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THE GENERALIZED COGOTTLIEB GROUPS, RELATED ACTIONS AND EXACT SEQUENCES

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Abstract. The generalized coGottlieb sets are not known to be groups in general. We study some conditions which make them groups. Moreover, there are actions on the generalized coGottlieb sets which are different from known actions up to now. We give related exact sequence of the generalized coGottlieb sets. Using them, we obtain certain results related to the maps which preserve generalized coGottlieb sets.

1. Introduction

The *Gottlieb group* $G_n(X)$ $(n \geq 1)$ was defined by Gottlieb [3] and the *coGottlieb group* $G^n(X;\mathbb{F})$ by Haslam [5, 6] for any abelian group $\mathbb{F}$. The purpose of this paper is to study some properties of the generalizations of the coGottlieb group, namely the homotopy set $DG(X,Z)$ of cocyclic maps by Varadarajan [12], the homotopy set $p^{\top}(X,Z)$ of p-cocyclic maps for a map $p : X \to A$ by Oda [10], and the subset $G_p^n(X;\mathbb{F})$ of p-cocyclic elements in $H^n(X;\mathbb{F})$ by Yoon [14]. We have relations : $DG(X,Z) = (1_X)^{\top}(X,Z)$ for the identity map $1_X : X \to X$ and $G_p^n(X;\mathbb{F}) = p^{\top}(X,K(\mathbb{F},n))$ for the Eilenberg-Mac Lane space $K(\mathbb{F},n)$ (see Section 2 for the definitions).

When Z is a grouplike space, the subset $p^{\top}(X,Z)$ is not known to be closed under the addition $\dot{+}$ in general, even if $Z = K(\mathbb{F},n)$, although the homotopy set $[X,Z]$ is a group. However, if Z is a grouplike space, then we see by Proposition 2.1 that $p^{\top}(X,Z)$ contains the unit 0 and the inverse $-\alpha \in p^{\top}(X,Z)$ for any $\alpha \in p^{\top}(X,Z)$ and, of course, satisfies the associativity; moreover, if $\alpha \in p^{\top}(X,Z)$, then $k\alpha \in p^{\top}(X,Z)$ for any integer k by Theorem 2.3. For further study we introduce the following terminology (Definition 2.4): Let n be a positive integer and $\mathbb{F}$ an abelian group. A map $p : X \to A$ is said to be an $(n,\mathbb{F})$-*essential map* of the coGottlieb group of X if the addition $+$ is

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closed in $G^n_p(X;\mathbb{F})$. A map $p : X \to A$ is said to be an *essential* map if it is an $(n,\mathbb{F})$-essential map of the coGottlieb group of X for every n and $\mathbb{F}$.

More generally, we call a map $p : X \to A$ a *strongly essential* (or *s-essential*) map if the addition $\dot{+}$ is closed in $p^\top(X,Y)$ for any grouplike spaces Y (Definition 4.5),

In Section 3 we consider an action of $H^n(A;\mathbb{F})$ on $H^n(X;\mathbb{F})$. For any elements $a \in H^n(A;\mathbb{F})$ and $f \in H^n(X;\mathbb{F})$, we define an element $a * f \in H^n(X;\mathbb{F})$ by $a*f = a\circ p + f$. Here, the symbol $+$ is the addition in $H^n(X;\mathbb{F})$ induced by the Hopf structure $m : K(\mathbb{F},n)\times K(\mathbb{F},n) \to K(\mathbb{F},n)$. Then we have a pairing

$$\mu : H^n(A;\mathbb{F}) \times H^n(X;\mathbb{F}) \to H^n(X;\mathbb{F})$$

given by $\mu(a,f) = a*f = a\circ p + f$ for any $a \in H^n(A;\mathbb{F})$ and $f \in H^n(X;\mathbb{F})$. We show that there exists an action of the coGottlieb group $G^n(A;\mathbb{F})$ on the set $G^n_p(X;\mathbb{F})$ defined by $\mu(f,a) = a*f$ (Theorem 3.4):

$$\mu : G^n(A;\mathbb{F}) \times G^n_p(X;\mathbb{F}) \to G^n_p(X;\mathbb{F}).$$

In Section 4 we prove that the following sequence is exact as sets for any spaces X, A, Y_1, Y_2 and any map $p : X \to A$ (Theorem 4.2):

$$0 \longrightarrow p^\top(X,Y_1) \xrightarrow{i_{1\sharp}} p^\top(X,Y_1\times Y_2) \xrightarrow{p_{2\sharp}} p^\top(X,Y_2) \longrightarrow 0.$$

The following result is proved (Theorem 4.7(3)): If Y_1 and Y_2 are grouplike spaces and $p : X \to A$ is strongly essential, then there exists an isomorphism of groups

$$p^\top(X,Y_1\times Y_2) \cong p^\top(X,Y_1)\times p^\top(X,Y_2).$$

Let $\mathbb{H}$ and $\mathbb{L}$ be any abelian groups and $p : X \to A$ a map. A homomorphism $h : \mathbb{H} \to \mathbb{L}$ induces a function $h_* : G^n_p(X;\mathbb{H}) \to G^n_p(X;\mathbb{L})$ (Proposition 5.1) and in Section 5 we study some properties of them. Consider the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ B & & A \end{array}$$

Let $m \geq 1$ be an integer. We define the following subset of the homotopy set $[Y,X]$:

$$DCP^m_{q,p}(Y,X;\mathbb{F}) = \{f \in [Y,X] \mid G^n_q(Y;\mathbb{F}) \supset f^*(G^n_p(X;\mathbb{F})) \quad \text{for all} \quad n \leq m\}.$$

A map $f : Y \to X$ is called an $\mathbb{F}$-(q,p)-*cocyclic element preserving map up to* m or an $\mathbb{F}$-$DCP^m_{q,p}$-*map* if $f \in DCP^m_{q,p}(Y,X;\mathbb{F})$ ([7]).

In Theorem 5.3 we prove that if $0 \to \mathbb{H} \xrightarrow{h} \mathbb{G} \xrightarrow{g} \mathbb{L} \to 0$ is a split short exact sequence of abelian groups, then the following relation holds:

$$DCP^m_{q,p}(Y,X;\mathbb{G}) \ \subset \ DCP^m_{q,p}(Y,X;\mathbb{H}) \cap DCP^m_{q,p}(Y,X;\mathbb{L}),$$

and if, in addition, q is an $(n, \mathbb{G})$-essential map for any $n \leq m$, then the equality $DCP^m_{q,p}(Y, X; \mathbb{G}) = DCP^m_{q,p}(Y, X; \mathbb{H}) \cap DCP^m_{q,p}(Y, X; \mathbb{L})$ holds (Theorem 5.4).

2. Certain properties of coGottlieb groups

We consider based topological spaces and based maps in this paper, therefore, a *space* is a based topological space and a *map* is a based map. The identity map for a space X is denoted by $1_X : X \to X$. The symbol $f \simeq g : X \to Y$ means a based homotopy and the homotopy class of a map $f : X \to Y$ is denoted by $[f] : X \to Y$. We use the same symbol f for a map f and its homotopy class $[f]$ in some cases for simplicity. The set of homotopy classes of maps from X to Y is denoted by $[X, Y]$.

For any maps $h : A \to B$ and $u : A \to C$, let

$$h\Delta u = (h \times u) \circ \Delta : A \xrightarrow{\Delta} A \times A \xrightarrow{h \times u} B \times C$$

be the composition of the diagonal map $\Delta : A \to A \times A$ and the product map $h \times u : A \times A \to B \times C$. Let $i_{X,Y} : X \vee Y \to X \times Y$ be the natural inclusion for any spaces X and Y.

Let $h : A \to B$ and $u : A \to C$ be any maps. If there exists a map $\theta : A \to B \vee C$ of type $h\Delta u : A \to B \times C$, namely

$$i_{B,C} \circ \theta \simeq h\Delta u : A \to B \times C,$$

then we write $h \top u$. If $h \top u$, then we have the relation

$$(h' \circ h \circ d) \top (u' \circ u \circ d)$$

for any $d : D \to A$, $h' : B \to B'$ and $u' : C \to C'$ by Theorems 3.3 and 3.4 of [10].

