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A NEW FAMILY OF FUBINI TYPE NUMBERS AND POLYNOMIALS ASSOCIATED WITH APOSTOL-BERNOULLI NUMBERS AND POLYNOMIALS

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Abstract. The purpose of this paper is to construct a new family of the special numbers which are related to the Fubini type numbers and the other well-known special numbers such as the Apostol-Bernoulli numbers, the Frobenius-Euler numbers and the Stirling numbers. We investigate some fundamental properties of these numbers and polynomials. By using generating functions and their functional equations, we derive various formulas and relations related to these numbers and polynomials. In order to compute the values of these numbers and polynomials, we give their recurrence relations. We give combinatorial sums including the Fubini type numbers and the others. Moreover, we give remarks and observation on these numbers and polynomials.

1. Introduction

In this section we introduce some generating functions for some special numbers and polynomials with their recurrence relations and other well-known properties. By using generating function and functional equation method, we derive our relations and identities.

We start to give some useful and well-known numbers and polynomials, which have many applications in almost all branches of mathematics and also mathematical physics.

The Bernoulli polynomials $B_n(x)$ are defined by means of the following generating functions (*cf.* [1]-[33]; see also the references cited in each of these earlier

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works):

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1}$$

where $|t|<2\pi$. By (1), one can easily deduce the following formula:

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}B_k, \tag{2}$$

where B_k denote the Bernoulli numbers, which are defined by the following recurrence relation:

$$B_0=1 \text{ and } \sum_{k=0}^{n-1}\binom{n}{k}B_k=0 \quad (n\in\mathbb{N}\setminus\{1\}).$$

One can also easily see that

$$B_n=B_n(0) \quad (n\in\mathbb{N}_0)$$

(*cf.* [1]-[33]; see also the references cited in each of these earlier works).

Here and in the following, let $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{N}_0$ be the sets of complex numbers, positive integers, and non-negative integers, respectively.

The other famous family is the Apostol-Bernoulli polynomials $\mathcal{B}_n(x,\lambda)$, which are defined by means of the following generating function ([2]):

$$\frac{te^{xt}}{\lambda e^t-1}=\sum_{n=0}^{\infty}\mathcal{B}_n(x,\lambda)\frac{t^n}{n!} \tag{3}$$

$$(|t|<2\pi \text{ if } \lambda=1;\ |t|<|\log\lambda| \text{ if } \lambda\neq 1 \text{ and } \lambda\in\mathbb{C}).$$

From this equation, one can easily see that

$$B_n(x)=\mathcal{B}_n(x,1)$$

and

$$\mathcal{B}_n(\lambda)=\mathcal{B}_n(0,\lambda),$$

where $\mathcal{B}_n(\lambda)$ denotes so-called the Apostol-Bernoulli numbers (*cf.* [1]-[33]; see also the references cited in each of these earlier works).

In work of Apostol [2] and by equation (3), we have the following properties: For $n\geq 1$, we have

$$\lambda\mathcal{B}_n(x+1,\lambda)-\mathcal{B}_n(x,\lambda)=nx^{n-1}.$$

Substituting $x=0$ and $n=1$ into the above equation, one can easily see that

$$\lambda\mathcal{B}_1(1,\lambda)=1+\mathcal{B}_1(\lambda). \tag{4}$$

If $n\geq 2$, one can see that

$$\lambda\mathcal{B}_n(1,\lambda)=\mathcal{B}_n(\lambda) \tag{5}$$

(*cf.* [2]).

For $x = 0$ and $n = 0, 1, 2, 3, 4, \ldots$, one can compute a few values of the Apostol-Bernoulli numbers given by equation (3) as follows:

$$\begin{aligned}
\mathcal{B}_0(\lambda) &= 0,\\
\mathcal{B}_1(\lambda) &= \frac{1}{\lambda-1},\\
\mathcal{B}_2(\lambda) &= -\frac{2\lambda}{(\lambda-1)^2},\\
\mathcal{B}_3(\lambda) &= \frac{3\lambda(\lambda+1)}{(\lambda-1)^3},\\
\mathcal{B}_4(\lambda) &= -\frac{4\lambda(\lambda^2+4\lambda+1)}{(\lambda-1)^4}.
\end{aligned}$$

Recently, not only the Bernoulli polynomials and numbers, but also the Apostol-Bernoulli polynomials and numbers have been studied by many authors with various methods and techniques. There are also many type generalizations of these polynomials and numbers.

