results and related estimates, see also ([12], [13], [14] and [15]).

Also, M. Jeong [8] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [7] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

2. Main Results

In this section, for holomorphic function $f(z) = z + c_{p+1}z^{p+1} + c_{p+2}z^{p+2} + ...$ belong to the class of $\mathcal{M}(\alpha)$, it has been estimated from below the modulus of the non-tangential derivative of the function on the boundary point of the unit disc.

Theorem 2.1. *Let $f(z) \in \mathcal{M}(\alpha)$. Assume that, for some $b \in \partial E$, f' has a non-tangential limit $f'(b)$ at b and $f'(b) = \alpha$. Then f has the second non-tangential derivative at $b \in \partial E$ and*

$$\left|f''(b)\right| \geq p(1-\alpha). \tag{2.1}$$

The equality in (2.1) *occurs for the function*

$$f(z) = z - \frac{1-\alpha}{1+p}z^{p+1}.$$

Proof. Consider the function

$$\varphi(z) = \frac{f'(z)-1}{1-\alpha}.$$

$\varphi(z)$ is a holomorphic function in the unit disc E and $\varphi(0) = 0$. From the Jack's lemma and since $f(z) \in \mathcal{M}(\alpha)$, we obtain $|\varphi(z)| < 1$ for $|z| < 1$. It can be easily shown a non-tangential derivative of φ at $b \in \partial E$ (see [18]). Thus, the second non-tangential derivative of $f(z)$ at $b \in \partial E$ is obtained. Also, we have $|\varphi(b)| = 1$ for $b \in \partial E$.

From (1.5), we obtain

$$p \leq \left|\varphi'(b)\right| = \frac{|f''(b)|}{1-\alpha}$$

and

$$\left|f''(b)\right| \geq p(1-\alpha).$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z) = z - \frac{1-\alpha}{1+p}z^{p+1}.$$

Then

$$f'(z) = 1 - \frac{1-\alpha}{1+p}(1+p)z^p = 1-(1-\alpha)z^p,$$
$$f''(z) = -(1-\alpha)pz^{p-1}$$

and

$$\left|f''(1)\right| = (1-\alpha)p.$$

□

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$|f''(b)| \geq (1-\alpha)\left(p + \frac{1-\alpha-(p+1)|c_{p+1}|}{1-\alpha+(p+1)|c_{p+1}|}\right). \tag{2.2}$$

The inequality (2.2) *is sharp with equality for the function*

$$f(z) = z - \frac{1-\alpha}{1+p} z^{p+1}.$$

Proof. Let $\varphi(z)$ be as in the proof of Theorem 2.1. Using the inequality (1.4) for the function $\varphi(z)$, we obtain

$$p + \frac{1-|d_p|}{1+|d_p|} \leq |\varphi'(b)| \leq \frac{|f''(b)|}{1-\alpha},$$

where $|d_p| = \frac{|\varphi^{(p)}(0)|}{p!} = \frac{1+p}{1-\alpha}|c_{p+1}|$. Thus, we take

$$p + \frac{1-\frac{1+p}{1-\alpha}|c_{p+1}|}{1+\frac{1+p}{1-\alpha}|c_{p+1}|} \leq \frac{|f''(b)|}{1-\alpha}$$

and

$$p + \frac{1-\alpha-(1+p)|c_{p+1}|}{1-\alpha+(1+p)|c_{p+1}|} \leq \frac{|f''(b)|}{1-\alpha}$$

So, we obtain the inequality (2.2).

Now, we shall show that the inequality (2.2) is sharp. Let

$$f(z) = z - \frac{1-\alpha}{1+p} z^{p+1}.$$

Then

$$f'(z) = 1 - \frac{1-\alpha}{1+p}(1+p) z^p = 1-(1-\alpha) z^p,$$

$$f''(z) = -(1-\alpha) p z^{p-1}$$

and

$$|f''(1)| = (1-\alpha) p.$$

Since $|c_{p+1}| = \frac{1-\alpha}{1+p}$, (2.2) is satisfied with equality. That is,

$$\begin{aligned} (1-\alpha)\left(p + \frac{1-\alpha-(p+1)|c_{p+1}|}{1-\alpha+(p+1)|c_{p+1}|}\right) &= (1-\alpha)\left(p + \frac{1-\alpha-(p+1)\frac{1-\alpha}{1+p}}{1-\alpha+(p+1)\frac{1-\alpha}{1+p}}\right) \\ &= (1-\alpha) p. \end{aligned}$$