For any map $p : X \to A$, we define

$$p^\top(X, Z) = \{[a] : X \to Z \mid p \top a\} \subset [X, Z]$$

as in [10]. If $p = 1_X$, then we recover the set

$$DG(X, Z) = \{[a] : X \to Z \mid 1_X \top a\} \subset [X, Z]$$

defined by Varadarajan [12]. Let $K(\mathbb{F}, n)$ be the Eilenberg-MacLane space. The *coGottlieb group* or the *coevaluation subgroup*

$$G^n(X; \mathbb{F}) = DG(X, K(\mathbb{F}, n)) = \{[a] : X \to K(\mathbb{F}, n) \mid 1_X \top a\} \subset H^n(X; \mathbb{F})$$

was defined by Haslam [5, 6] for any abelian group $\mathbb{F}$. Yoon [14] defined *the generalized coGottlieb set* $G^n_p(X; \mathbb{F}) = G^n(X, p, A; \mathbb{F})$ of $H^n(X; \mathbb{F})$ by

$$G^n_p(X; \mathbb{F}) = p^\top(X, K(\mathbb{F}, n)) = \{[a] : X \to K(\mathbb{F}, n) \mid p \top a\} \subset H^n(X; \mathbb{F})$$

for any map $p : X \to A$.

We begin by studying some properties of the subset $p^\top(X, Z)$ of $[X, Z]$, where Z is a *grouplike space* of Whitehead [13]:

Proposition 2.1. *If Z is a grouplike space, then the set $p^\top(X, Z)$ satisfies the following*:

(1) *If* $\alpha, \beta, \gamma \in p^{\top}(X, Z)$, *then* $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$.
(2) $0 \in p^{\top}(X, Z)$.
(3) *If* $\alpha \in p^{\top}(X, Z)$, *then* $-\alpha \in p^{\top}(X, Z)$.
(4) (*The case* $p = 1_X$) $\alpha, \beta \in DG(X, Z)$ *implies* $\alpha+\beta \in DG(X, Z)$.

Proof. (1) The associativity holds in $[X, Z]$. However, the addition $+$ is not always closed in $p^{\top}(X, Z)$.

(2) Since $p\top 0$ holds, we have $0 \in p^{\top}(X, Z)$.

(3) If $\alpha \in p^{\top}(X, Z)$, then $p\top\alpha$ and $-\alpha = \nu \circ \alpha$ for the inversion $\nu : Z \to Z$. Since $p\top(\nu \circ \alpha)$, we have $-\alpha \in p^{\top}(X, Z)$.

(4) This is the result of Theorem 4.2 of Lim [8]. □

(*Remark*: If $Z = K(\mathbb{F}, n)$, then (4) is a result of Section 5 of Haslam [6]. However, it is not known when $p^{\top}(X, Z)$ is a group for $p \neq 1_X, *$.)

The following is the case where $Z = K(\mathbb{F}, n)$ in Proposition 2.1.

Corollary 2.2. *The set* $G_p^n(X; \mathbb{F})$ *satisfies the following*:
(1) *If* $\alpha, \beta, \gamma \in G_p^n(X; \mathbb{F})$, *then* $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$.
(2) $0 \in G_p^n(X; \mathbb{F})$.
(3) *If* $\alpha \in G_p^n(X; \mathbb{F})$, *then* $-\alpha \in G_p^n(X; \mathbb{F})$.
(4) $\alpha, \beta \in G^n(X; \mathbb{F})$ *implies* $\alpha+\beta \in G^n(X; \mathbb{F})$.

Theorem 2.3. *Let* $p : X \to A$ *be a map and* Z *a grouplike space. If* $\alpha \in p^{\top}(X, Z)$, *then* $k\alpha \in p^{\top}(X, Z)$ *for any integer* k.

Proof. If $k = 0, \pm 1$, the result is clear by Proposition 2.1. Let $k \geq 2$ be a natural number. Let $\Delta_k : Z \to \prod_k Z$ be the k-hold diagonal map, and $m_k : \prod_k Z \to Z$ the k-hold multiplication. If $f \in p^{\top}(X, Z)$, then $p\top f$ and hence $p\top(m_k \circ \Delta_k \circ f)$. We see

$$m_k \circ \Delta_k \circ f = m_k \circ (\prod_k f) \circ \Delta_k = kf.$$

It follows that $kf \in p^{\top}(X, Z)$. □

By Corollary 2.2, the subset $G_p^n(X; \mathbb{F})$ of $H^n(X : \mathbb{F})$ contains the *unit* 0 and the *inverse* $-\alpha \in G_p^n(X; \mathbb{F})$ for any $\alpha \in G_p^n(X; \mathbb{F})$ and of course satisfies the *associativity*, although $G_p^n(X; \mathbb{F})$ is not proved to be closed under the addition $+$ in general. Therefore we define the following:

Definition 2.4. Let n be a positive integer and $\mathbb{F}$ an abelian group. A map $p : X \to A$ is said to be an $(n, \mathbb{F})$-*essential map* of the coGottlieb group of X if the addition $+$ is closed in $G_p^n(X; \mathbb{F})$. A map $p : X \to A$ is said to be an *essential* map if it is an $(n, \mathbb{F})$-essential map of the coGottlieb group of X for every n and $\mathbb{F}$.

We see that a map $p : X \to A$ is an $(n, \mathbb{F})$-essential map if $p\top\alpha$ and $p\top\beta$ implies $p\top(\alpha+\beta)$ for any $\alpha, \beta \in [X, K(\mathbb{F}, n)]$. Clearly p is $(n, \mathbb{F})$-essential if and only if $G_p^n(X; \mathbb{F})$ is a subgroup of $H^n(X; \mathbb{F})$.

Example 2.5. The identity map $1_X : X \to X$ is an essential map, since the coGottlieb set $G^n_{1_X}(X;\mathbb{F})$ is an abelian group by Theorem 4.2 of [8]. If $p = *$ is the constant map, then $G^n_*(X;\mathbb{F}) = H^n(X;\mathbb{F})$; and hence $*$ is also an essential map. Any cocyclic map $p : X \to A$ is essential, since a map $p : X \to A$ is a cocyclic map if and only if $G^n_p(X;\mathbb{F}) = H^n(X;\mathbb{F})$ for every abelian group $\mathbb{F}$ ([14]).

Example 2.6. The inclusion relation $G^n_p(X;\mathbb{F}) \subset G^n_{r\circ p}(X;\mathbb{F})$ holds for any maps $p : X \to A$ and $r : A \to B$. Hence if $r : A \to B$ is a map with a left homotopy inverse map $\ell : B \to A$ and $p : X \to A$ is an $(n,\mathbb{F})$-essential map, then we see

$$G^n_p(X;\mathbb{F}) \subset G^n_{r\circ p}(X;\mathbb{F}) \subset G^n_{\ell\circ r\circ p}(X;\mathbb{F}) \subset G^n_{1_A\circ p}(X;\mathbb{F}) \subset G^n_p(X;\mathbb{F}),$$

or $G^n_{r\circ p}(X;\mathbb{F}) = G^n_p(X;\mathbb{F})$. Hence $r \circ p : X \to B$ is also $(n,\mathbb{F})$-essential.

Any map $p : X \to A$ with a left homotopy inverse $\ell : A \to X$ is essential, since $G^n_{1_X}(X;\mathbb{F}) \subset G^n_p(X;\mathbb{F}) \subset G^n_{\ell\circ p}(X;\mathbb{F}) = G^n_{1_X}(X;\mathbb{F})$ or $G^n_{1_X}(X;\mathbb{F}) = G^n_p(X;\mathbb{F})$.

Any homotopy equivalence is also an essential map.

