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WEAK SOLUTIONS AND ENERGY ESTIMATES FOR A DEGENERATE NONLOCAL PROBLEM INVOLVING SUB-LINEAR NONLINEARITIES

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Abstract. This paper deals with the existence and energy estimates of solutions for a class of degenerate nonlocal problems involving sub-linear nonlinearities, while the nonlinear part of the problem admits some hypotheses on the behavior at origin or perturbation property. In particular, for a precise localization of the parameter, the existence of a non-zero solution is established requiring the sublinearity of nonlinear part at origin and infinity. We also consider the existence of solutions for our problem under algebraic conditions with the classical Ambrosetti-Rabinowitz. In what follows, by combining two algebraic conditions on the nonlinear term which guarantees the existence of two solutions as well as applying the mountain pass theorem given by Pucci and Serrin, we establish the existence of the third solution for our problem. Moreover, concrete examples of applications are provided.

1. Introduction

The goal of this paper is to study the existence and the qualitative properties of nontrivial weak solutions for nonlocal degenerate problems of the form

$$(1)\qquad \begin{cases} -M\left(\int_\Omega |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x\right) \operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) \\ = \lambda |x|^{-p(a+1)+c} f(u) \qquad \text{in } \Omega, \\ u=0 \qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $0 \le a < \frac{N-p}{p}$, $1 < p < N$, $c > 0$, $M : \mathbb{R}^+ \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are two continuous functions.

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Equations such as (1) initially appeared in [30] as a model in describing free transverse vibrations of a clamped string in which the tension depends non-negligibly on deformation. Because the function M contains an integral over Ω, such equations are not pointwise identities any longer; therefore, they are frequently called nonlocal problems.

The Kirchhoff model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example in [3, 13]. On the other hand, Kirchhoff-type boundary value problems model several physical and biological systems, for example the population density. It received great attention only after Lions [33] proposed an abstract framework for the problem. The solvability of the Kirchhoff type problems has been studied by many researchers. Some earlier classical investigations of Kirchhoff equations can be seen in the papers [1, 25, 27, 28, 29, 34, 37, 40] and the references therein. The existence and multiplicity of solutions for stationary higher order problems of Kirchhoff type (in n-dimensional domains, $n \geq 1$) were also treated in some recent papers, via variational methods like the symmetric mountain pass theorem in [21] and a three critical point theorem in [6]. Moreover, in [4, 5], some evolutionary higher order Kirchhoff problems were treated, mainly focusing on the qualitative properties of the solutions. During the last few decades, the existence of multiple non-zero solutions for p-Laplacian type equations, which corresponds to the special case $a = 0$ and $c = p$, has been studied by many researchers using different methods. See for instance [10, 18, 19, 23, 31, 32, 35]. For example, using the minimization technique and maximum principle, Brézis and Oswald in [10] obtained an existence and uniqueness result for the p-Laplacian type problem

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

when the behaviour of $f(s)/s$ is suitably controlled at infinity. In [19], Chang and Toan used variational methods to study of the problem (2) while $f(x, u) = h(x)|u|^{q-2}+g(x)$ and $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^N$ is a bounded domain having cylindrical symmetry, $\Omega_1 \subset \mathbb{R}^m$ is a bounded regular domain and Ω_2 is a k-dimensional ball of radius R, centered in the origin and $m + k = N$, $m \geq 1$, $k \geq 2$. They proved that (2) in this case, has at least one/two solutions when $g = 0/g \neq 0$ and some suitable conditions on the functions a and h are imposed. Note that, a special case of the operator $\operatorname{div}(a(x, \nabla u))$ is the p-Laplacian.

On the other hand, during the last two decades, many works are devoted to the study of the existence of nontrivial solutions for different kinds of singular problems. See [7, 14, 16, 17, 20, 26] and the references therein. Some classical tools have been used to study singular equations in the literature, including variational methods, the method of upper and lower solutions, degree theory and some fixed point theorems. For the results related to our problems, we mention the following two results. Tyagi in [41] considered a singular quasilinear equation with sign changing nonlinearity and used a three critical point theorem to established the existence results. In [43], Yang et al. proved that (1) has least three distinct weak solutions under some mild assumptions about a and some growth and singularity conditions of f.

However, as far as we know, there are very few existence results for singular p-Laplacian type equations in which there are singularities not only in nonlinear term, but also in diverge term. This is the aim of the present work, because, it is clear that in problem (1) there exists singularity not only in nonlinear term, but also in diverge term $\operatorname{div}(|x|^{-ap}|\nabla u(x)|^{p-2}\nabla u)$, which will lead to some difficulties in the proof. In this paper, we are interested in the existence results and energy estimates of solutions for problem (1). The main result of this paper ensures the existence of exact values of the parameter λ for which problem (1) admits at least one/two/three non-zero weak solutions. Several special cases of the main results and illustrating examples are also given. We also refer the reader to [9, 15, 22, 24] for some related results in this subject.

The paper is organized as follows. In Section 2, we recall some basic definitions and our main tool. In Section 3, we are devoted to prove the main results of this paper.