□

Theorem 2.3. *Let $f(z) \in \mathcal{M}(\alpha)$ and $p \geq 2$. Assume that, for some $b \in \partial E$, f' has a non-tangential limit $f'(b)$ at b and $f'(b) = \alpha$. Then f has the second non-tangential derivative at $b \in \partial E$ and*

$$\left|f''(b)\right| \geq (1-\alpha)\left(p + \frac{2\left(1-\alpha-(1+p)\left|c_{p+1}\right|\right)^2}{(1-\alpha)^2 - \left((p+1)\left|c_{p+1}\right|\right)^2 + (1-\alpha)(p+2)\left|c_{p+2}\right|}\right). \tag{2.3}$$

The inequality (2.3) *is sharp with equality for the function*

$$f(z) = z - \frac{1-\alpha}{1+p} z^{p+1}.$$

Proof. Let $\varphi(z)$ be as in the proof of Theorem 2.1. By the maximum principle for each $z \in E$, we have $|\varphi(z)| \leq |z^p|$. So,

$$\omega(z) = \frac{\varphi(z)}{z^p}$$

is a holomorphic function in E and $|\omega(z)| < 1$ for $|z| < 1$.

In particular, we have

$$|\omega(0)| = \frac{1+p}{1-\alpha}\left|c_{p+1}\right| \leq 1 \tag{2.4}$$

and

$$\left|\omega'(0)\right| = \frac{p+2}{1-\alpha}\left|c_{p+2}\right|.$$

Moreover, it can be seen that

$$\frac{b\varphi'(b)}{\varphi(b)} = \left|\varphi'(b)\right| \geq \left|(b^p)'\right| = \frac{b\,(b^p)'}{b^p}.$$

The function

$$\Upsilon(z) = \frac{\omega(z)-\omega(0)}{1-\overline{\omega(0)}\omega(z)}$$

is a holomorphic in the unit disc E, $|\Upsilon(z)| < 1$ for $|z| < 1$, $\Upsilon(0) = 0$ and $|\Upsilon(b)| = 1$ for $b \in \partial E$. It can be easily shown that the function Υ has a non-tangential derivative at $b \in \partial E$ (see [18]). Therefore, the second non-tangential derivative of f at $b \in \partial E$ is obtained.

From (1.4), we obtain

$$\begin{aligned}
\frac{2}{1+|\Upsilon'(0)|} &\leq \left|\Upsilon'(b)\right| = \frac{1-|\omega(0)|^2}{\left|1-\overline{\omega(0)}\omega(b)\right|^2}\left|\omega'(b)\right| \leq \frac{1+|\omega(0)|}{1-|\omega(0)|}\left|\omega'(b)\right| \\
&= \frac{1+|\omega(0)|}{1-|\omega(0)|}\left\{\left|\varphi'(b)\right| - p\right\}.
\end{aligned}$$

Since

$$\Upsilon'(z)=\frac{1-|\omega(0)|^2}{\left(1-\overline{\omega(0)}\omega(z)\right)^2}\omega'(z),$$

$$\left|\Upsilon'(0)\right|=\frac{|\omega'(0)|}{1-|\omega(0)|^2}=\frac{(1-\alpha)\,(p+2)\,|c_{p+2}|}{(1-\alpha)^2-((p+1)\,|c_{p+1}|)^2},$$

we take

$$\begin{aligned}\frac{2}{1+\frac{(1-\alpha)(p+2)|c_{p+2}|}{(1-\alpha)^2-((p+1)|c_{p+1}|)^2}} &\leq \frac{1+\frac{1+p}{1-\alpha}\,|c_{p+1}|}{1-\frac{1+p}{1-\alpha}\,|c_{p+1}|}\left\{\frac{|f''(b)|}{1-\alpha}-p\right\}\\ &= \frac{1-\alpha+(1+p)\,|c_{p+1}|}{1-\alpha-(1+p)\,|c_{p+1}|}\left\{\frac{|f''(b)|}{1-\alpha}-p\right\}.\end{aligned}$$

Therefore, we obtain

$$p+\frac{2}{1+\frac{(1-\alpha)(p+2)|c_{p+2}|}{(1-\alpha)^2-((p+1)|c_{p+1}|)^2}}\frac{1-\alpha-(1+p)\,|c_{p+1}|}{1-\alpha+(1+p)\,|c_{p+1}|}\leq\frac{|f''(b)|}{1-\alpha}.$$

So, we obtain the inequality (2.3).