Proposition 2.7. *If $p : X \to X$ is a homotopy idempotent map, that is, $p^2 = p \circ p \simeq p$, then $\alpha \circ p + \beta \in G^n_p(X;\mathbb{F})$ for any α, $\beta \in G^n_p(X;\mathbb{F})$.*

Proof. Let α, $\beta \in G^n_p(X;\mathbb{F})$. Then $p\top\alpha$ and $p\top\beta$ hold and we have $(p\circ p)\top(\alpha \circ p + \beta)$ by Theorem 3.9(2) of [10]. It follows that $p\top(\alpha \circ p + \beta)$ or $\alpha \circ p + \beta \in G^n_p(X;\mathbb{F})$. $\square$

Corollary 2.8. *If there exists $p : X \to X$ such that $p^2 \simeq p$ and $\beta \circ p \simeq \beta$ for every $\beta \in G^n_p(X;\mathbb{F})$, then p is an essantial map.*

Example 2.9. Assume that $H^n(X;\mathbb{F}) \cong 0$ or $\mathbb{Z}_{q_n}$ where q_n is a prime number for any $n \geq 1$. Then $G^n_p(X;\mathbb{F}) \cong 0$ or $\mathbb{Z}_{q_n}$ by Theorem 2.3. It follows that all $p : X \to A$ is an essential map of coGottlieb group of X.

Proposition 2.10. *If $p : X \to A$ is an essential map, then the following hold.*

(1) *The induced function $k : G^n_p(X;\mathbb{F}) \to G^n_p(X;\mathbb{F})$ defined by $k(\alpha) = k\alpha$ for any integer k is a homomorphism.*
(2) *There exists a bilinear multiplication $\mu : \mathbb{Z} \times G^n_p(X;\mathbb{F}) \to G^n_p(X;\mathbb{F})$ given by $\mu(k,\alpha) = k\alpha$.*

Proof. (1) is obtained by Theorem 2.3.

(2) Define $\mu : \mathbb{Z} \times G^n_p(X;\mathbb{F}) \to G^n_p(X;\mathbb{F})$ by $\mu(k,\alpha) = k\alpha$. Then we have

$$\mu(h,\mu(k,\alpha)) = h(k\alpha),\ \mu(k+k',\alpha) = \mu(k,\alpha) + \mu(k',\alpha),$$
$$\mu(k,\alpha+\beta) = \mu(k,\alpha) + \mu(k,\beta).$$

$\square$

3. Action $H^n(A;\mathbb{F})$ on $G_p^n(X;\mathbb{F})$ for a given map $p : X \to A$

Let $\nabla = \nabla_Z : Z \vee Z \to Z$ the folding map. For any map $\beta : B \to Z$ and $\gamma : C \to Z$, we define a map

$$\beta\nabla\gamma = \nabla \circ (\beta \vee \gamma) : B \vee C \xrightarrow{\beta\vee\gamma} Z \vee Z \xrightarrow{\nabla} Z.$$

A map $\theta : A \to B \vee C$ defines an addition

$$\beta \dot{+} \gamma = (\beta\nabla\gamma) \circ \theta : A \to Z$$

for any map $\beta : B \to Z$ and $\gamma : C \to Z$; dually, a map $\mu : X \times Y \to Z$ defines an addition

$$\chi \underset{\cdot}{+} \eta = \mu \circ (\chi\Delta\eta) : A \to Z$$

for any map $\chi : A \to X$ and $\eta : A \to Y$ (see [11]).

Now we consider an action of $H^n(A;\mathbb{F})$ on $H^n(X;\mathbb{F})$. For any elements $a \in H^n(A;\mathbb{F})$ and $f \in H^n(X;\mathbb{F})$, we define an element $a * f \in H^n(X;\mathbb{F})$ by

$$a * f = a \circ p + f.$$

Here, the symbol $+$ is the addition in $H^n(X;\mathbb{F})$ induced by the Hopf structure $m : K(\mathbb{F}, n) \times K(\mathbb{F}, n) \to K(\mathbb{F}, n)$. Then we have a pairing

$$\mu : H^n(A;\mathbb{F}) \times H^n(X;\mathbb{F}) \to H^n(X;\mathbb{F})$$

given by $\mu(a, f) = a * f = a \circ p + f$ for any $a \in H^n(A;\mathbb{F})$ and $f \in H^n(X;\mathbb{F})$. We note that μ is a surjective homomorphims of groups.

Proposition 3.1. *The pairing $\mu : H^n(A;\mathbb{F}) \times H^n(X;\mathbb{F}) \to H^n(X;\mathbb{F})$ satisfies the following relations*:

(1) *$a * 0 = a \circ p$ for any $a \in H^n(A;\mathbb{F})$ and $0 * f = f$ for any $f \in H^n(X;\mathbb{F})$.*
(2) *$a * (b * f) = (a + b) * f$ for any a, $b \in H^n(A;\mathbb{F})$ and any $f \in H^n(X;\mathbb{F})$.*
(3) *$(h \circ a) * (h \circ f) = h \circ (a * f)$ for any $a \in H^n(A;\mathbb{F})$, $f \in H^n(X;\mathbb{F})$ and any map $h : K(\mathbb{F}, n) \to K(\mathbb{G}, n)$.*

Proof. (1) For any $a \in H^n(A;\mathbb{F})$ and $f \in H^n(X;\mathbb{F})$, we have

$$a * 0 = a \circ p + 0 = a \circ p, \quad 0 * f = 0 \circ p + f = 0 + f = f.$$

(2) For any a, $b \in H^n(A;\mathbb{F})$, $f \in H^n(X;\mathbb{F})$ and any $f \in H^n(X;\mathbb{F})$, we have

$$\begin{aligned} a * (b * f) &= a \circ p + (b * f) = a \circ p + (b \circ p + f) \\ &= (a \circ p + b \circ p) + f = (a + b) \circ p + f = (a + b) * f. \end{aligned}$$

(3) For any $a \in H^n(A;\mathbb{F})$ and any $h : K(\mathbb{F}, n) \to K(\mathbb{G}, n)$ we have

$$(h \circ a) * (h \circ f) = h \circ a \circ p + h \circ f = h \circ (a \circ p + f) = h \circ (a * f). \qquad \square$$

Corollary 3.2. *For any a, $b \in H^n(A;\mathbb{F})$ and any $f, g \in H^n(X;\mathbb{F})$, we have the following relations*:

$$a * (f + g) = a \circ p + (f + g) = a * f + g = f + a * g,$$

$$a * f = a \circ p + f = a * 0 + f,$$

$$(a+b)*(f+g) = (a+b)*f+g = a*f+b*g = b*(a*f+g).$$

Now we consider an action of $H^n(A;\mathbb{F})$ on $G_p^n(X;\mathbb{F})$ for a given map $p : X \to A$. For any element $f \in G_p^n(X;\mathbb{F})$, there exists a coaffiliated map $\psi_{p,f} : X \to A \vee K(\mathbb{F},n)$ such that $j \circ \psi_{p,f} \simeq (p \times f) \circ \Delta$ as in the following diagram:

$$\begin{tikzcd} X \arrow[r, "\psi_{p,f}"] \arrow[d, "\Delta"'] & A \vee K(\mathbb{F},n) \arrow[d, "j"] \\ X \times X \arrow[r, "p\times f"] & A \times K(\mathbb{F},n) \end{tikzcd}$$

For $a \in H^n(A;\mathbb{F})$ and $b \in H^n(K(\mathbb{F},n);\mathbb{F})$, we define a map $a \dot{+} b : X \to K(\mathbb{F},n)$ by the composition:

$$a \dot{+} b : X \xrightarrow{\psi_{p,f}} A \vee K(\mathbb{F},n) \xrightarrow{a \vee b} K(\mathbb{F},n) \vee K(\mathbb{F},n) \xrightarrow{\nabla} K(\mathbb{F},n).$$

Then $a \dot{+} b$ is an element of $H^n(X;\mathbb{F})$. Moreover, by Theorem 2.7(2) of [11] (set $h = p$, $r = f$, $f = g = 1_{K(\mathbb{F},n)}$, $\alpha = a$ and $\delta = b$ in the theorem), we have

$$a \dot{+} b = a \circ p + b \circ f,$$

where $\dot{+}$ is the addition induced by $\psi_{p,f} : X \to A \vee K(\mathbb{F},n)$ as above and $+$ is the addition in $H^n(X;\mathbb{F})$ which is denoted by $\underset{\cdot}{+}$ in [11]. Therefore, we have a pairing

$$\mu : H^n(A;\mathbb{F}) \times G_p^n(X;\mathbb{F}) \to H^n(X;\mathbb{F})$$

given by $\mu(a,f) = a \dot{+} \iota_{K(\mathbb{F},n)}\,(= a \circ p + f = a * f)$ for any $a \in H^n(A;\mathbb{F})$ and $f \in G_p^n(X;\mathbb{F})$.