2. Preliminaries

In this section, we state some preliminary results, which can be found in [11, 12, 42]. First, for all $u \in \mathrm{C}_0^\infty(\mathbb{R}^N)$, there exists a constant $C_{a,b} > 0$ such that

$$\left(\int_{\mathbb{R}^N} |x|^{-bq}|u(x)|^q \mathrm{d}x\right)^{\frac{p}{q}} \leq C_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x, \tag{3}$$

where

$$-\infty < a < \frac{N-p}{p}, \quad a \leq b \leq a+1, \quad q = p^*(a,b) = \frac{Np}{N-dp}, \quad d = 1+a-b.$$

Let $\mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ be the completion of $\mathrm{C}_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{a,p} = \left(\int_\Omega |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

Then $\mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ is a reflexive Banach space. From the boundedness of Ω and the standard approximation argument, it is easy to see that (3) holds for any $u \in \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ in the sense that

$$\left(\int_{\mathbb{R}^N} |x|^{-\alpha}|u(x)|^r \mathrm{d}x\right)^{\frac{p}{r}} \leq C_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x$$

for $1 \leq r \leq p^* = \frac{Np}{N-p}$, $\alpha \leq (1+a)r + N(1-\frac{r}{p})$, that is, the embedding

$$\mathrm{W}_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow \mathrm{L}^r(\Omega, |x|^{-\alpha})$$

is continuous, where $\mathrm{L}^r(\Omega,|x|^{-\alpha})$ is the weighted $\mathrm{L}(\Omega,|x|^{-\alpha})$ space with the norm

$$|u|_{r,\alpha} := |u|_{\mathrm{L}^r(\Omega,|x|^{-\alpha})} = \left(\int_\Omega |x|^{-\alpha}|u(x)|^r \mathrm{d}x\right)^{\frac{1}{r}}.$$

In fact, we have the following compact embedding result which is an extension of the classical Rellich-Kondrachov compactness theorem [42].

Lemma 2.1. *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary, $0 \in \Omega$ and $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $1 \leq r < \frac{Np}{N-p}$, $\alpha < (1+a)r + N(1-\frac{r}{p})$. Then the embedding $\mathrm{W}_0^{1,p}(\Omega,|x|^{-ap}) \hookrightarrow \mathrm{L}^r(\Omega,|x|^{-\alpha})$ is compact.*

Definition ([20, Definition 2.1])**.** We say that $u \in \mathrm{W}_0^{1,p}(\Omega,|x|^{-ap})$ is a weak solution of (1) if

$$M\left(|x|^{-ap}|\nabla u(x)|^p \mathrm{d}x\right)\int_\Omega |x|^{-ap}|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\mathrm{d}x$$

$$-\lambda\int_\Omega |x|^{-(a+1)p+c} f(u(x))v(x)\mathrm{d}x = 0$$

for all $v \in \mathrm{C}_0^\infty(\Omega)$.

We refer the reader to [36, 39] for the following notations and results. Let X be a real Banach space. We say that a continuously Gâteaux differentiable functional $J : X \to \mathbb{R}$ satisfies the *Palais-Smale condition* (in short (PS)-condition) if any sequence $\{u_n\}$ such that

(j_1) $\{J(u_n)\}$ is bounded,
(j_2) $\lim\limits_{n\to\infty} \|J'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

Let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Set

$$J = \Phi - \Psi,$$

and fix $r_1, r_2 \in [-\infty,+\infty]$ with $r_1 < r_2$. We say that J satisfies the *Palais-Smale condition cut off lower at r_1 and upper at r_2* (in short ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition) if any sequence $\{u_n\}$ satisfying (j_1), (j_2) and

(j_3) $r_1 < \Phi(u_n) < r_2$, $\forall\, n \in \mathbb{N}$,

has a convergent subsequence.

Clearly, if $r_1 = -\infty$ and $r_2 = +\infty$ it coincides with the classical (PS)-condition. Moreover, if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ it is denoted by $(\mathrm{PS})^{[r_2]}$, while if $r_1 \in \mathbb{R}$ and $r_2 = +\infty$ it is denoted by ${}^{[r_1]}(\mathrm{PS})$. Furthermore, if J satisfies ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition, then it satisfies ${}^{[\varrho_1]}(\mathrm{PS})^{[\varrho_2]}$-condition for all $\varrho_1, \varrho_2 \in [-\infty,+\infty]$ such that $r_1 \leq \varrho_1 < \varrho_2 \leq r_2$.

In particular, we deduce that if J satisfies the classical (PS)-condition, then it satisfies ${}^{[\varrho_1]}(\mathrm{PS})^{[\varrho_2]}$-condition for all $\varrho_1, \varrho_2 \in [-\infty,+\infty]$ with $\varrho_1 < \varrho_2$. For

$r \in \mathbb{R}$ and $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, Set

$$\rho(r) = \sup_{v\in\Phi^{-1}(r,+\infty)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u)}{\Phi(v) - r}, \tag{4}$$

$$\beta(r_1, r_2) := \inf_{v\in\Phi^{-1}(r_1,r_2)} \frac{\sup\limits_{u\in\Phi^{-1}(r_1,r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}, \tag{5}$$

and

$$\rho_2(r_1, r_2) := \sup_{v\in\Phi^{-1}(r_1,r_2)} \frac{\Psi(v) - \sup\limits_{u\in\Phi^{-1}(-\infty,r_1]} \Psi(u)}{\Phi(v) - r_1}. \tag{6}$$

In the proof of our main results, we will apply the following two theorems.

Theorem 2.2 ([8, Theorem 5.1])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Assume that there exist r_1, $r_2 \in \mathbb{R}$ with $r_1 < r_2$, such that$\beta(r_1, r_2) < \rho_2(r_1, r_2)$, where β and ρ_2 are given by* (5) *and* (6)*, and for each*

$$\lambda \in \Big(\frac{1}{\rho_2(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)}\Big),$$

the function $J_\lambda := \Phi - \lambda\Psi$ satisfies ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition.