The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(m)}(x,\lambda)$ of order m are defined by means of the following generating function:

$$\left(\frac{t}{\lambda e^t - 1}\right)^m e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(m)}(x,\lambda)\frac{t^n}{n!}, \tag{6}$$

where $\mid t + \ln\lambda \mid < 2\pi$; $1^m = 1$. We also see that

$$B_n^{(m)}(x) = \mathcal{B}_n^{(m)}(x,1)$$

and

$$\mathcal{B}_n^{(m)}(\lambda) = \mathcal{B}_n^{(m)}(0,\lambda),$$

where $\mathcal{B}_n^{(m)}(\lambda)$ denote the so-called Apostol-Bernoulli numbers of order m ([16], [17], [23]). By using (6), one can see that

$$\lambda\mathcal{B}_n^{(m)}(x+1,\lambda) - \mathcal{B}_n^{(m)}(x,\lambda) = n\mathcal{B}_{n-1}^{(m-1)}(x,\lambda) \tag{7}$$

and also

$$\mathcal{B}_n^{(m)}(x,\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k^{(m)}(\lambda)x^{n-k}. \tag{8}$$

The Eulerian numbers or *Frobenius-Euler numbers* are defined by means of the following generating function:

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u)\frac{t^n}{n!}, \tag{9}$$

where $u \neq 1$ (*cf.* [7], [11], [13], [14], [15], [25], [26], [27], [29], [30], [31], [34]; see also the references cited in each of these earlier works). Substituting $u = -1$, into (9), we have

$$H_n(-1) = E_n,$$

where E_n denote Euler numbers of the first kind defined by means of the following generating function:

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}, \tag{10}$$

where $|t|<\pi$ (*cf.* [7], [11], [13], [14], [15], [25], [26], [27], [29], [30], [31], [34]; see also the references cited in each of these earlier works).

The Stirling numbers of the second kind are defined by means of the following generating function:

$$F_s(t,m)=\frac{\left(e^t-1\right)^m}{m!}=\sum_{n=0}^{\infty}S_2\left(n,m\right)\frac{t^n}{n!}. \tag{11}$$

By using this generating function, we have

$$S_2\left(n,m\right)=\frac{1}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(m-k\right)^n,$$

$$S_2\left(n,n\right)=1 \quad \left(n\in\mathbb{N}_0\right),$$

$$S_2\left(n,0\right)=0 \quad \left(n\in\mathbb{N}\right)$$

and

$$S_2\left(n,m\right)=0 \quad \left(m>n\right).$$

(*cf.* [1]-[33]; see also the references cited in each of these earlier works).

Now we recall another numbers which are related to the Apostol-Bernoulli numbers. The Fubini numbers $w_g(n)$ are defined by means of the following generating function [12]:

$$\frac{1}{2-e^t}=\sum_{n=0}^{\infty}w_g(n)\frac{t^n}{n!}, \tag{12}$$

where $w_g(0)=1$ and $|t|<\ln 2$.

The nth Fubini numbers were defined in [10], [12], and [19] as the number of ways of writing an nth multiple integral. Or, in combinatoric applications, equivalently, these numbers can be defined as the number of order partition of nonempty subset of $\{1,2,3,\ldots,n\}$.

Muresan [19, p. 397] defined the Fubini type numbers by means of the following generating function:

$$F_M(t)=\frac{e^t-1}{2-e^t}=\sum_{n=0}^{\infty}w_M(n)\frac{t^n}{n!} \tag{13}$$

(*cf.* [10], [12], [19]; see also the references cited in each of these earlier works). In (13), we see that

$$w_M(0)=0.$$

Belbachir *et al.* [4] modified the numbers $w_g(n)$. They defined a new family of the numbers, a_n. These numbers are related to the Apostol-Bernoulli numbers of order 2 and the Fubini numbers $w_g(n)$. A generating function for the numbers a_n have been defined by

$$\frac{2}{(2-e^t)^2} = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}. \tag{14}$$

By using (14), we have a few values as follows:

$$a_0 = 2,\ a_1 = 4,\ a_2 = 16,\ a_3 = 88,\ a_4 = 616, \ldots.$$

The generalized Fubini numbers $f_{n,k}$ are defined by means of following generating function (*cf.* [19]):

$$F_k(t) = \frac{e^t - 1}{k + 1 - ke^t} = \sum_{n=1}^{\infty} f_{n,k} \frac{t^n}{n!}. \tag{15}$$

It obvious that $f_{n,1} = w_M(n)$ for all $n \in \mathbb{N}_0$ and $f_{0,k} = 1$.