To show that the inequality (2.3) is sharp, take the holomorphic function

$$f(z)=z-\frac{1-\alpha}{1+p}z^{p+1}.$$

Then

$$f''(z)=-\left(1-\alpha\right)pz^{p-1}$$

and

$$\left|f''(1)\right|=(1-\alpha)\,p.$$

Since $|c_{p+1}|=\frac{1-\alpha}{1+p}$, (2.3) is satisfied with equality. □

If $f(z)-z$ has no critical points different from $z=0$ in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

Theorem 2.4. *Let $f(z)\in\mathcal{M}(\alpha)$ and $f(z)-z$ has no critical points in E except $z=0$, $c_{p+1}>0$, and $p\geq 2$. Assume that, for some $b\in\partial E$, f' has a non-tangential limit $f'(b)$ at b and $f'(b)=\alpha$. Then f has the second non-tangential derivative at $b\in\partial E$ and we have the inequality*

$$\left|f''(b)\right|\geq(1-\alpha)\left(p-\frac{2(p+1)\,|c_{p+1}|\ln^2\left(\frac{1+p}{1-\alpha}\,|c_{p+1}|\right)}{2(p+1)\,|c_{p+1}|\ln\left(\frac{1+p}{1-\alpha}\,|c_{p+1}|\right)-(p+2)\,|c_{p+2}|}\right)\tag{2.5}$$

and

$$|c_{p+2}| \le \frac{2}{p+2}\left|(p+1)c_{p+1}\ln\left(\frac{(p+1)\,|c_{p+1}|}{1+\alpha}\right)\right|. \tag{2.6}$$

In addition, the equality in (2.5) *occurs for the function* $f(z) = z - \frac{1-\alpha}{1+p}z^{p+1}$ *and the equality in* (2.6) *occurs for the function*

$$f(z) = \int_0^z \frac{1+\alpha t^p e^Q}{1-t^p e^Q}dt,$$

where $0 < c_{p+1} < 1,\ \ln\left(\frac{1+p}{1-\alpha}\,|c_{p+1}|\right) < 0$ *and* $Q = \frac{1+t}{1-t}\ln\left(\frac{p+1}{1-\alpha}c_{p+1}\right)$.

Proof. Let $c_{p+1} > 0$ in the expression of the function $f(z)$. Having in mind the inequality (2.4) and the function $f(z)-z$ has no critical points in E except $E-\{0\}$, we denote by $\ln\omega(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln\omega(0) = \ln\left(\frac{1+p}{1-\alpha}\,|c_{p+1}|\right) < 0.$$

The auxiliary function

$$\Phi(z) = \frac{\ln\omega(z) - \ln\omega(0)}{\ln\omega(z) + \ln\omega(0)}$$

is a holomorphic in the unit disc E, $|\Phi(z)| < 1, \Phi(0) = 0$ and $|\Phi(b)| = 1$ for $b \in \partial E$. It can be easily shown a non-tangential derivative of Φ at $b \in \partial E$ (see [18]). Therefore, the second non-tangential derivative of f at $b \in \partial E$ is obtained.

From (1.4), we obtain

$$\begin{aligned}\frac{2}{1+|\Phi'(0)|} &\le \left|\Phi'(b)\right| = \frac{|2\ln\omega(0)|}{|\ln\omega(b)+\ln\omega(0)|^2}\left|\frac{\omega'(b)}{\omega(b)}\right| \\ &= \frac{-2\ln\omega(0)}{\ln^2\omega(0)+\arg^2\omega(b)}\left\{\left|\varphi'(b)\right| - p\right\}.\end{aligned}$$

Replacing $\arg^2\omega(b)$ by zero, then

$$\frac{1}{1-\frac{(p+2)|c_{p+2}|}{2(p+1)|c_{p+1}|\ln\left(\frac{1+p}{1-\alpha}|c_{p+1}|\right)}} \le \frac{-1}{\ln\left(\frac{1+p}{1-\alpha}\,|c_{p+1}|\right)}\left\{\frac{|f''(b)|}{1-\alpha} - p\right\}.$$

Thus, we obtain the inequality (1.15) with an obvious equality case.