Then for all $\lambda \in \Big(\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\Big)$ there exists $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $J_\lambda(u_{0,\lambda}) \le J_\lambda(u)$ for all $u \in \Phi^{-1}(r_1, r_2)$ and $J'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.3 ([8, Corolary 5.1])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable function. Put*

$$\beta^* := \liminf_{r\to+\infty} \frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}{r}$$

and assume that there is $\bar{r} \in \mathbb{R}$ such that

$$\rho(\bar{r}) > \beta^*,$$

where ρ is given by (4)*. Moreover, assume that for each $\lambda \in (\frac{1}{\rho(\bar{r})}, \frac{1}{\beta^*})$ the function $J_\lambda := \Phi - \lambda\Psi$ satisfies ${}^{[\bar{r}]}(\mathrm{PS})^{[r]}$-condition for all $r > \bar{r}$.*

Then there is $r_2 > \bar{r}$ such that for each $\lambda \in (\frac{1}{\rho(\bar{r})}, \frac{1}{\beta^*})$, there is $u_{0,\lambda} \in \Phi^{-1}(\bar{r}, r_2)$ such that $J_\lambda(u_{0,\lambda}) \le J_\lambda(u)$ for all $u \in \Phi^{-1}(\bar{r}, r_2)$ and $J'_\lambda(u_{0,\lambda}) = 0$.

3. Main results

First we introduce the following assumptions:

($\mathcal{M}$) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function and satisfies

$$m_0 t^{\alpha-1} \le M(t) \le m_1 t^{\alpha-1} \quad \text{for all}\, t \in \mathbb{R}^+,$$

where $m_1 > m_0 > 0$ and $1 < \alpha < \min\Big\{\frac{N}{N-p}, \frac{N-p(a+1)+c}{N-p(a+1)}\Big\}$;

($\mathcal{F}$) $\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{\alpha p-1}} = 0$, i.e., f is $(\alpha p-1)$-sublinear at infinity.

Corresponding to the function M, we introduce the function

$$\widehat{M}(t) = \int_0^t M(\xi)d\xi \quad \text{for all } t \geq 0.$$

Let $B(\xi, r)$ denote the N-dimensional open ball with center ξ and radius $r > 0$, $x_0 \in \Omega$, $R_0 > 0$ is small such that $|x_0| > R_0$ and $B(x_0, R_0) \subset \Omega$, ω_N be the volume of the N-dimensional unit ball and $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^N$. For every $\mathcal{A} \subset \Omega$ and $\gamma \in \mathbb{R}$ set

$$\mu(\mathcal{A}, \gamma) := \int_{\mathcal{A}} |x|^{-\gamma} \mathrm{d}x.$$

Now put

$$\Pi(f, \gamma, \sigma) = \lambda_1^p \gamma \sqrt[\alpha p]{\frac{m_0^2}{\alpha^2 p}} \max_{|t|\leq\sigma} |f(t)| \big(\mu(\Omega, (a+1)p - c)\big)^{\frac{p-1}{p}},$$

where

$$\lambda_1 := \inf_{u \in X\setminus\{0\}} \frac{\int_\Omega |x|^{-ap} |\nabla u(x)|^p \mathrm{d}x}{\int_\Omega |x|^{-p(a+1)+c} |u(x)|^p \mathrm{d}x}.$$

For given nonnegative constant γ and positive constant $\sigma \in (0, 1)$ with

$$\gamma \neq \frac{\sqrt[\alpha p]{m_1} \sqrt[p]{\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap)} |t_0|}{\sqrt[\alpha p]{m_0} R_0(1-\sigma)},$$

put

$$b_\gamma(\sigma) := \frac{\Pi(f, \gamma, \sigma) - \Upsilon(F, \sigma)}{m_0 \gamma^{\alpha p} - m_1 |t_0|^{\alpha p} R_0^{-\alpha p} (1-\sigma)^{-\alpha p} \big(\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap)\big)^{\alpha}}, \tag{7}$$

where

$$\begin{aligned} \Upsilon(F, \sigma) &= F(t_0)(|x_0| + R_0)^{-p(a+1)+c} \sigma^N R_0^N \omega_N \\ &\quad - \max_{|t|\leq R_0} |F(t)| (|x_0| - R_0)^{-p(a+1)+c} (1-\sigma)^N R_0^N \omega_N. \end{aligned}$$

Theorem 3.1. *Suppose that the assumptions* ($\mathcal{M}$) *and* ($\mathcal{F}$) *are satisfied and assume that there exist three real constants* γ_1, γ_2 *and* $\sigma \in (0, 1)$, *with*

$$\gamma_1 < \frac{\sqrt[\alpha p]{m_1} \sqrt[p]{\big(\mu\left(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap\right)\big)} |t_0|}{\sqrt[\alpha p]{m_1} R_0(1-\sigma)} < \gamma_2, \tag{8}$$

such that

$$b_{\gamma_2}(\sigma) < b_{\gamma_1}(\sigma). \tag{9}$$

Then for each parameter $\lambda \in \left(\frac{\alpha p}{b_{\gamma_1}(\sigma)}, \frac{\alpha p}{b_{\gamma_2}(\sigma)}\right)$, *problem* (1) *possesses at least one non-zero weak solution* $u_{0,\lambda} \in \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$, *such that*

$$\sqrt[\alpha p]{\frac{m_0}{m_1}}\gamma_1 < \|u_{0,\lambda}\|_{a,p} < \gamma_2.$$