Some values of generalized Fubini numbers are listed in the following table (*cf.* [19]):

TABLE 1. Generalized Fubini numbers $f_{n,k}$

$n \diagdown k$	1	2	3	4
1	1	1	1	1
2	3	5	7	9
3	13	37	73	121
4	75	365	1015	2169
5	541	4501	17641	48601

We now summarize our paper as follows:

In Section 1, we look for some properties of the numbers $w_g(n)$ and $w_M(n)$. We give some formulas and relations related to these numbers and the other special numbers. We also give relationships between the numbers $w_g(n)$, $w_M(n)$, Bernoulli numbers, Apostol-Bernoulli numbers, Apostol-Bernoulli polynomials, Frobenius-Euler numbers and Stirling numbers.

In Section 2, we define a new family of polynomials $a_n^{(l)}(x)$ and new family of numbers. We investigate their properties and relations.

In Section 3, by using generating functions and their functional equations, we derive many novel identity and relations including the Apostol-Bernoulli numbers and polynomials, the Stirling numbers of the second kind and the Fubini type numbers and polynomials.

In Section 4, by applying the Riemann integral to our identities and relations in Section 2, we derive combinatorial sums including the Fubini type numbers and the Apostol-Bernoulli numbers.

2. Relationship between the numbers $w_g(n)$ and $w_M(n)$

In this section we investigate some properties of the numbers $w_g(n)$ and $w_M(n)$. We give some formulas and relation related to these numbers and the other special numbers.

We give a relationship between the numbers $w_g(n)$ and $w_M(n)$. Combining (12) with (13), we get

$$\sum_{n=0}^{\infty} w_M(n)\frac{t^n}{n!} = \left(e^t - 1\right)\sum_{n=0}^{\infty} w_g(n)\frac{t^n}{n!}.$$

From this equation, we obtain

$$\sum_{n=0}^{\infty} w_M(n)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{n=0}^{\infty} w_g(n)\frac{t^n}{n!} - \sum_{n=0}^{\infty} w_g(n)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} w_M(n)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} w_g(k) - w_g(n)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain an interesting identity asserted by the following theorem.

Theorem 2.1. *Let n be a positive integer. Then we have*

$$w_M(n) = \sum_{k=0}^{n-1}\binom{n}{k} w_g(k). \tag{16}$$

By using generating functions (3) with $x = 0$ and (12), we derive a relation between the Apostol-Bernoulli numbers, $\mathcal{B}_n(\lambda)$ and the Fubini numbers $w_g(n)$ are given as follows:

$$\mathcal{B}_n\left(\frac{1}{2}\right) = -2n w_g(n-1). \tag{17}$$

We observe that the Fubini numbers $w_g(n)$ are related to the Frobenius-Euler numbers. That is

$$w_g(n) = H_n(2).$$

By substituting (17) into (16), we obtain a relation between the numbers $w_M(n)$ and $\mathcal{B}_n\left(\frac{1}{2}\right)$ by the following proposition:

Proposition 2.2.

$$w_M(n) = -\frac{1}{2}\sum_{k=0}^{n-1}\binom{n}{k}\frac{1}{k+1}\mathcal{B}_{k+1}\left(\frac{1}{2}\right).$$

Comtet [10] raised the following question concerning the Fubini numbers and the Stirling numbers of the second kind:

$$w_g(n) = \sum_{k=0}^{n} k! S_2(n,k).$$

We briefly glance exercise from Comtet's [10, p. 228, Exercise 20]. In order to solve this exercise, we use (12). Firstly, we assume that $|e^t - 1| < 1$ in (12), thus we get

$$\sum_{n=0}^{\infty} w_g(n) \frac{t^n}{n!} = \sum_{k=0}^{\infty} (e^t - 1)^k.$$

Combining (11) with the above equation, we have

$$\sum_{n=0}^{\infty} w_g(n) \frac{t^n}{n!} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} k! S_2(n,k) \frac{t^n}{n!}.$$