Likewise, $\Phi(z)$ function satisfies the assumptions of the Schwarz lemma, we obtain

$$\begin{aligned}1 &\ge \left|\Phi'(0)\right| = \frac{|2\ln\omega(0)|}{|\ln\omega(0)+\ln\omega(0)|^2}\left|\frac{\omega'(0)}{\omega(0)}\right| \\ &= \frac{-1}{2\ln\left(\frac{1+p}{1-\alpha}\,|c_{p+1}|\right)}\frac{\frac{p+2}{1-\alpha}\,|c_{p+2}|}{\frac{1+p}{1-\alpha}\,|c_{p+1}|}\end{aligned}$$

and

$$1 \geq \frac{-1}{2\ln\left(\frac{1+p}{1-\alpha}\left|c_{p+1}\right|\right)} \frac{(p+2)\left|c_{p+2}\right|}{(p+1)\left|c_{p+1}\right|}.$$

Therefore, we have

$$\left|c_{p+2}\right| \leq \frac{2}{p+2}\left|(p+1)c_{p+1}\ln\left(\frac{p+1}{1-\alpha}\left|c_{p+1}\right|\right)\right|.$$

We shall show that the inequality (2.6) is sharp. Let

$$f(z) = \int_0^z \frac{1+\alpha t^p e^Q}{1-t^p e^Q} dt.$$

Thus, we get

$$f'(z) = \frac{1+\alpha z^p e^Q}{1-z^p e^Q}$$

and

$$f'(z) = 1 + z^p \mathrm{k(z)},$$

where

$$\mathrm{k(z)} = (1+\alpha)\frac{e^{\frac{1+z}{1-z}\ln\left(\frac{p+1}{1-\alpha}c_{p+1}\right)}}{1-z^p e^{\frac{1+z}{1-z}\ln\left(\frac{p+1}{1-\alpha}c_{p+1}\right)}}.$$

Then

$$\mathrm{k}'(0) = (p+2)\, c_{p+2}.$$

Under the simple calculations, we obtain

$$(p+2)\, c_{p+2} = 2\ln\left(\frac{p+1}{1-\alpha}c_{p+1}\right)(p+1)\, c_{p+1}$$

and

$$\left|c_{p+2}\right| = \frac{2}{p+2}\left|(p+1)\, c_{p+1}\ln\left(\frac{p+1}{1-\alpha}\left|c_{p+1}\right|\right)\right|.$$

□

The following inequality (2.7) is weaker, but is simpler than (2.5) and does not contain the coeffient c_{p+2}.

Theorem 2.5. *Let $f(z) \in \mathcal{M}(\alpha)$ and $f(z) - z$ has no critical points in E except $z = 0$, $c_{p+1} > 0$, and $p \geq 2$. Assume that, for some $b \in \partial E$, f' has a non-tangential limit $f'(b)$ at b and $f'(b) = \alpha$. Then f has the second non-tangential derivative at $b \in \partial E$ and we have the inequality*

$$\left|f''(b)\right| \geq (1-\alpha)\left(p - \frac{1}{2}\ln\left(\frac{(p+1)}{1-\alpha}\left|c_{p+1}\right|\right)\right). \tag{2.7}$$

The inequality (2.7) *is sharp and the equality is achieved if and only if* $f(z)$ *is the function of the form*

$$f(z)=\int_0^z \frac{1+\alpha t^p e^Q}{1-t^p e^Q}dt,$$

where $0<c_{p+1}<1$, $\ln\left(\frac{(p+1)c_{p+1}}{1-\alpha}\right)<0$, $Q=\frac{1+te^{i\theta}}{1-te^{i\theta}}\ln\left(\frac{(p+1)c_{p+1}}{1-\alpha}\right)$ *and* θ *is a real number.*