Proof. We will apply Theorem 2.2. Let $X := \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ and consider the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{p}\widehat{M}\left(\int_\Omega |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x\right) \tag{10}$$

and

$$\Psi(u) = \int_\Omega |x|^{-p(a+1)+c}F(u(x))\mathrm{d}x. \tag{11}$$

Since

$$\Phi(u) \geq \frac{m_0}{p\alpha}\left(\int_\Omega |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x\right)^\alpha = \frac{m_0}{p\alpha}\|u\|_{a,p}^{\alpha p},$$

the functional $\Phi : X \to \mathbb{R}$ is coercive. On the other hand, Φ and Ψ are continuously Gâteaux differentiable. More precisely, we have

$$\Phi'(u)(v) = M\left(\int_\Omega |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x\right)\int_\Omega |x|^{-ap}|\nabla u(x)|^{p-2}\nabla u.\nabla v \mathrm{d}x$$

and

$$\Psi'(u)(v) = \int_\Omega |x|^{-(a+1)p+c}f(u(x))v(x)\mathrm{d}x$$

for every $u, v \in X$. Fix $\lambda > 0$. A critical point of the functional $J_\lambda := \Phi - \lambda\Psi$ is a function $u \in X$ such that

$$\Phi'(u)(v) - \lambda\Psi'(u)(v) = 0$$

for every $v \in X$. Hence, the critical points of the functional J_λ are weak solutions of problem (1). At this point, let us observe that $\Phi(0_X) = \Psi(0_X) = 0$. Moreover, for every $u \in X$ with $\Phi(u) \leq r$, we have

$$\Psi(u) \leq \Pi(f, m_0, r).$$

Indeed, since $1 < \alpha < \min\left\{\frac{N}{N-p}, \frac{N-p(a+1)+c}{N-p(a+1)}\right\}$, the embedding

$$X \hookrightarrow \mathrm{L}^{\alpha p}(\Omega, |x|^{-p(a+1)+c})$$

is compact (see Lemma 2.1). Thus there exists $C_1 > 0$ such that

$$C_1\|u\|_{\mathrm{L}^{\alpha p}(\Omega, |x|^{-p(a+1)+c})} \leq \|u\|_{a,p} \quad \text{for all}\, u \in X,$$

or

$$C_1^{\alpha p}\int_\Omega |x|^{-p(a+1)+c}|u(x)|^{\alpha p}\mathrm{d}x \leq \left(\int_\Omega |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x\right)^a \quad \text{for all}\, u \in X.$$

It follows that

$$\lambda_\alpha := \inf_{u\in X\setminus\{0\}} \frac{\int_\Omega |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x}{\int_\Omega |x|^{-p(a+1)+c}|u(x)|^{\alpha p}\mathrm{d}x} > 0.$$

Now we fix $\lambda \in \mathbb{R}$ arbitrary. By $(\mathcal{F})$ we have

$$|f(t)| \leq \frac{m_0\lambda_\alpha}{1+|\lambda|}|t|^{\alpha p-1} \quad \text{for every } |t| \geq \sigma,$$

where m_0 is given in the assumption $(\mathcal{M})$. By integrating the sides of the above inequality, we have

$$|F(t)| \leq \frac{m_0\lambda_\alpha}{\alpha p(1+|\lambda|)}|t|^{\alpha p} + \max_{|t|\leq\sigma}|f(t)||t| \quad \text{for every } t \in \mathbb{R}.$$

Thus, for every $u \in X$ we obtain

$$\begin{aligned}
\Psi(u) &\leq \frac{m_0\lambda_\alpha}{\alpha p(1+|\lambda|)}\int_\Omega |x|^{-p(a+1)+c}|u(x)|^{\alpha p}\mathrm{d}x \\
&\quad + \max_{|t|\leq\sigma}|f(t)|\int_\Omega |x|^{-p(a+1)+c}|u|\mathrm{d}x \\
&\leq \max_{|t|\leq\sigma}|f(t)|\left(\int_\Omega |x|^{-(a+1)p+c}|u(x)|^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_\Omega |x|^{-(a+1)p+c}\mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\leq \lambda_1^p \max_{|t|\leq\sigma}|f(t)|\|u\|_{ap}\left(\int_\Omega |x|^{-(a+1)p+c}\mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\leq \lambda_1^p \sqrt[\alpha p]{\frac{rm_0}{\alpha}}\max_{|t|\leq\sigma}|f(t)|\left(\int_\Omega |x|^{-(a+1)p+c}\mathrm{d}x\right)^{\frac{p-1}{p}} \\
&= \lambda_1^p \sqrt[\alpha p]{\frac{rm_0}{\alpha}}\max_{|t|\leq\sigma}|f(t)|\big(\mu(\Omega,(a+1)p-c)\big)^{\frac{p-1}{p}}.
\end{aligned}$$