Remark 3.3. For a given map $g : K(\mathbb{G},n) \to K(\mathbb{F},n)$ by replacing $a \in H^n(A;\mathbb{F})$ by $a = g \circ l \in H^n(A;\mathbb{F})$ for $l \in H^n(A;\mathbb{G})$ in the above pairing, we can get a pairing

$$\mu : H^n(A;\mathbb{G}) \times G_p^n(X;\mathbb{F}) \to H^n(X;\mathbb{F})$$

defined by $\mu(l,f) = g \circ l \circ p + f$.

Theorem 3.4. *There exists an action of the coGottlieb group $G^n(A;\mathbb{F})$ on the set $G_p^n(X;\mathbb{F})$, that is, the function*

$$\mu : G^n(A;\mathbb{F}) \times G_p^n(X;\mathbb{F}) \to G_p^n(X;\mathbb{F})$$

defined by $\mu(a,f) = a * f$ for any $a \in G^n(A;\mathbb{F})$ and $f \in G_p^n(X;\mathbb{F})$ is well defined.

Proof. If $a \in G^n(A;\mathbb{F})$ and $f \in G_p^n(X;\mathbb{F})$, then $1_A \top a$ and $p \top f$. It follows that $(1_A \circ p) \top (a \circ p + f)$ or $p \top (a \circ p + f)$ by Theorem 3.9(2) of [10]. Hence $a * f = a \circ p + f \in G_p^n(X;\mathbb{F})$. □

Proposition 3.5. (1) *For any $a \in H^n(A;\mathbb{F})$ and $f \in H^n(X;\mathbb{F})$, the induced homomorphism $(a * f)_\# : \pi_i(X) \to \pi_i(K(\mathbb{F},n))$ satisfies*

$$(a * f)_\#(x) = a_\#(p_\#(x)) + f_\#(x)$$

for any $x \in \pi_i(X)$ and $i \geq 0$.

(2) *For any $f \in G_p^n(X;\mathbb{F})$ and $a \in H^n(A;\mathbb{F})$, the induced homomorphism $(a * f)^* : H^i(K(\mathbb{F},n);\mathbb{G}) \to H^i(X;\mathbb{G})$ satisfies*

$$(a * f)^*(x) = (a \circ p)^*(x) + f^*(x)$$

for any $x \in H^i(K(\mathbb{F},n);\mathbb{G})$ and $i \geq 1$.

Proof. (1) Since $a * f = a \circ p + f$, we have

$$(a * f)_\#(x) = (a \circ p + f) \circ x = (a \circ p \circ x) + (f \circ x) = a_\#(p_\#(x)) + f_\#(x).$$

(2) Since $f \in G_p^n(X;\mathbb{F})$, we have $a * f = a \circ p + f = a \dotplus \iota_{K(\mathbb{F},n)}$. Therefore,

$$\begin{aligned}(a * f)^*(x) &= x \circ (a \dotplus 1_{K(\mathbb{F},n)}) = (x \circ a) \dotplus (x \circ 1_{K(\mathbb{F},n)}) \\ &= (x \circ a) \dotplus x = (x \circ a \circ p) \underset{\cdot}{+} (x \circ f) = (a \circ p)^*(x) + f^*(x)\end{aligned}$$

by Theorem 2.7(2) of [11]. □

Arkowitz, Lupton and Murillo [2] defined

$$\mathcal{E}^*(X) = \{f \in \mathcal{E}(X) \mid f^* = 1 : H^i(X;\mathbb{Z}) \to H^i(X;\mathbb{Z}) \text{ for all } i\}$$

for any space X, and

$$\mathcal{E}_\#(X) = \{f \in \mathcal{E}(X) \mid f_\# = 1 : \pi_i(X) \to \pi_i(X) \text{ for all } i \leq N = \dim(X)\}$$

for any CW-complex X. We define

$$\mathcal{E}^*_{n,\mathbb{F}}(X) = \{f \in \mathcal{E}(X) \mid f^* = 1 : H^i(X;\mathbb{F}) \to H^i(X;\mathbb{F}) \text{ for all } i \leq n\}.$$

Example 3.6. Let $X = K(\mathbb{F},n)$. Let $\iota : X \to X$ be the identity map and $a \in H^n(A;\mathbb{F})$. Then we have the following results:

(1) $a * \iota \in \mathcal{E}_\#(X)$ if and only if $a_\# \circ p_\# = 0 : \pi_n(X) \to \pi_n(A) \to \pi_n(K(\mathbb{F},n))$.

(2) Suppose that $\iota \in G_p^n(X,\mathbb{F})$. Then, $a * \iota \in \mathcal{E}^*_{n,\mathbb{F}}(X)$ if and only if $p^* \circ a^* = 0 : H^n(X;\mathbb{F}) \to H^n(A;\mathbb{F}) \to H^n(X;\mathbb{F})$.

(1) is obtained by Proposition 3.5(1).

(2) First assume that $a * \iota \in \mathcal{E}^*_{n,\mathbb{F}}(X)$. By Proposition 3.5(2) and the condition $a * \iota \in \mathcal{E}^*_{n,\mathbb{F}}(X)$, we have

$$p^* \circ a^* = 0 : H^n(X;\mathbb{F}) \to H^n(A;\mathbb{F}) \to H^n(X;\mathbb{F}).$$

Conversely assume that $p^* \circ a^* = 0 : H^n(X;\mathbb{F}) \to H^n(A;\mathbb{F}) \to H^n(X;\mathbb{F})$. Then for any $x \in H^n(X;\mathbb{F})$, we get

$$(a * \iota)^*(x) = (a \circ p)^*(x) + \iota^*(x) = \iota^*(x) = x.$$

It follows that

$$((-a) * \iota) \circ (a * \iota) = (a * \iota)^*((-a) * \iota) = (-a) * \iota = ((-a) * \iota)^*(\iota) = \iota.$$

Similarly, we have $((-a)*\iota)^*(x)=x$ and $(a*\iota)\circ((-a)*\iota)=\iota$. Hence $a*\iota$ is a homotopy equivalence.

Remark 3.7. If $1_{K(\mathbb{F},n)}\in G_p^n(K(\mathbb{F},n),\mathbb{F})$, then $G_p^n(K(\mathbb{F},n),\mathbb{F})=H^n(K(\mathbb{F},n),\mathbb{F})$. Let $X=K(\mathbb{F},n)$ and $p:X\to A$. Suppose that the induced cohomology homomorphism $p^*=0:H^n(A;\mathbb{F})\to H^n(X;\mathbb{F})$ in Example 3.6(2). Then for $a\in H^n(A;\mathbb{F})$ we have $a\circ p=p^*(a)=0$, and

$$a*\iota=(a\circ p)\dot{+}1_{K(\mathbb{F},n)}=0\dot{+}1_{K(\mathbb{F},n)}=\iota\in[X,X].$$

If A is a co-Hopf space, then we have the following sufficient conditions for the map $p:X\to A$ to be an $(n,\mathbb{F})$-essential map of the coGottlieb group of X.

Theorem 3.8. *Let A be co-Hopf space. Then the map $p:X\to A$ is an $(n,\mathbb{F})$-essential map of the coGottlieb group of X if one of the following conditions is satisfied*:

(1) $p^*:H^n(A;\mathbb{F})\to H^n(X;\mathbb{F})$ *is surjective.*
(2) $p:X\to A$ *is an $(n+1)$-equivalence.*

Proof. (1) Consider the function $\mu:G^n(A;\mathbb{F})\times G_p^n(X;\mathbb{F})\to G_p^n(X;\mathbb{F})$ in Theorem 3.4. Since $0\in G_p^n(X;\mathbb{F})$, we have $\mu(a,0)=ap+0=ap$ for all $a\in G^n(A;\mathbb{F})$. Then $p^*(G^n(A;\mathbb{F}))$ is a subset of $G_p^n(X;\mathbb{F})$.