Since $n < k$, $S_2(n,k) = 0$, we deduce the above equation

$$\sum_{n=0}^{\infty} w_g(n) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} k! S_2(n,k) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we a solution of the exercise from Comtet's:

$$w_g(n) = \sum_{k=0}^{n} k! S_2(n,k).$$

By using this identity, we also give an explicit formula for the numbers $w_M(n)$ as follows:

Theorem 2.3.

$$w_M(n) = \sum_{k=0}^{n-1} \sum_{j=0}^{k} \sum_{l=0}^{j} (-1)^l \binom{n}{k} \binom{j}{l} (j-l)^k .$$

3. A new family of polynomials and numbers

In this section, we define a new family of polynomials $a_n^{(l)}(x)$ by means of the following generating function:

$$F_a(x;t,l) = \frac{2^l}{(2-e^t)^{2l}} e^{xt} = \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!}, \tag{18}$$

where $l \in \mathbb{N}_0$ and $\mid t \mid < \ln 2$.

By (18), we have

$$a_n^{(l)}(0) = a_n^{(l)}$$

which denote *Fubini type numbers of order l*. In next section, we show that these numbers are related to the Apostol-Bernoulli numbers. Substituting $x = 0$ and $l = 1$ into (18), we get

$$a_n^{(1)} = a_n.$$

The numbers $a_n^{(l)}$ are defined by means of the following generating function:

$$F_a(t,l) = \frac{2^l}{(2-e^t)^{2l}} = \sum_{n=0}^{\infty} a_n^{(l)} \frac{t^n}{n!}. \tag{19}$$

Theorem 3.1. *The following identity holds true*:

$$a_n^{(u+v)} = \sum_{k=0}^{n} \binom{n}{k} a_k^{(u)} a_{n-k}^{(v)} \quad (u, v \in \mathbb{N}). \tag{20}$$

Proof. By (19), we derive the following functional equation:

$$F_a(t, u+v) = F_a(t,u)F_a(t,v).$$

From this equation, we obtain

$$\sum_{n=0}^{\infty} a_n^{(u+v)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} a_n^{(u)} \frac{t^n}{n!} \sum_{n=0}^{\infty} a_n^{(v)} \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} a_n^{(u+v)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k^{(u)} a_{n-k}^{(v)} \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

For $n = 0, 1, 2, 3, 4, 5, \ldots$, we compute a few values of the numbers $a_n^{(l)}$ given by equation (20) as follows:

TABLE 2. Some values of the numbers $a_n^{(l)}$

$n \diagdown l$	1	2	3	4	5
0	2	4	8	16	32
1	4	16	48	128	320
2	16	96	384	1280	3840
3	88	736	3744	15104	53120
4	616	6816	42720	204032	827520
5	5224	73696	556128	3093248	14288000

Theorem 3.2. *The following identity holds true*:

$$a_n^{(l)}(x) = \sum_{k=0}^{n} \binom{n}{k} a_k^{(l)} x^{n-k} \quad (l \in \mathbb{N}_0). \tag{21}$$

Proof. By (18) and (19), we get

$$\sum_{n=0}^{\infty} a_n^{(l)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} a_n^{(l)}\frac{t^n}{n!}\sum_{n=0}^{\infty} x^n\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} a_n^{(l)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a_k^{(l)}x^{n-k}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

We now define the generalized Fubini numbers $f_{n,k}^{(m)}$ of order m by means of the following generating function:

$$F_k^{(m)}(t) = \sum_{n=1}^{\infty} f_{n,k}^{(m)}\frac{t^n}{n!} = \left(\frac{e^t-1}{k+1-ke^t}\right)^m. \tag{22}$$

From this equation, we get the following theorem:

Theorem 3.3.

$$f_{n,k}^{(m)} = \sum_{j=1}^{n}\binom{n}{j}f_{j,k}^{(l)}f_{n-j,k}^{(m-l)}.$$

Proof. The proof is similar to that of Theorem 3.1. We omit the details. □

4. Identities and relations

In this section, by using generating functions, we derive various novel identity and relations including the Apostol-Bernoulli numbers and polynomials, the Stirling numbers of the second kind and the Fubini type numbers and polynomials.