Therefor, for every $u \in \Phi^{-1}(-\infty, r)$ we have

$$\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u) \leq \Pi(f,\gamma,\sigma), \tag{12}$$

where $\gamma = \sqrt[\alpha p]{\frac{\alpha pr}{m_0}}$. Now we define r_1, r_2 and w_σ by

$$r_1 := \frac{m_0}{\alpha p}\gamma_1^{\alpha p}, \qquad r_2 := \frac{m_0}{\alpha p}\gamma_2^{\alpha p}$$

and

$$w_\sigma(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R}^N\setminus B_{R_0}(x_0), \\ t_0 & \text{for } x \in B_{\sigma R_0}(x_0), \\ \frac{t_0}{R_0(1-\sigma)}(R_0-|x|) & \text{for } x \in B_{R_0}(x_0)\setminus B_{\sigma R_0}(x_0), \end{cases}$$

with $\sigma \in (0,1)$. Simple computations show that

$$\|w_\sigma\|_{a,p}^p = \int_\Omega |x|^{-ap}|\nabla w_\sigma(x)|^p\mathrm{d}x = \frac{|t_0|^p}{R_0^P(1-\sigma)^p}\mu\left(B_{R_0}(x_0)\setminus B_{\sigma R_0}(x_0), ap\right)$$

$$\geq |t_0|^p(|x_0|+R_0)^{-ap}(1-\sigma)^{-p}(1-\sigma^N)\omega_N R_0^{N-2}.$$

Clearly, $w_\sigma \in \mathrm{H}_0^1(\Omega)$ and by the assumption $(\mathcal{M})$, we have

$$\Phi(w_\sigma) \geq \frac{m_0|t_0|^{\alpha p}}{p\alpha R_0^{\alpha p}(1-\sigma)^{\alpha p}}\big(\mu\left(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap\right)\big)^{\alpha},$$
$$\Phi(w_\sigma) \leq \frac{m_1|t_0|^{\alpha p}}{p\alpha R_0^{\alpha p}(1-\sigma)^{\alpha p}}\big(\mu\left(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap\right)\big)^{\alpha}.$$

Taking (8) into account, by a direct computation, one has $r_1 < \Phi(w_\sigma) < r_2$. Moreover, we have that

$$\begin{aligned}\Psi(w_\sigma) &= \int_{B_{\sigma R_0}(x_0)} F(w_\sigma(x))\mathrm{d}x + \int_{B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0)} F(w_\sigma(x))\mathrm{d}x \\ &\geq F(t_0)(|x_0|+R_0)^{-p(a+1)+c}\sigma^N R_0^N\omega_N \\ &\quad - \max_{|t|\leq R_0}|F(t)|(|x_0|-R_0)^{-p(a+1)+c}(1-\sigma)^N R_0^N\omega_N.\end{aligned} \tag{13}$$

From (12) we have

$$\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u) \leq \Pi(f,\gamma_1,\sigma), \tag{14}$$

and

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u) \leq \Pi(f,\gamma_2,\sigma). \tag{15}$$

On the other hand,

$$\begin{aligned}\beta(r_1,r_2) &:= \inf_{v\in\Phi^{-1}(r_1,r_2)}\frac{\sup\limits_{u\in\Phi^{-1}(r_1,r_2)}\Psi(u)-\Psi(v)}{r_2-\Phi(v)} \\ &\leq \frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)-\Psi(w_\delta)}{r_2-\Phi(w_\delta)} \\ &\leq \frac{\alpha p\left(\Pi(f,\gamma_2,\sigma)-\Upsilon(F,\sigma)\right)}{m_0\gamma_2^{\alpha p}-m_1|t_0|^{\alpha p}R_0^{-\alpha p}(1-\sigma)^{-\alpha p}\big(\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0),ap)\big)^{\alpha}},\end{aligned}$$

and

$$\begin{aligned}\rho_2(r_1,r_2) &:= \sup_{v\in\Phi^{-1}(r_1,r_2)}\frac{\Psi(v)-\sup\limits_{u\in\Phi^{-1}(-\infty,r_1]}\Psi(u)}{\Phi(v)-r_1} \\ &\geq \frac{\Psi(w_\sigma)-\sup\limits_{u\in\Phi^{-1}(-\infty,r_1]}\Psi(u)}{\Phi(w_\sigma)-r_1} \\ &\geq \frac{\alpha p\left(\Pi(f,\gamma_1,\sigma)-\Upsilon(F,\sigma)\right)}{m_0\gamma_1^{\alpha p}-m_1|t_0|^{\alpha p}R_0^{-\alpha p}(1-\sigma)^{-\alpha p}\big(\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0),ap)\big)^{\alpha}}.\end{aligned}$$

Hence, by using the notation (7), from (13) and (14) together with (15), it follows that

$$\beta(r_1, r_2) \leq \frac{b_{\gamma_2}(\delta)}{\alpha p} \quad \text{and} \quad \rho_2(r_1, r_2) \geq \frac{b_{\gamma_1}(\delta)}{\alpha p}.$$

Finally, the assumption (9) yields

$$\beta(r_1, r_2) < \rho_2(r_1, r_2).$$

Now, from [20, Lemma 2.4], the functional J_λ satisfies the classical (PS)-condition, and so it satisfies $^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition for all r_1 and r_2 with $r_1 < r_2 < +\infty$. Therefore, by Theorem 2.2, for each $\lambda \in \left(\frac{\alpha p}{a_{\gamma_1}(\delta)}, \frac{\alpha p}{a_{\gamma_2}(\delta)}\right)$, the functional J_λ possesses at least one critical point $u_{0,\lambda}$ such that $r_1 < \Phi(u_{0,\lambda}) < r_2$, that is

$$\sqrt[\alpha p]{\frac{m_0}{m_1}}\gamma_1 < \|u_{0,\lambda}\|_{a,p} < \gamma_2.$$

This completes the proof. $\square$

Remark 3.2. The results of Theorem 3.1 hold if condition $(\mathcal{F})$ is replaced by

$(\mathcal{F}')$ there exist three positive constants b_1, b_2 and α with

$$1 < \alpha < \min\left\{\frac{N}{N-p}, \frac{N - p(a+1) + c}{N - p(a+1)}\right\}$$

such that

$$|f(t)| \leq b_1 + b_2|t|^{\alpha p - 1} \quad \text{for all}\, t \in \mathbb{R}.$$

See the similar argument in [9, Theorem 2.1].