Proof. Let $c_{p+1}>0$. Using the inequality (1.5) for the function $\Phi(z)$, we obtain

$$1\leq\left|\Phi'(b)\right|=\frac{|2\ln\omega(0)|}{|\ln\omega(b)+\ln\omega(0)|^2}\left|\frac{\omega'(b)}{\omega(b)}\right|=\frac{-2\ln\omega(0)}{\ln^2\omega(0)+\arg^2\omega(b)}\left\{\left|\varphi'(b)\right|-p\right\}.$$

Replacing $\arg^2\varphi(b)$ by zero, then

$$1\leq\left|\Phi'(b)\right|\leq\frac{-2}{\ln\left(\frac{1+p}{1-\alpha}\left|c_{p+1}\right|\right)}\left\{\frac{|f''(b)|}{1-\alpha}-p\right\}. \tag{2.8}$$

Therefore, we obtain the inequality (2.8).

If $|f''(b)|=(1-\alpha)\left(p-\frac{1}{2}\ln\left(\frac{(p+1)}{1-\alpha}\left|c_{p+1}\right|\right)\right)$ from (2.8) and $|\Phi'(b)|=1$, we obtain

$$f(z)=\int_0^z \frac{1+\alpha t^p e^{\frac{1+te^{i\theta}}{1-te^{i\theta}}\ln\left(\frac{(p+1)c_{p+1}}{1-\alpha}\right)}}{1-t^p e^{\frac{1+te^{i\theta}}{1-te^{i\theta}}\ln\left(\frac{(p+1)c_{p+1}}{1-\alpha}\right)}}dt.$$

□

In the following Theorem, we shall give an estimate below $|f''(b)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z=0$ and $z_0\neq 0$.

Theorem 2.6. *Let* $f(z)=z+c_2z^2+c_3z^3+...\in\mathcal{M}(\alpha)$ *and* $f'(z_0)=1$ *for* $0<|z_0|<1$. *Assume that, for some* $b\in\partial E$, f' *has a non-tangential limit* $f'(b)$ *at* b *and* $f'(b)=\alpha$. *Then* f *has the second non-tangential derivative at* $b\in\partial E$ *and*

$$|f''(b)|\geq(1-\alpha)\Bigg(1+\frac{1-|z_0|^2}{|1-z_0|^2}+\frac{(1-\alpha)|z_0|-|f''(0)|}{(1-\alpha)|z_0|+|f''(0)|}\times \left[1+\frac{(1-\alpha)^2|z_0|^2+|f''(0)|\left(1-|z_0|^2\right)|f''(z_0)|-(1-\alpha)\left(1-|z_0|^2\right)|f''(z_0)|-(1-\alpha)|f''(0)|}{(1-\alpha)^2|z_0|^2+|f''(0)|\left(1-|z_0|^2\right)|f''(z_0)|+(1-\alpha)\left(1-|z_0|^2\right)|f''(z_0)|+(1-\alpha)|f''(0)|}\frac{1-|z_0|^2}{|1-z_0|^2}\right]\Bigg). \tag{2.9}$$

The inequality (2.9) *is sharp with equality for each possible values* $|f''(0)|=(1-\alpha)c$ *and* $|f''(z_0)|=(1-\alpha)d$ $\left(0\leq c\leq(1-\alpha)|z_0|,\ 0\leq d\leq(1-\alpha)\frac{|z_0|}{1-|z_0|^2}\right)$.

Proof. Let

$$p(z)=\frac{z-z_0}{1-\overline{z_0}z}.$$

Let $k : E \to E$ be a holomorphic function and a point $z_0 \in E$. Therefore, we have

$$|k(z)| \le \frac{|k(z_0)| + |p(z)|}{1 + |k(z_0)|\,|p(z)|}. \tag{2.10}$$

If $w : E \to E$ is holomorphic function and $0 < |z_0| < 1$, letting

$$k(z) = \frac{w(z) - w(0)}{z\left(1 - \overline{w(0)}w(z)\right)}$$

in (2.10), we obtain

$$\left|\frac{w(z) - w(0)}{\left(1 - \overline{w(0)}w(z)\right)}\right| \le |z| \frac{\left|\frac{w(z_0)-w(0)}{z_0\left(1-\overline{w(0)}w(z_0)\right)}\right| + |p(z)|}{1 + \left|\frac{w(z_0)-w(0)}{z_0\left(1-\overline{w(0)}w(z_0)\right)}\right| |p(z)|}$$