By using (11), (13) and (19), we derive the following functional equation:

$$\frac{(2k)!}{2^k}F_s(t,2k)F_a(t,k) = F_M^{2k}(t).$$

By using this functional equation, we get

$$\sum_{n=0}^{\infty} w_M^{(2k)}(n)\frac{t^n}{n!} = \frac{(2k)!}{2^k}\sum_{n=0}^{\infty} S_2(n,2k)\frac{t^n}{n!}\sum_{n=0}^{\infty} a_n^{(k)}\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} w_M^{(2k)}(n)\frac{t^n}{n!} = \frac{(2k)!}{2^k}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}S_2(j,2k)\,a_{n-j}^{(k)}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem.

Theorem 4.1.

$$w_M^{(2k)}(n) = \frac{(2k)!}{2^k} \sum_{j=0}^{n} \binom{n}{j} S_2(j, 2k)\, a_{n-j}^{(k)}. \tag{23}$$

Substituting $k = 1$ into (23), we get the following alternative formulas for the numbers $w_M^{(2)}(n)$ as follows:

$$w_M^{(2)}(n) = \sum_{j=0}^{n} \binom{n}{j} w_M(j) w_M(n-j),$$

$$w_M^{(2)}(n) = \sum_{j=0}^{n} \binom{n}{j} S_2(j, 2)\, a_{n-j},$$

and since $j < 2$, $S_2(j, 2) = 0$, we have

$$w_M^{(2)}(n) = \sum_{j=2}^{n} \binom{n}{j} S_2(j, 2)\, a_{n-j}.$$

Proposition 4.2 (A recurrence relation for the numbers a_n)**.** *Let* $a_0 = 2$. *Then we have*

$$a_n = \sum_{k=0}^{n} \binom{n}{k} a_k - \sum_{k=0}^{n} \binom{n}{k} 2^{n-k-2} a_k. \tag{24}$$

Proof. By (14), we get

$$4 \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} - 4e^t \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} + e^{2t} \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} = 2.$$

Therefore

$$\sum_{n=0}^{\infty} \left(4a_n - 4 \sum_{k=0}^{n} \binom{n}{k} a_k + \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} a_k \right) \frac{t^n}{n!} = 2.$$

For $n \in \mathbb{N}$, by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

A relation between the numbers a_n and the Apostol-Bernoulli numbers of order 2 is given by the following proposition:

Proposition 4.3. *The following identity holds true*:

$$a_n = \frac{1}{2(n+1)(n+2)} \mathcal{B}_{n+2}^{(2)}\left(\frac{1}{2}\right) \quad (n \in \mathbb{N}_0). \tag{25}$$

Proof. By combining (6) with (14), we obtain

$$\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} = \frac{1}{2t^2} \sum_{n=0}^{\infty} \mathcal{B}_n^{(2)}\left(\frac{1}{2}\right) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} 2n\,(n-1)\,a_{n-2}\frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(2)}\left(\frac{1}{2}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Substituting $l = 1$ into (21), we obtain

$$a_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k}. \tag{26}$$

Combining (25) with (26), we arrive at the following corollary:

Corollary 4.1.

$$a_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\mathcal{B}_{k+2}^{(2)}(\frac{1}{2})}{2(k+1)(k+2)} x^{n-k}. \tag{27}$$

Proposition 4.4. *Let $l, b \in \mathbb{N}$ with $l \geq b$. Then*

$$a_n^{(l)}(x) = \sum_{k=0}^{n} \binom{n}{k} a_k^{(b)} a_{n-k}^{(l-b)}(x). \tag{28}$$

Proof. By (18), we define the following functional equation:

$$F_a(x; t, l) = F_a(t, b) F_a(x; t, l-b).$$

From this equation, we get

$$\sum_{n=0}^{\infty} a_n^{(l)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} a_n^{(b)}\frac{t^n}{n!} \sum_{n=0}^{\infty} a_n^{(l-b)}(x)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} a_n^{(l)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} a_k^{(b)} a_{n-k}^{(l-b)}(x)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Proposition 4.5. *We have*

$$a_n^{(l)}(x+1) = \sum_{k=0}^{n} \binom{n}{k} a_k^{(l)}(x). \tag{29}$$

Proof. By using (18), we get

$$\sum_{n=0}^{\infty} a_n^{(l)}(x+1)\frac{t^n}{n!} = \sum_{n=0}^{\infty} a_n^{(l)}(x)\frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} a_n^{(l)}(x+1)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}a_k^{(l)}(x)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Proposition 4.6. *Let $m, n \in \mathbb{N}$ with $n \geq m$. Then*

$$\frac{\partial^m}{\partial x^m} a_n^{(l)}(x) = m!\binom{n}{m} a_{n-m}^{(l)}(x). \tag{30}$$