Now, we point out a particular case of Theorem 3.1.

Theorem 3.3. *Suppose that the assumptions* $(\mathcal{M})$ *and* $(\mathcal{F})$ *are satisfied and assume that there exist two positive constants* γ *and* $\sigma \in (0,1)$ *with*

$$\gamma > \frac{\sqrt[\alpha p]{m_1}\,\sqrt[p]{\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap)}|t_0|}{\sqrt[\alpha p]{m_0}R_0(1-\sigma)},$$

such that

$$\Pi(f, \gamma, \sigma) < \Upsilon(F, \sigma). \tag{16}$$

Then for each parameter

$$\lambda \in \left(\frac{\alpha p m_1 |t_0|^{\alpha p}\big(\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap)\big)^{\alpha}}{R_0^{\alpha p}(1-\sigma)^{\alpha p}\Upsilon(F,\sigma)}, \frac{\alpha p m_0 \gamma^{\alpha p}}{\Pi(f,\gamma,\sigma)}\right),$$

problem (1) *possesses at least one non-zero weak solution* $u_{0,\lambda} \in \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ *such that* $\|u_{0,\lambda}\|_{a,p} < \gamma$.

Proof. Taking $\gamma_1 = 0$ and $\gamma_2 = \gamma$ and bearing (7) in mind, we obtain

$$\begin{aligned} b_\gamma(\sigma) &= \frac{\Pi(f,\gamma,\sigma) - \Upsilon(F,\sigma)}{m_0\gamma^{\alpha p} - m_1|t_0|^{\alpha p} R_0^{-\alpha p}(1-\sigma)^{-\alpha p}\big(\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap)\big)^{\alpha}} \\ &\le \frac{\Pi(f,\gamma,\sigma)}{m_0\gamma^{\alpha p}} \le \frac{R_0^{\alpha p}(1-\sigma)^{\alpha p}\Upsilon(F,\sigma)}{m_1|t_0|^{\alpha p}\big(\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap)\big)^{\alpha}} = b_0(\sigma). \end{aligned}$$

Hence, Theorem 3.1 ensures the conclusion. □

Now, we give an application of Theorem 2.3 which will be used later to obtain multiple solutions for the problem (1).

Theorem 3.4. *Suppose that the assumptions* $(\mathcal{M})$ *and* $(\mathcal{F})$ *are satisfied and assume that there exist two positive constants* $\bar\gamma$ *and* $\bar\sigma \in (0,1)$ *with*

$$\bar\gamma < \frac{\sqrt[\alpha p]{m_1}\,\sqrt[p]{\mu(B_{R_0}(x_0)\backslash B_{\bar\sigma R_0}(x_0), ap)}\,|t_0|}{\sqrt[\alpha p]{m_0}\,R_0(1-\bar\sigma)},$$

such that

$$\Pi(f,\bar\gamma,\bar\sigma) < \Upsilon(F,\bar\sigma). \tag{17}$$

Then for each $\lambda > \tilde\lambda$, *where*

$$\tilde\lambda := \frac{m_0\bar\gamma^{\alpha p} - m_1|t_0|^{\alpha p}R_0^{-\alpha p}(1-\sigma)^{-\alpha p}\big(\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap)\big)^{\alpha}}{\alpha p\,(\Pi(f,\bar\gamma,\bar\sigma) - \Upsilon(F,\bar\sigma))},$$

problem (1) *possesses at least one non-trivial weak solution* $\bar u_{0,\lambda} \in \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ *such that*

$$\|\bar u_{0,\lambda}\|_{a,p} > \sqrt[\alpha p]{\frac{m_0}{m_1}}\,\bar\gamma. \tag{18}$$

Proof. Take $X = \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ and put $I_\lambda = \Phi - \lambda\Psi$, where Φ and Ψ are given as in (10) and (11), respectively. The functionals Φ and Ψ satisfy all assumptions requested in Theorem 2.3. Put

$$\bar r := \frac{m_0}{\alpha p}\bar\gamma_1^{\alpha p}.$$

From [20, Lemma 2.4], the functional J_λ satisfies the classical (PS)-condition, and so it satisfies ${}^{[\bar r]}(\mathrm{PS})^{[r]}$-condition for all r with $r > \bar r$. Arguing as in the proof of Theorem 3.1, we obtain that

$$\begin{aligned} \rho(\bar r) &= \sup_{v\in\Phi^{-1}(\bar r,+\infty)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(-\infty,\bar r]}\Psi(u)}{\Phi(v) - \bar r} \\ &\ge \frac{\Psi(w_\sigma) - \sup\limits_{u\in\Phi^{-1}(-\infty,\bar r]}\Psi(u)}{\Phi(w_\sigma) - \bar r} \\ &\ge \frac{\alpha p\,(\Pi(f,\bar\gamma,\bar\sigma) - \Upsilon(F,\bar\sigma))}{m_0\bar\gamma^{\alpha p} - m_1|t_0|^{\alpha p}R_0^{-\alpha p}(1-\sigma)^{-\alpha p}\big(\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap)\big)^{\alpha}}. \end{aligned}$$

Hence, from our assumption it follows that $\rho(\bar{r}) > 0$. Therefore, it follows from Theorem 2.3 with $\beta^* = 0$, for each $\lambda > \tilde{\lambda}$, the functional J_λ admits at least one local minimum $\bar{u}_{0,\lambda} \in \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ such that $\Phi(\bar{u}_{0,\lambda}) > \bar{r}$, which is just (18). Thus the conclusion is obtained. □

The following result is a straight consequence of Theorem 3.3.