and

$$|w(z)| \le \frac{|w(0)| + |z|\,\frac{|C|+|p(z)|}{1+|C||p(z)|}}{1 + |w(0)|\,|z|\,\frac{|C|+|p(z)|}{1+|C||p(z)|}}, \tag{2.11}$$

where

$$C = \frac{w(z_0) - w(0)}{z_0\left(1 - \overline{w(0)}w(z_0)\right)}.$$

Without loss of generality, we will assume that $b = 1$. If we take

$$w(z) = \frac{\varphi(z)}{z\frac{z-z_0}{1-\overline{z_0}z}},$$

then

$$w(0) = \frac{\varphi'(0)}{-z_0},\ w(z_0) = \frac{\varphi'(z_0)\left(1 - |z_0|^2\right)}{z_0}$$

and

$$C = \frac{\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0} + \frac{\varphi'(0)}{z_0}}{z_0\left(1 + \overline{\frac{\varphi'(0)}{z_0}}\,\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right)},$$

where $|C| \le 1$. Let $|w(0)| = \beta$ and

$$D = \frac{\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right| + \left|\frac{\varphi'(0)}{z_0}\right|}{|z_0|\left(1 + \left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|\right)}.$$

From (2.11), we take

$$|\varphi(z)| \leq |z|\,|p(z)|\,\frac{\beta + |z|\,\frac{D+|p(z)|}{1+D|p(z)|}}{1+\beta\,|z|\,\frac{D+|p(z)|}{1+D|p(z)|}}$$

and

$$\frac{1-|\varphi(z)|}{1-|z|} \geq \frac{1+\beta\,|z|\,\frac{D+|p(z)|}{1+D|p(z)|} - \beta\,|z|\,|p(z)| - |z|^2\,|p(z)|\,\frac{D+|p(z)|}{1+D|p(z)|}}{(1-|z|)\left(1+\beta\,|z|\,\frac{D+|p(z)|}{1+D|p(z)|}\right)} = \psi.$$

Let $\vartheta(z) = 1+\beta\,|z|\,\frac{D+|p(z)|}{1+D|p(z)|}$ and $q(z) = 1 + D\,|p(z)|$. Then
(2.12)

$$\psi = \frac{1-|z|^2\,|p(z)|^2}{(1-|z|)\,\vartheta(z)q(z)} + D\,|p(z)|\,\frac{1-|z|^2}{(1-|z|)\,\vartheta(z)q(z)} + D\beta\,|z|\,\frac{1-|p(z)|^2}{(1-|z|)\,\vartheta(z)q(z)}.$$

Since

$$\lim_{z\to 1}\vartheta(z) = 1+\beta,\ \lim_{z\to 1}q(z) = 1+D$$

and

$$1-|p(z)|^2 = 1-\left|\frac{z-z_0}{1-\overline{z_0}z}\right|^2 = \frac{\left(1-|z_0|^2\right)\left(1-|z|^2\right)}{|1-\overline{z_0}z|^2},$$

passing to the non-tangential limit in (2.12) gives

$$\begin{aligned}\left|\varphi'(1)\right| &\geq \frac{2}{(1+\beta)\,(1+D)}\left(1+\frac{1-|z_0|^2}{|1-z_0|^2} + D + \beta D\frac{1-|z_0|^2}{|1-z_0|^2}\right)\\ &= 1+\frac{1-|z_0|^2}{|1-z_0|^2} + \frac{1-\beta}{1+\beta}\left(1+\frac{1-D}{1+D}\frac{1-|z_0|^2}{|1-z_0|^2}\right).\end{aligned}$$