Proof. By applying $\frac{\partial^m}{\partial x^m}$ derivative operator to equation (18), we obtain

$$\sum_{n=0}^{\infty}\frac{\partial^m}{\partial x^m} a_n^{(l)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} a_n^{(l)}(x)\frac{t^{n+m}}{n!}.$$

Hence

$$\sum_{n=0}^{\infty}\frac{\partial^m}{\partial x^m} a_n^{(l)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} m!\binom{n}{m} a_{n-m}^{(l)}(x)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

A relationship between the numbers a_n and $w_g(n)$ is given by the following theorem:

Theorem 4.7.

$$a_n = 2\sum_{k=0}^{n}\binom{n}{k} w_g(k) w_g(n-k).$$

Proof. By combining (12) with (14), we get

$$\frac{1}{2}\sum_{n=0}^{\infty} a_n\frac{t^n}{n!} = \sum_{n=0}^{\infty} w_g(n)\frac{t^n}{n!}\sum_{n=0}^{\infty} w_g(n)\frac{t^n}{n!}.$$

Therefore

$$\frac{1}{2}\sum_{n=0}^{\infty} a_n\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} w_g(k) w_g(n-k)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 4.8.

$$\sum_{m=0}^{n}\sum_{j=0}^{2l}(-1)^j\binom{2l}{j}\binom{n}{m} j! S_2(m,j) a_{n-m}^{(l)}(x) = 2^l x^n. \tag{31}$$

Proof. By using (18), we get

$$2^l \sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = \left(e^t - 2\right)^{2l} \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!}.$$

After some elementary calculation in the above equation, we get

$$\sum_{n=0}^{\infty} 2^l x^n \frac{t^n}{n!} = \sum_{j=0}^{2l} (-1)^j \binom{2l}{j} j! \frac{(e^t-1)^j}{j!} \sum_{n=0}^{\infty} a_n^{(l)}(x) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} 2^l x^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \sum_{j=0}^{2l} (-1)^j \binom{2l}{j} j! S_2(m,j) a_{n-m}^{(l)}(x) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Substituting $x = 0$ into (31), we get the following corollary:

Corollary 4.2.

$$\sum_{m=0}^{n} \sum_{j=0}^{2l} (-1)^j \binom{n}{m} \binom{2l}{j} j! S_2(m,j) a_{n-m}^{(l)} = 0.$$

Theorem 4.9. *The following identity holds true*:

$$f_{n-1,k} = -\frac{1}{k(k+1)n} \mathcal{B}_n \left(\frac{k}{k+1} \right) \quad (n \in \mathbb{N}).$$

Proof. By using (15), we get

$$\sum_{n=1}^{\infty} f_{n,k} \frac{t^n}{n!} = \frac{1}{t(k+1)} \left(\frac{t}{\frac{k}{k+1} e^t - 1} - \frac{t e^t}{\frac{k}{k+1} e^t - 1} \right).$$

Now combining (3) with the above equation, we obtain

$$(k+1) \sum_{n=1}^{\infty} f_{n,k} \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} \left(\mathcal{B}_n \left(\frac{k}{k+1} \right) - \mathcal{B}_n \left(1, \frac{k}{k+1} \right) \right) \frac{t^n}{n!}.$$

Since

$$\mathcal{B}_0 \left(\frac{k}{k+1} \right) - \mathcal{B}_0 \left(1, \frac{k}{k+1} \right) + \mathcal{B}_1 \left(\frac{k}{k+1} \right) - \mathcal{B}_1 \left(1, \frac{k}{k+1} \right) = 0,$$

we have

$$\sum_{n=2}^{\infty} (k+1)\, n f_{n-1,k} \frac{t^n}{n!} = \sum_{n=2}^{\infty} \left(\mathcal{B}_n \left(\frac{k}{k+1} \right) - \mathcal{B}_n \left(1, \frac{k}{k+1} \right) \right) \frac{t^n}{n!}.$$

Combining (5) with the above equation, we have

$$\sum_{n=2}^{\infty}(k+1)\,nf_{n-1,k}\frac{t^n}{n!}=\frac{-1}{k}\sum_{n=2}^{\infty}\mathcal{B}_n\left(\frac{k}{k+1}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 4.10. *We have*

$$f_{n-m,k}^{(m)}=\frac{1}{m!\binom{n}{m}(k+1)^m}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\mathcal{B}_n^{(m)}\left(j,\frac{k}{k+1}\right). \tag{32}$$