Since A be co-Hopf space, we have $G^n(A;\mathbb{F})=H^n(A;\mathbb{F})$. If $p^*:H^n(A;\mathbb{F})\to H^n(X;\mathbb{F})$ is surjective, then $p^*(G^n(A;\mathbb{F}))=H^n(X;\mathbb{F})$. Since $p^*(G^n(A;\mathbb{F}))\subseteq G_p^n(X;\mathbb{F})$, we get

$$G_p^n(X;\mathbb{F})\subseteq H^n(X;\mathbb{F})=p^*(G^n(A;\mathbb{F}))\subseteq G_p^n(X;\mathbb{F}).$$

It follows that $G_p^n(X;\mathbb{F})=H^n(X;\mathbb{F})$.

(2) If the map $p:X\to A$ is an $(n+1)$-equivalence, then we have the result by virtue of the Proposition 8.2.2 in Arkowitz [1]: *Let $f:X\to Y$ be an n-equivalence, let Z be a space, and let $f^*:[Y,Z]\to[X,Z]$ be the induced map. If $\pi_i(Z)=0$ for $i\geq n$, then f^* is a surjection. If $\pi_i(Z)=0$ for $i\geq n+1$, then f^* is an injection.* □

Example 3.9. Let $X=\mathbb{C}P^n$, the complex projective n-space and $A=S^{2n}$, the $2n$-sphere. Let $p:\mathbb{C}P^n\to S^{2n}$ be the natural projection. Then $p^*:H^n(S^{2n};\mathbb{Z})\to H^n(\mathbb{C}P^n;\mathbb{Z})$ is an isomorphism and satisfies the condition of Theorem 3.8(1).

4. Exact sequences

Let $i_1:Y_1\to Y_1\times Y_2$, $i_2:Y_1\to Y_1\times Y_2$ be the inclusions and $p_1:Y_1\times Y_2\to Y_1$, $p_2:Y_1\times Y_2\to Y_2$ be the projections as is shown in the following diagram.

$$\begin{tikzcd} Y_1 \arrow[r, shift right, "i_1"'] & Y_1\times Y_2 \arrow[l, shift right, "p_1"'] \arrow[r, shift left, "p_2"] & Y_2 \arrow[l, shift left, "i_2"] \end{tikzcd}$$

Let $\alpha\Delta\beta \in [X, Y_1 \times Y_2]$ be the element defined by elements $\alpha \in [X, Y_1]$ and $\beta \in [X, Y_2]$. We define a set function

$$\Phi : p^{\top}(X, Y_1 \times Y_2) \to p^{\top}(X, Y_1) \times p^{\top}(X, Y_2),$$

by $\Phi(\alpha\Delta\beta) = (p_1 \circ (\alpha\Delta\beta), p_2 \circ (\alpha\Delta\beta)) = (\alpha, \beta)$ for any $\alpha\Delta\beta \in p^{\top}(X, Y_1 \times Y_2)$; if $p\top(\alpha\Delta\beta)$, then we see $p\top(p_1 \circ (\alpha\Delta\beta))$ and $p\top(p_2 \circ (\alpha\Delta\beta))$.

Proposition 4.1. *The set function $\Phi : p^{\top}(X, Y_1 \times Y_2) \to p^{\top}(X, Y_1) \times p^{\top}(X, Y_2)$ is a monomorphism for any X, A, Y_1, Y_2 and $p : X \to A$.*

Proof. We have the result by the universality of the product space $Y_1 \times Y_2$. $\square$

Theorem 4.2. *The following sequences are exact as sets for any spaces X, A, Y_1, Y_2 and any map $p : X \to A$.*

$$0 \longrightarrow p^{\top}(X, Y_1) \xrightarrow{i_{1\sharp}} p^{\top}(X, Y_1 \times Y_2) \xrightarrow{p_{2\sharp}} p^{\top}(X, Y_2) \longrightarrow 0;$$

$$0 \longrightarrow p^{\top}(X, Y_2) \xrightarrow{i_{2\sharp}} p^{\top}(X, Y_1 \times Y_2) \xrightarrow{p_{1\sharp}} p^{\top}(X, Y_1) \longrightarrow 0.$$

Proof. We prove the first exact sequence; the second one is proved similarly.

For any $\alpha_2 \in p^{\top}(X, Y_2)$, we have $*\Delta\alpha_2 = i_2 \circ \alpha_2 \in p^{\top}(X, Y_1 \times Y_2)$ and $p_2 \circ (*\Delta\alpha_2) = \alpha_2$.

Assume that $\alpha_1\Delta\alpha_2 \in p^{\top}(X, Y_1 \times Y_2)$ and $p_{2\sharp}(\alpha_1\Delta\alpha_2) = *$. Then we have $\alpha_2 \simeq *$ and hence

$$\alpha_1\Delta\alpha_2 = \alpha_1\Delta* = i_1 \circ \alpha_1,$$

where $\alpha_1 = p_1 \circ (\alpha_1\Delta\alpha_2) \in p^{\top}(X, Y_1)$.

$$\begin{tikzcd} & X \arrow[r, "p"] \arrow[dl, dotted, "\alpha_1"'] \arrow[d, "\alpha_1\Delta\alpha_2"] & A \\ Y_1 \arrow[r, shift right, "i_1"'] & Y_1 \times Y_2 \arrow[l, shift right, "p_1"'] \arrow[r, shift left, "p_2"] & Y_2 \arrow[l, shift left, "i_2"] \end{tikzcd}$$

Finally, the inclusion $i_{1\sharp} : p^{\top}(X, Y_1) \to p^{\top}(X, Y_1 \times Y_2)$ is a monomorphism by the universality of the product space $Y_1 \times Y_2$. $\square$

Proposition 4.3. *If $0 \to \mathbb{H} \to \mathbb{G} \to \mathbb{L} \to 0$ is a split short exact sequence of abelian groups, then there exists the following short exact sequence of sets*:

$$0 \to G^n_p(X; \mathbb{H}) \to G^n_p(X; \mathbb{G}) \to G^n_p(X; \mathbb{L}) \to 0.$$

Proof. The sequence $0 \to \mathbb{H} \to \mathbb{G} \to \mathbb{L} \to 0$ is a split short exact sequence if and only if $\mathbb{G} \cong \mathbb{H} \oplus \mathbb{L}$. The product space $K(\mathbb{H}, n) \times K(\mathbb{L}, n)$ is an Eilenberg-Mac Lane space of type $(\mathbb{H} \oplus \mathbb{L}, n)$. Hence the upper exact sequence in Theorem 4.2 becomes the following exact sequence:

$$0 \to p^{\top}(X, K(\mathbb{H}, n)) \xrightarrow{i_{1\sharp}} p^{\top}(X, K(\mathbb{H}, n) \times K(\mathbb{L}, n)) \xrightarrow{p_{2\sharp}} p^{\top}(X, K(\mathbb{L}, n)) \to 0,$$

which is the exact sequence in question. $\square$

Corollary 4.4. *Let* $0 \to \mathbb{H} \to \mathbb{G} \to \mathbb{L} \to 0$ *be a split short exact sequence of abelian groups. If* p *is an* $(n, \mathbb{F})$*-essential map for any* $\mathbb{F} = \mathbb{H}, \mathbb{G}$ *and* $\mathbb{L}$*, then there exists the following isomorphism of groups*:

$$G_p^n(X : \mathbb{G}) \cong G_p^n(X : \mathbb{H}) \oplus G_p^n(X : \mathbb{L}).$$

Proof. If p is an $(n, \mathbb{F})$-essential map for any $\mathbb{F} = \mathbb{H}, \mathbb{G}$ and $\mathbb{L}$, then the short exact sequence of sets in Proposition 4.3 becomes the short exact sequence of the groups with a cross section, and hence the result follows. □

Definition 4.5. A map $p : X \to A$ is said to be a *strongly essential* (or *s-essential*) map if the addition $\dot{+}$ is closed in $p^\top(X, Y)$ for any grouplike spaces Y.