Moreover, since

$$\begin{aligned}\frac{1-\beta}{1+\beta} &= \frac{1-|w(0)|}{1+|w(0)|} = \frac{1-\frac{|\varphi'(0)|}{|z_0|}}{1+\frac{|\varphi'(0)|}{|z_0|}} = \frac{|z_0|-|\varphi'(0)|}{|z_0|+|\varphi'(0)|} = \frac{|z_0| - \frac{|f''(0)|}{1-\alpha}}{|z_0|+\frac{|f''(0)|}{1-\alpha}}\\ &= \frac{(1-\alpha)\,|z_0| - |f''(0)|}{(1-\alpha)\,|z_0| + |f''(0)|}\end{aligned}$$

and

$$\frac{1-D}{1+D} = \frac{1-\frac{\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|+\left|\frac{\varphi'(0)}{z_0}\right|}{|z_0|\left(1+\left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)(1-|z_0|^2)}{z_0}\right|\right)}}{1+\frac{\left|\frac{\varphi'(z_0)(1-|z_0|^2)}{z_0}\right|+\left|\frac{\varphi'(0)}{z_0}\right|}{|z_0|\left(1+\left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)(1-|z_0|^2)}{z_0}\right|\right)}}$$

$$= \frac{|z_0|\left(1+\left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|\right)-\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|-\left|\frac{\varphi'(0)}{z_0}\right|}{|z_0|\left(1+\left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|\right)+\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|+\left|\frac{\varphi'(0)}{z_0}\right|}$$

$$= \frac{|z_0|\left(1+\frac{|f''(0)|}{(1-\alpha)|z_0|}\frac{\frac{|f''(z_0)|}{1-\alpha}\left(1-|z_0|^2\right)}{|z_0|}\right)-\frac{\frac{|f''(z_0)|}{1-\alpha}\left(1-|z_0|^2\right)}{|z_0|}-\frac{|f''(0)|}{(1-\alpha)|z_0|}}{|z_0|\left(1+\frac{|f''(0)|}{(1-\alpha)|z_0|}\frac{\frac{|f''(z_0)|}{1-\alpha}\left(1-|z_0|^2\right)}{|z_0|}\right)+\frac{\frac{|f''(z_0)|}{1-\alpha}\left(1-|z_0|^2\right)}{|z_0|}+\frac{|f''(0)|}{(1-\alpha)|z_0|}}$$

$$= \frac{(1-\alpha)^2|z_0|^2+|f''(0)|\left(1-|z_0|^2\right)|f''(z_0)|}{(1-\alpha)^2|z_0|^2+|f''(0)|\left(1-|z_0|^2\right)|f''(z_0)|}\times$$

$$\frac{-(1-\alpha)\left(1-|z_0|^2\right)|f''(z_0)|-(1-\alpha)|f''(0)|}{+(1-\alpha)\left(1-|z_0|^2\right)|f''(z_0)|+(1-\alpha)|f''(0)|}$$

we obtain

$$|\varphi'(1)| \geq 1+\frac{1-|z_0|^2}{|1-z_0|^2}+\frac{(1-\alpha)|z_0|-|f''(0)|}{(1-\alpha)|z_0|+|f''(0)|}\times$$

$$\left[1+\frac{(1-\alpha)^2|z_0|^2+|f''(0)|\left(1-|z_0|^2\right)|f''(z_0)|-(1-\alpha)\left(1-|z_0|^2\right)|f''(z_0)|-(1-\alpha)|f''(0)|}{(1-\alpha)^2|z_0|^2+|f''(0)|\left(1-|z_0|^2\right)|f''(z_0)|+(1-\alpha)\left(1-|z_0|^2\right)|f''(z_0)|+(1-\alpha)|f''(0)|}\frac{1-|z_0|^2}{|1-z_0|^2}\right].$$

From (1.2), we have

$$\varphi'(z)=\frac{f''(z)}{1-\alpha}$$

and

$$\left|\varphi'(1)\right|=\frac{|f''(1)|}{1-\alpha}.$$

Thus, we obtain the inequality (2.9).

Now, we shall show that the inequality (2.9) is sharp.