Theorem 3.5. *Suppose that the assumptions* $(\mathcal{M})$ *and* $(\mathcal{F})$ *are satisfied and assume that*

$$\lim_{\xi \to 0^+} \frac{F(\xi)}{\xi^{\alpha p}} = +\infty. \tag{19}$$

Furthermore, let $\gamma > 0$ *and set*

$$\lambda_\gamma^\star := \frac{\alpha p m_0 \gamma^{\alpha p}}{\Pi(f, \gamma, \sigma)}.$$

Then for every $\lambda \in (0, \lambda_\gamma^\star)$, *problem* (1) *admits at least one non-zero weak solution* $u_{0,\lambda} \in \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ *such that* $\|u_{0,\lambda}\|_{a,p} < \gamma$.

Proof. Fix $\lambda \in (0, \lambda_\gamma^\star)$. From (19) there exists a constant $\sigma \in (0,1)$ with

$$\gamma > \frac{\sqrt[\alpha p]{m_1}\,\sqrt[p]{\mu(B_{R_0}(x_0) \backslash B_{\sigma R_0}(x_0), ap)}|t_0|}{\sqrt[\alpha p]{m_0} R_0(1-\sigma)}$$

such that

$$\frac{\alpha p m_1 |t_0|^{\alpha p}\big(\mu(B_{R_0}(x_0) \backslash B_{\sigma R_0}(x_0), ap)\big)^\alpha}{R_0^{\alpha p}(1-\sigma)^{\alpha p}\Upsilon(F, \sigma)} < \lambda < \frac{\alpha p m_0 \gamma^{\alpha p}}{\Pi(f, \gamma, \sigma)}.$$

Hence, by Theorem 3.3, problem (1) possesses at least one non-zero weak solution $u_{0,\lambda}$ such that $\|u_{0,\lambda}\|_{a,p} < \gamma$. □

Example 3.6. Let $N = 3$, $p = \frac{5}{2}$, $a = \frac{1}{6}$, $c = \frac{59}{12}$, $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ x_1^2 + x_2^2 + x_3^2 \leq 4\} \subset \mathbb{R}^3$, $x_0 = (1,1,1) \in \Omega$, $R_0 = 1$, $M(t) = t$ for all $t \in \mathbb{R}$ and consider the problem

$$\begin{cases} -\mathrm{div}\left(\frac{\sqrt{|\nabla u|}\nabla u}{\sqrt[12]{|x|^5}}\right) \int_\Omega \frac{\sqrt{|\nabla u(x)|^5}}{\sqrt[12]{|x|^5}} \mathrm{d}x = \lambda |x|^3 f(u) & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{20}$$

and

$$f(t) = \begin{cases} e^t, & t \in (-\infty, -1], \\ e^{\sin(\frac{\pi}{2}t)}, & t \in (-1, 1), \\ e^{-\cos(\pi t)}, & t \in [1, \infty). \end{cases}$$

We observe that $\min\left\{\frac{N}{N-p}, \frac{N-p(a+1)+c}{N-p(a+1)}\right\} = \min\{6, 60\} = 6$, thus M satisfies the condition $(\mathcal{M})$ with $m_0 = m_1 = 1$ and $\alpha = 2$, and we have

$$\mu\big(\Omega, (a+1)p - c\big) = \mu(\Omega, -2) = \int_0^{2\pi}\int_0^{\pi}\int_0^2 \rho^4 \sin\phi \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta = \frac{128\pi}{5}.$$

Moreover, by choosing $t_0 = 1$, $\gamma = 1$ and $\sigma = \frac{1}{2}$ we have

$$w_\sigma(x_1, x_2, x_3) = \begin{cases} 0 & \text{for } x \in \mathbb{R}^3 \backslash B_1(1,1,1), \\ 1 & \text{for } x \in B_{\frac{1}{2}}(1,1,1), \\ 2(1 - \sqrt{x_1^2 + x_2^2 + x_3^2}) & \text{for } (x_1, x_2, x_3) \in B_1(1,1,1) \backslash B_{\frac{1}{2}}(1,1,1), \end{cases}$$

$$\lim_{\xi \to 0^+} \frac{f(\xi)}{\xi^{\alpha p - 1}} = \lim_{\xi \to 0^+} \frac{e^{\sin(\frac{\pi}{2}\xi)}}{\xi^4} = +\infty,$$

$$\lim_{\xi \to +\infty} \frac{f(\xi)}{\xi^{\alpha p - 1}} = \lim_{\xi \to +\infty} \frac{e^{-\cos(\pi \xi)}}{\xi^4} = 0,$$