Remark 4.6. The term '$(n, \mathbb{F})$-essential map' is defined for the case where $Y = K(\mathbb{F}, n)$. The term 'strongly essential map' is used for grouplike spaces Y.

Theorem 4.7. *Let* X *and* A *be any spaces and* $p : X \to A$ *any map. If* $\alpha_1 \in p^\top(X, Y_1)$ *and* $\alpha_2 \in p^\top(X, Y_2)$*, then the following hold.*

(1) $\alpha_1\Delta*$, $*\Delta\alpha_2 \in p^\top(X, Y_1 \times Y_2)$.

(2) *If* Y_1 *and* Y_2 *are Hopf spaces, then* $\alpha_1\Delta\alpha_2 = (\alpha_1\Delta*) \dot{+} (*\Delta\alpha_2)$*, where* $\dot{+}$ *is the addition in* $[X, Y_1 \times Y_2]$.

(3) *If* Y_1 *and* Y_2 *are grouplike spaces and* $p : X \to A$ *is strongly essential, then* $\alpha_1\Delta\alpha_2 \in p^\top(X, Y_1 \times Y_2)$*, and hence there exists an isomorphism of groups*

$$p^\top(X, Y_1 \times Y_2) \cong p^\top(X, Y_1) \times p^\top(X, Y_2).$$

Proof. (1) If $p\top\alpha_1$ and $p\top\alpha_2$, then $p\top(i_1 \circ \alpha_1)$ and $p\top(i_2 \circ \alpha_2)$. Since $\alpha_1\Delta* = i_1 \circ \alpha_1$ and $*\Delta\alpha_2 = i_2 \circ \alpha_2$, we have $p\top(\alpha_1\Delta*)$ and $p\top(*\Delta\alpha_2)$.

(2) We see $(\alpha_1\Delta*) \dot{+} (*\Delta\alpha_2) = (\alpha_1 \dot{+} *)\Delta(* \dot{+} \alpha_2) = \alpha_1\Delta\alpha_2$.

(3) If $p : X \to A$ is strongly essential, then $p^\top(X, Y_1 \times Y_2)$ is a group. It follows then that $p\top\alpha_1$ and $p\top\alpha_2$ imply $p\top(\alpha_1\Delta\alpha_2)$ by the results of Parts (1) and (2).

$$\begin{tikzcd}
p^\top(X, Y_1 \times Y_2) \arrow[r, "\cong"] \arrow[d, "\cap"] & p^\top(X, Y_1) \times p^\top(X, Y_2) \arrow[d, "\cap"'] \\
{[X, Y_1 \times Y_2]} \arrow[r, "\cong"] & {[X, Y_1] \times [X, Y_2]}
\end{tikzcd}$$

□

Remark 4.8. Suppose that Y_1 and Y_2 are Hopf spaces with multiplications $m_1 : Y_1 \times Y_1 \to Y_1$ and $m_2 : Y_2 \times Y_2 \to Y_2$ respectively. Then the multiplication of $Y_1 \times Y_2$ in Theorem 4.7(2) is given by

$$(m_1 \times m_2) \circ (1_{Y_1} \times T \times 1_{Y_2}) : (Y_1 \times Y_2) \times (Y_1 \times Y_2) \to (Y_1 \times Y_1) \times (Y_2 \times Y_2) \to Y_1 \times Y_2,$$

where $T : Y_2 \times Y_1 \to Y_1 \times Y_2$ is the switching map.

Remark 4.9. Let $m : Y_1 \times Y_2 \to Y$ be a pairing. Then for any $\alpha_1 \in [X, Y_1]$ and $\alpha_2 \in [X, Y_2]$, the 'addition' $\dot{+}$ is defined by

$$\alpha_1 \dot{+} \alpha_2 = m \circ (\alpha_1 \Delta \alpha_2) = m \circ (\alpha_1 \times \alpha_2) \circ \Delta.$$

If $p \top (\alpha_1 \Delta \alpha_2)$, then we see $p \top (\alpha_1 \dot{+} \alpha_2)$.

Remark 4.10. Let $\mathbf{Top}_*$ be the category of small topological spaces with base point and **Set** the category of small sets. We note that

$$p^\top(X, \bullet) : \mathbf{Top}_* \to \mathbf{Set}$$

is a functor. If $p : X \to A$ is strongly essential, then $p^\top(X, \bullet)$ preserves products for grouplike spaces by Theorem 4.7(3).

5. Long sequences of coGottlieb sets

Proposition 5.1. *Let $\mathbb{H}$ and $\mathbb{L}$ be any abelian groups and $p : X \to A$ a map. A homomorphism $h : \mathbb{H} \to \mathbb{L}$ induces a function*

$$h_* : G_p^n(X; \mathbb{H}) \to G_p^n(X; \mathbb{L}).$$

Proof. Let $\overline{h} : K(\mathbb{H}, n) \to K(\mathbb{L}, n)$ be the continuous map which induces the homomorphism of homotopy groups

$$\pi_n(\overline{h}) = h : \mathbb{H} = \pi_n(K(\mathbb{H}, n)) \to \pi_n(K(\mathbb{L}, n)) = \mathbb{L}.$$

We have the following commutative diagram:

$$\begin{array}{ccccccc} G_p^n(X;\mathbb{H}) & \subset & [X, K(\mathbb{H}, n)] & \xrightarrow{\overline{h}_\sharp} & [X, K(\mathbb{L}, n)] & \supset & G_p^n(X;\mathbb{L}) \\ & & \| & & \| & & \\ & & H^n(X;\mathbb{H}) & \xrightarrow{h_*} & H^n(X;\mathbb{L}) & & \end{array}$$

If $\alpha \in G_p^n(X; \mathbb{H}) \subset [X, K(\mathbb{H}, n)] = H^n(X; \mathbb{H})$, then $p \top \alpha$ and hence $p \top (\overline{h} \circ \alpha)$. It follows that $h_*(\alpha) = \overline{h} \circ \alpha \in G_p^n(X; \mathbb{L})$. □

We have the following long graded sequence of coGottlieb sets.

Theorem 5.2. *Let $0 \to \mathbb{H} \xrightarrow{h} \mathbb{G} \xrightarrow{g} \mathbb{L} \to 0$ be a short exact sequence of abelian groups. Then, there exists the following long graded sequence of sets*:

$$G_p^1(X;\mathbb{H}) \xrightarrow{h_*} G_p^1(X;\mathbb{G}) \xrightarrow{g_*} \cdots \xrightarrow{g_*} G_p^{n-1}(X;\mathbb{L}) \xrightarrow{\partial_*}$$
$$G_p^n(X;\mathbb{H}) \xrightarrow{h_*} G_p^n(X;\mathbb{G}) \xrightarrow{g_*} G_p^n(X;\mathbb{L}) \xrightarrow{\partial_*} G_p^{n+1}(X;\mathbb{H}) \xrightarrow{h_*} \cdots$$

Proof. Since $0 \to \mathbb{H} \xrightarrow{h} \mathbb{G} \xrightarrow{g} \mathbb{L} \to 0$ is a short exact sequence of abelian groups, we have the following fibration sequence (see p. 167 [4]):

$$K(\mathbb{H},1) \xrightarrow{\overline{h}} K(\mathbb{G},1) \xrightarrow{\overline{g}} K(\mathbb{L},1) \xrightarrow{\overline{\partial}} K(\mathbb{H},2) \xrightarrow{\overline{h}} K(\mathbb{G},2) \to \cdots$$
$$\xrightarrow{\overline{g}} K(\mathbb{L},n-1) \xrightarrow{\overline{\partial}} K(\mathbb{H},n) \xrightarrow{\overline{h}} K(\mathbb{G},n) \xrightarrow{\overline{g}} K(\mathbb{L},n) \xrightarrow{\overline{\partial}} K(\mathbb{H},n+1) \xrightarrow{\overline{h}} \cdots$$

where $\overline{\partial} : \Omega K(\mathbb{L}, n) = K(\mathbb{L}, n-1) \to K(\mathbb{H}, n)$ is the induced map (see Theorem 6.4.14 and Theorem 6.5.7 of [9]). By Theorem 6.4.14, Theorem 6.5.7 and Corollary 6.5.8 of [9], we have an exact sequence

$$\to [X, \Omega K(\mathbb{L}, n)] \xrightarrow{\overline{\partial}_*} [X, K(\mathbb{H}, n)] \xrightarrow{\overline{h}_*} [X, K(\mathbb{G}, n)] \xrightarrow{\overline{g}_*} [X, K(\mathbb{L}, n)] \xrightarrow{\overline{\partial}_*} \cdots$$

which is the long exact sequence

$$\cdots \to H^{n-1}(X; \mathbb{L}) \xrightarrow{\partial_*} H^n(X; \mathbb{H}) \xrightarrow{h_*} H^n(X; \mathbb{G}) \xrightarrow{g_*} H^n(X; \mathbb{L}) \xrightarrow{\partial_*} \cdots.$$