Proof. By using (22), we get

$$\sum_{n=1}^{\infty}f_{n,k}^{(m)}\frac{t^n}{n!}=\frac{1}{(k+1)^m\left(1-\frac{k}{k+1}e^t\right)^m t^m}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}e^{jt}t^m.$$

After some elementary calculations, we get

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(m!\binom{n}{m}f_{n-m,k}^{(m)}\right)\frac{t^n}{n!}\\&=\sum_{n=1}^{\infty}\left(\frac{1}{(k+1)^m}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\mathcal{B}_n^{(m)}\left(j,\frac{k}{k+1}\right)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 4.11. *We have*

$$\begin{aligned}S_2(n,m)=&\frac{k^m}{m!}\sum_{v=0}^{n}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{n}{v}\left(\frac{k+1}{k}\right)^{m-j}j^v f_{n-v,k}^{(m)}\\&-\frac{k^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}j^n\left(\frac{k+1}{k}\right)^{m-j}.\end{aligned}$$

Proof. By using (22), we get

$$\frac{1}{m!}(k+1-ke^t)^m\sum_{n=1}^{\infty}f_{n,k}^{(m)}\frac{t^n}{n!}=\sum_{n=0}^{\infty}S_2\left(n,m\right)\frac{t^n}{n!}.$$

From this equation, we have

$$\begin{aligned}\sum_{n=0}^{\infty}S_2\left(n,m\right)\frac{t^n}{n!}=&-\frac{k^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(\frac{k+1}{k}\right)^{m-j}\sum_{n=0}^{\infty}j^n\frac{t^n}{n!}\\&+\frac{k^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(\frac{k+1}{k}\right)^{m-j}\sum_{n=0}^{\infty}j^n\frac{t^n}{n!}\sum_{n=0}^{\infty}f_{n,k}^{(m)}\frac{t^n}{n!}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} S_2\left(n,m\right)\frac{t^n}{n!} = &-\sum_{n=0}^{\infty}\frac{k^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}j^n\left(\frac{k+1}{k}\right)^{m-j}\frac{t^n}{n!}\\ &+\sum_{n=0}^{\infty}\frac{k^m}{m!}\sum_{v=0}^{n}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{n}{v}\left(\frac{k+1}{k}\right)^{m-j}j^v f_{n-v,k}^{(m)}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

4.1. Combinatorial sums including the Fubini numbers

Here, by using the Riemann integral, we derive combinatorial sums including the Fubini type numbers and the Apostol-Bernoulli numbers.

Integrating both sides of (26) from 0 to 1 with respect to x, we arrive at the following result:

$$\int_0^1 a_n(x)dx = \sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{n+1-k}. \tag{33}$$

Combining the above result with (27), we get the following combinatorial sum:

Theorem 4.12. *The following identity holds true*:

$$\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{n+1-k} = \sum_{k=0}^{n}\binom{n}{k}\frac{\mathcal{B}_{k+2}^{(2)}(\frac{1}{2})}{2(k+1)(k+2)(n+1-k)}.$$

Theorem 4.13. *The following identity holds true*:

$$\sum_{m=0}^{n}\sum_{j=0}^{2l}\sum_{k=0}^{n-m}(-1)^j\binom{n-m}{k}\binom{2l}{j}\binom{n}{m}\frac{j!S_2(m,j)a_k^{(l)}}{n+1-k-m} = \frac{2^l}{n+1}.$$

Proof. Integrating both sides of (31) from 0 to 1 with respect to x, we arrive at the following result:

$$\sum_{m=0}^{n}\sum_{j=0}^{2l}(-1)^j\binom{2l}{j}\binom{n}{m}j!S_2(m,j)\int_0^1 a_{n-m}^{(l)}(x)dx = \frac{2^l}{n+1}. \tag{34}$$

And also by integrating both sides of (21) from 0 to 1 with respect to x, we get

$$\int_0^1 a_{n-m}^{(l)}(x)dx = \sum_{k=0}^{n-m}\binom{n-m}{k}\frac{a_k^{(l)}}{n+1-k-m}.$$

By substituting the above equation into (34), we get the desired result. □

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