Since $w(z)=\frac{\varphi(z)}{z\frac{z-z_0}{1-\overline{z_0}z}}$ is holomorphic function in the unit disc and $|w(z)|\leq 1$ for $|z|<1$, we obtain

$$\left|\varphi'(0)\right| \leq |z_0|$$

and

$$\left|\varphi'(z_0)\right| \leq \frac{|z_0|}{1-|z_0|^2}.$$

We take $z_0 \in (-1,0)$ and arbitrary two numbers c and d, such that $0 \leq c \leq (1-\alpha)|z_0|$, $0 \leq d \leq (1-\alpha)\frac{|z_0|}{1-|z_0|^2}$. Let

$$K = \frac{\frac{d\left(1-|z_0|^2\right)}{z_0}+\frac{c}{z_0}}{z_0\left(1+cd\frac{1-|z_0|^2}{z_0^2}\right)} = \frac{1}{z_0^2}\frac{d\left(1-|z_0|^2\right)+c}{1+cd\frac{1-|z_0|^2}{z_0^2}}.$$

The composite function

$$\upsilon(z) = -z\frac{z-z_0}{1-\overline{z_0}z}\frac{-\frac{c}{z_0}+z\frac{K+\frac{z-z_0}{1-\overline{z_0}z}}{1+K\frac{z-z_0}{1-\overline{z_0}z}}}{1-\frac{c}{z_0}z\frac{K+\frac{z-z_0}{1-\overline{z_0}z}}{1+K\frac{z-z_0}{1-\overline{z_0}z}}}$$

is holomorphic in E and $|\upsilon(z)|<1$ for $|z|<1$. Let

$$\text{(2.13)} \qquad \frac{f'(z)-1}{1-\alpha} = -z\frac{z-z_0}{1-\overline{z_0}z}\frac{-\frac{c}{z_0}+z\frac{K+\frac{z-z_0}{1-\overline{z_0}z}}{1+K\frac{z-z_0}{1-\overline{z_0}z}}}{1-\frac{c}{z_0}z\frac{K+\frac{z-z_0}{1-\overline{z_0}z}}{1+K\frac{z-z_0}{1-\overline{z_0}z}}}.$$

Therefore, we take $|f''(0)| = (1-\alpha)c$ and

$$\begin{aligned}
\frac{|f''(z_0)|}{1-\alpha} &= -\frac{z_0}{1-z_0^2}\frac{-\frac{c}{z_0}+Kz_0}{1-\frac{c}{z_0}z_0K} = \frac{z_0}{1-z_0^2}\frac{-\frac{c}{z_0}+\frac{1}{z_0^2}\frac{d\left(1-|z_0|^2\right)+c}{1+cd\frac{1-|z_0|^2}{z_0^2}}z_0}{1-\frac{c}{z_0}z_0\frac{1}{z_0^2}\frac{d\left(1-|z_0|^2\right)+c}{1+cd\frac{1-|z_0|^2}{z_0^2}}} \\
\left|f''(z_0)\right| &= (1-\alpha)\,d.
\end{aligned}$$

From (2.13), with the simple calculations, we obtain

$$\begin{aligned}
\frac{|f''(1)|}{1-\alpha} &= 1+\frac{1-z_0^2}{(1-z_0)^2} \\
&\quad + \frac{\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{1-K^2}{(1+K)^2}\right)\left(1-\frac{c}{z_0}\right)+\frac{c}{z_0}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{1-K^2}{(1+K)^2}\right)\left(-\frac{c}{z_0}+1\right)}{\left(-\frac{c}{z_0}+1\right)^2} \\
&= 1+\frac{1-z_0^2}{(1-z_0)^2}+\frac{1+\frac{c}{z_0}}{1-\frac{c}{z_0}}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{1-K}{1+K}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 1+\frac{1-z_0^2}{(1-z_0)^2} \\
&\quad +\frac{c+z_0}{-c+z_0}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{z_0^2+cd\left(1-z_0^2\right)-d\left(1-z_0^2\right)-c}{z_0^2+cd\left(1-z_0^2\right)+d\left(1-z_0^2\right)+c}\right)
\end{aligned}
$$

and

$$
\frac{|f''(1)|}{1-\alpha}=1+\frac{1-z_0^2}{(1-z_0)^2}+\frac{c+z_0}{-c+z_0}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{z_0^2+cd\left(1-z_0^2\right)-d\left(1-z_0^2\right)-c}{z_0^2+cd\left(1-z_0^2\right)+d\left(1-z_0^2\right)+c}\right).
$$

Since $z_0 \in (-1,0)$, the last equality show that (2.9) is sharp. □

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