If $\alpha \in G_p^{n-1}(X; \mathbb{L})$, then $p \top \alpha$ and consequently $p \top (\overline{\partial} \circ \alpha)$. This implies $\partial_*(\alpha) = (\overline{\partial} \circ \alpha) \in G_p^n(X; \mathbb{H})$. Hence we have the a graded sequence

$$\cdots \to G_p^{n-1}(X; \mathbb{L}) \xrightarrow{\partial_*} G_p^n(X; \mathbb{H}) \xrightarrow{h_*} G_p^n(X; \mathbb{G}) \xrightarrow{g_*} G_p^n(X; \mathbb{L}) \xrightarrow{\partial_*} \cdots.$$

□

For any map $f : Y \to X$, we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{1*}"] & H^m(X; \mathbb{H}) \arrow[r, "h_{1*}"] \arrow[d, "f^*_{\mathbb{H}}"'] & H^m(X; \mathbb{G}) \arrow[r, "g_{1*}"] \arrow[d, "f^*_{\mathbb{G}}"'] & H^m(X; \mathbb{L}) \arrow[r, "\partial_{1*}"] \arrow[d, "f^*_{\mathbb{L}}"'] & \cdots \\
\cdots \arrow[r, "\partial_{2*}"] & H^m(Y; \mathbb{H}) \arrow[r, "h_{2*}"] & H^m(Y; \mathbb{G}) \arrow[r, "g_{2*}"] & H^m(Y; \mathbb{L}) \arrow[r, "\partial_{2*}"] & \cdots
\end{tikzcd}$$

Consider the following diagram:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "q"'] & X \arrow[d, "p"] \\
B & A
\end{tikzcd}$$

Let $m \geq 1$ be an integer. We define the following subset of the homotopy set $[Y, X]$:

$$DCP^m_{q,p}(Y, X; \mathbb{F}) = \{f \in [Y, X] \mid G^n_q(Y; \mathbb{F}) \supset f^*(G^n_p(X; \mathbb{F})) \quad \text{for all} \quad n \leq m\}.$$

A map $f : Y \to X$ is called an $\mathbb{F}$-(q, p)-*cocyclic element preserving map up to* m or an $\mathbb{F}$-$DCP^m_{q,p}$-*map* if $f \in DCP^m_{q,p}(Y, X; \mathbb{F})$ (see [7]).

By Theorem 5.2, we have the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{1*}"] & G^m_p(X; \mathbb{H}) \arrow[r, "h_{1*}"] \arrow[d, "f^*_{\mathbb{H}}"'] & G^m_p(X; \mathbb{G}) \arrow[r, "g_{1*}"] \arrow[d, "f^*_{\mathbb{G}}"'] & G^m_p(X; \mathbb{L}) \arrow[r, "\partial_{1*}"] \arrow[d, "f^*_{\mathbb{L}}"'] & \cdots \\
\cdots \arrow[r, "\partial_{2*}"] & G^m_q(Y; \mathbb{H}) \arrow[r, "h_{2*}"] & G^m_q(Y; \mathbb{G}) \arrow[r, "g_{2*}"] & G^m_q(Y; \mathbb{L}) \arrow[r, "\partial_{2*}"] & \cdots
\end{tikzcd}$$

If $f \in DCP^m_{q,p}(Y, X; \mathbb{H}) \cap DCP^m_{q,p}(Y, X; \mathbb{G}) \cap DCP^m_{q,p}(Y, X; \mathbb{L})$, then the above diagram is commutative. In general, the homomorphism $f^*_{\mathbb{F}} : G^m_p(X; \mathbb{F}) \to G^m_q(Y; \mathbb{F})$ is not well defined for $\mathbb{F} = \mathbb{H}$, $\mathbb{G}$ and $\mathbb{L}$.

The relation of $DCP^m_{q,p}(Y,X;\mathbb{H})$, $DCP^m_{q,p}(Y,X;\mathbb{G})$ and $DCP^m_{q,p}(Y,X;\mathbb{L})$ is not clear, but if $0\to\mathbb{H}\to\mathbb{G}\to\mathbb{L}\to 0$ is a split short exact sequence of abelian groups, then, the following inclusions of sets exist in $[Y,X]$ by Theorem 5.3 below:

$$\begin{array}{ccc} DCP^m_{q,p}(Y,X;\mathbb{G}) & \subset & DCP^m_{q,p}(Y,X;\mathbb{L}) \\ \cap & & \cap \\ DCP^m_{q,p}(Y,X;\mathbb{H}) & \subset & DCP^m_{q,p}(Y,X;\mathbb{L})\cup DCP^m_{q,p}(Y,X;\mathbb{H}) \end{array}$$

Theorem 5.3. *Let* $0\to\mathbb{H}\xrightarrow{h}\mathbb{G}\xrightarrow{g}\mathbb{L}\to 0$ *be a split short exact sequence of abelian groups. Then the following relation holds*:

$$DCP^m_{q,p}(Y,X;\mathbb{G})\ \subset\ DCP^m_{q,p}(Y,X;\mathbb{H})\cap DCP^m_{q,p}(Y,X;\mathbb{L}).$$

Proof. We have homomorphisms $r:\mathbb{L}\to\mathbb{G}$ and $\ell:\mathbb{G}\to\mathbb{H}$ such that $g\circ r=1_{\mathbb{L}}$, $\ell\circ h=1_{\mathbb{H}}$ and $h\circ\ell+r\circ g=1_{\mathbb{G}}$.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{H} \arrow[r, shift left, "h"] & \mathbb{G} \arrow[l, shift left, "\ell"] \arrow[r, shift left, "g"] & \mathbb{L} \arrow[l, shift left, "r"] \arrow[r] & 0 \end{tikzcd}$$

We then have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^n(X;\mathbb{H}) \arrow[r, shift left, "h_{1*}"] \arrow[d, "f^*_{\mathbb{H}}"'] & H^n(X;\mathbb{G}) \arrow[l, shift left, "\ell_{1*}"] \arrow[r, shift left, "g_{1*}"] \arrow[d, "f^*_{\mathbb{G}}"'] & H^n(X;\mathbb{L}) \arrow[l, shift left, "r_{1*}"] \arrow[r] \arrow[d, "f^*_{\mathbb{L}}"'] & 0 \\ 0 \arrow[r] & H^n(Y;\mathbb{H}) \arrow[r, shift left, "h_{2*}"] & H^n(Y;\mathbb{G}) \arrow[l, shift left, "\ell_{2*}"] \arrow[r, shift left, "g_{2*}"] & H^n(Y;\mathbb{L}) \arrow[l, shift left, "r_{2*}"] \arrow[r] & 0 \end{tikzcd}$$

for any map $f:Y\to X$ and any $n\le m$, where the rows are short exact sequences.

By the above commutative diagram, we know that

$$f^*_{\mathbb{H}}=\ell_{2*}\circ f^*_{\mathbb{G}}\circ h_{1*} \text{ and } f^*_{\mathbb{L}}=g_{2*}\circ f^*_{\mathbb{G}}\circ r_{1*}.$$

By Proposition 5.1, the homomorphisms h_{k*}, g_{k*}, r_{k*} and ℓ_{k*} are induced by maps for $k=1,2$. If $f\in DCP^m_{q,p}(Y,X;\mathbb{G})$ then $f^*_{\mathbb{G}}:G^n_p(X;\mathbb{G})\to G^n_q(Y;\mathbb{G})$ is well defined for any $n\le m$. Hence, we have $f\in DCP^m_{q,p}(Y,X;\mathbb{H})\cap DCP^m_{q,p}(Y,X;\mathbb{L})$. □

Theorem 5.4. *Let* $0\to\mathbb{H}\xrightarrow{h}\mathbb{G}\xrightarrow{g}\mathbb{L}\to 0$ *be a split short exact sequence of abelian groups. Suppose that q is an $(n,\mathbb{G})$-essential map for any $n\le m$. Then the following equality holds*:

$$DCP^m_{q,p}(Y,X;\mathbb{G})=DCP^m_{q,p}(Y,X;\mathbb{H})\cap DCP^m_{q,p}(Y,X;\mathbb{L}).$$

Proof. We have the following diagram where two rows are split exact sequences by Proposition 4.3:

$$
\begin{tikzcd}
0 \arrow[r] & G_p^n(X;\mathbb{H}) \arrow[r, shift left, "h_{1*}"] \arrow[d, "f_{\mathbb{H}}^*"'] & G_p^n(X;\mathbb{G}) \arrow[l, shift left, "\ell_{1*}"] \arrow[r, shift left, "g_{1*}"] \arrow[d, "f_{\mathbb{G}}^*"'] & G_p^n(X;\mathbb{L}) \arrow[l, shift left, "r_{1*}"] \arrow[r] \arrow[d, "f_{\mathbb{L}}^*"'] & 0 \\
0 \arrow[r] & G_q^n(Y;\mathbb{H}) \arrow[r, shift left, "h_{2*}"] & G_q^n(Y;\mathbb{G}) \arrow[l, shift left, "\ell_{*2}"] \arrow[r, shift left, "g_{2*}"] & G_q^n(Y;\mathbb{L}) \arrow[l, shift left, "r_{2*}"] \arrow[r] & 0
\end{tikzcd}
$$

for any map $f : Y \to X$. By Proposition 5.1, the homomorphisms h_{k*}, g_{k*}, r_{k*} and ℓ_{k*} are well defined for $k = 1, 2$. By the commutativity of the diagram in the proof of Theorem 5.3, we know that

$$
f_{\mathbb{H}}^* = \ell_{2*} \circ f_{\mathbb{G}}^* \circ h_{1*} \text{ and } f_{\mathbb{L}}^* = g_{2*} \circ f_{\mathbb{G}}^* \circ r_{1*};
$$
$$
f_{\mathbb{G}}^* = h_{2*} \circ f_{\mathbb{H}}^* \circ \ell_{1*} + r_{2*} \circ f_{\mathbb{L}}^* \circ g_{1*}.
$$

Since q is an $(n, \mathbb{G})$-essential map for any $n \leq m$, the set $G_q^n(Y;\mathbb{G})$ is a group and hence we see that $f \in DCP_{q,p}^m(Y, X;\mathbb{G})$ if and only if $f \in DCP_{q,p}^m(Y, X;\mathbb{H}) \cap DCP_{q,p}^m(Y, X;\mathbb{L})$. □

Lemma 5.5. *$DCP_{q,p}^m(Y, X; \oplus^s\mathbb{L}) \subset DCP_{q,p}^m(Y, X;\mathbb{L})$ for any $s \geq 2$.*

Proof. Let $0 \to \mathbb{L} \to \mathbb{L} \oplus \mathbb{L} \to \mathbb{L} \to 0$ be the short exact sequence for the direct sum $\mathbb{L} \oplus \mathbb{L}$. By Theorem 5.3, we have

$$
DCP_{q,p}^m(Y, X;\mathbb{L} \oplus \mathbb{L}) \subset DCP_{q,p}^m(Y, X;\mathbb{L}).
$$

Furthermore, let $0 \to \oplus^j\mathbb{L} \to \oplus^{j+k}\mathbb{L} \to \oplus^k\mathbb{L} \to 0$ be the short exact sequence for the direct sum $\oplus^{j+k}\mathbb{L}$ for $j,\ k \geq 1$. Then by induction we have

$$
DCP_{q,p}^m(Y, X; \oplus^{j+k}\mathbb{L}) \subset DCP_{q,p}^m(Y, X;\mathbb{L}).
$$

□

Proposition 5.6. *Let $\boldsymbol{G}$ be a finitely generated abelian group. Assume that $\boldsymbol{G} = \boldsymbol{F} \oplus \boldsymbol{T}$ where $\boldsymbol{F} \neq 0$ is the free part and $\boldsymbol{T} \neq 0$ is the torsion part. Let $\boldsymbol{F} = \oplus^s\mathbb{Z}$ and $\boldsymbol{T} = \oplus^t\mathbb{Z}_{p_t^{a_t}}$ where p_t is a prime number and a_t is a positive integer for any t. Let M_T of $\boldsymbol{T}$ be a subgroup of $\boldsymbol{T}$ defined by $M_T = \oplus^i\mathbb{Z}_{q_i^{b_i}}$ such that $q_i^{b_i} \neq q_j^{b_j}$ for $i \neq j$ (that is, M_T is defined making use of all the different direct summands in $\boldsymbol{T}$). Then the following inclusion holds*:

$$
DCP_{q,p}^m(Y, X; \boldsymbol{G}) \subset DCP_{q,p}^m(Y, X;\mathbb{Z} \oplus M_T).
$$

Proof. By Lemma 5.5, we have the result. □

Corollary 5.7. *Assume the same conditions as in Proposition 5.6. Suppose that q is an essential map. Then $DCP_{q,p}^m(Y, X; \boldsymbol{G}) = DCP_{q,p}^m(Y, X;\mathbb{Z} \oplus M_T)$.*

Proposition 5.8. *If the homomorphism $g : \mathbb{G} \to \mathbb{L}$ has a right inverse homomorphism $r : \mathbb{L} \to \mathbb{G}$, then*

$$DCP^m_{q,p}(Y, X; \mathbb{G}) \subset DCP^m_{q,p}(Y, X; \mathbb{L}).$$

If $g : \mathbb{G} \to \mathbb{L}$ is an isomorphism, then $DCP^m_{q,p}(Y, X; \mathbb{G}) = DCP^m_{q,p}(Y, X; \mathbb{L})$.

Proof. Let $f \in DCP^m_{q,p}(Y, X; \mathbb{G})$ and let $\alpha \in G^n_p(X; G)$. From the induced maps

$$K(\mathbb{L}, n) \xrightarrow{\overline{r}} K(\mathbb{G}, n) \xrightarrow{\overline{g}} K(\mathbb{L}, n),$$

we have the following commutative diagram:

$$\begin{tikzcd} H^n(X;\mathbb{G}) \arrow[r, shift left, "\overline{g}_{1*}"] \arrow[d, "f^*_{\mathbb{G}}"] & H^n(X;\mathbb{L}) \arrow[l, shift left, "\overline{r}_{1*}"] \arrow[d, "f^*_{\mathbb{L}}"] \\ H^n(Y;\mathbb{G}) \arrow[r, shift left, "\overline{g}_{2*}"] & H^n(Y;\mathbb{L}) \arrow[l, shift left, "\overline{r}_{2*}"] \end{tikzcd}$$

It follows that $\overline{g}_{2*} \circ f^*_{\mathbb{G}} = f^*_{\mathbb{L}} \circ \overline{g}_{1*}$. Hence composing the induced right inverse homotopy map $\overline{r}_*$, we have

$$f^*_{\mathbb{L}} = f^*_{\mathbb{L}} \circ \overline{g}_{1*} \circ \overline{r}_{1*} = \overline{g}_{2*} \circ f^*_{\mathbb{G}} \circ \overline{r}_{1*}.$$

If $\alpha \in G^n_p(X : \mathbb{L})$, then we have $f^*_{\mathbb{L}}(\alpha) = (\overline{g}_{2*} \circ f^*_{\mathbb{G}} \circ \overline{r}_{1*})(\alpha) \in G^n_q(Y : \mathbb{L})$ by the definition of f. This completes the proof. $\square$

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