and

$$\begin{aligned} \Pi(f, \gamma, \sigma) = \Pi(f, 1, \frac{1}{2}) &= \lambda_1^{\frac{5}{2}} \sqrt[5]{\frac{1}{10}} \max_{|t| \le \frac{1}{2}} |f(t)| \big(\mu(\Omega, -2)\big)^{\frac{3}{5}} \\ &= \frac{16 \sqrt[5]{\pi^3} e^{\frac{\sqrt{2}}{2}}}{\sqrt[5]{5^4}} \left(\inf_{u \in \mathrm{W}_0^{1,\frac{5}{2}}(\Omega, |x|^{-\frac{5}{12}}) \backslash \{0\}} \frac{\int_\Omega |x|^{-\frac{5}{12}} |\nabla u(x)|^{\frac{5}{2}} \mathrm{d}x}{\int_\Omega |x|^2 |u|^{\frac{5}{2}} \mathrm{d}x} \right)^{\frac{5}{2}} \\ &\le \frac{16 \sqrt[5]{\pi^3} e^{\frac{\sqrt{2}}{2}}}{\sqrt[5]{5^4}} \left(\frac{\int_\Omega |x|^{-\frac{5}{12}} |\nabla w_{\frac{1}{2}}|^{\frac{5}{2}} \mathrm{d}x}{\int_\Omega |x|^2 |w_{\frac{1}{2}}|^{\frac{5}{2}} \mathrm{d}x} \right)^{\frac{5}{2}} \\ &\le \frac{16 \sqrt[5]{\pi^3} e^{\frac{\sqrt{2}}{2}}}{\sqrt[5]{5^4}} \left(\frac{2 \int_0^{2\pi} \int_0^\pi \int_{\frac{1}{2}}^1 \rho^{\frac{19}{12}} \sin \phi \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta}{\int_0^{2\pi} \int_0^\pi \int_0^{\frac{1}{2}} \rho^4 \sin \phi \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta} \right)^{\frac{5}{2}} \\ &= \frac{16 \sqrt[5]{\pi^3} e^{\frac{\sqrt{2}}{2}}}{\sqrt[5]{5^4}} \left(\frac{240 \sqrt[12]{2^{29}} (\sqrt[12]{2^{31}} - 1)}{31} \right)^{\frac{5}{2}}. \end{aligned}$$

Hence, applying Theorem 3.5 for every

$$\lambda \in \left(0, \frac{\sqrt[5]{5^4} \times \sqrt{31^5}}{16 \sqrt[5]{\pi^3} e^{\frac{\sqrt{2}}{2}} \big(240 \sqrt[12]{2^{29}} (\sqrt[12]{2^{31}} - 1)\big)^{\frac{5}{2}}} \right),$$

then problem (20) possesses at least one non-zero weak solution

$$u_{0,\lambda} \in \mathrm{W}_0^{1,\frac{5}{2}}(\Omega, |x|^{-\frac{5}{12}})$$

such that $\|u_{0,\lambda}\|_{\frac{1}{6},\frac{5}{2}} < 1$ and $\lim_{\lambda \to 0^+} \|u_{0\lambda}\|_{\frac{1}{6},\frac{5}{2}} = 0$.

Theorem 3.7. *Suppose that the assumptions* $(\mathcal{M})$ *and* $(\mathcal{F})$ *are satisfied. Then the mapping* $\lambda \mapsto J_\lambda(u_{0,\lambda})$ *is negative and strictly decreasing in* $(0, \lambda_\gamma^\star)$.

Proof. The restriction of the functional J_λ to $\Phi^{-1}(0, r_2)$, where

$$r_2 := \frac{m_0}{\alpha p} \gamma_2^{\alpha p},$$

admits a global minimum, which is a critical point (local minimum) of J_λ in $\mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$. Moreover, since $w_\sigma \in \Phi^{-1}(0, r_2)$ and

$$\frac{\Phi(w_\sigma)}{\Psi(w_\sigma)} \leq \frac{m_1|t_0|^{\alpha p}\big(\mu\left(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap\right)\big)^\alpha}{p\alpha R_0^{\alpha p}(1-\sigma)^{\alpha p}\Upsilon(F,\sigma)} < \lambda,$$

we have

$$J_\lambda(u_{0,\lambda}) \leq J_\lambda(w_\sigma) = \Phi(w_\sigma) - \lambda\Psi(w_\sigma) < 0.$$

Next, we observe that

$$J_\lambda(u) = \lambda\left(\frac{\Phi(u)}{\lambda} - \Psi(u)\right)$$

for every $u \in X$ and fix $0 < \lambda_1 < \lambda_2 < \lambda_\gamma^\star$. Set

$$m_{\lambda_1} := \left(\frac{\Phi(u_{0,\lambda_1})}{\lambda_1} - \Psi(u_{0,\lambda_1})\right) = \inf_{u\in\Phi^{-1}(0,r_2)}\left(\frac{\Phi(u)}{\lambda_1} - \Psi(u)\right),$$

and

$$m_{\lambda_2} := \left(\frac{\Phi(u_{0,\lambda_2})}{\lambda_2} - \Psi(u_{0\lambda_2})\right) = \inf_{u\in\Phi^{-1}(0,r_2)}\left(\frac{\Phi(u)}{\lambda_2} - \Psi(u)\right).$$

Clearly, as claimed before, $m_{\lambda_i} < 0$ (for $i = 1, 2$), and $m_{\lambda_2} \leq m_{\lambda_1}$ thanks to $\lambda_1 < \lambda_2$. Then the mapping $\lambda \mapsto J_\lambda(u_{0,\lambda})$ is strictly decreasing in $(0, \lambda_\gamma^\star)$ owing to

$$J_{\lambda_2}(u_{0,\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = J_{\lambda_1}(u_{0,\lambda_1}).$$

□

Remark 3.8. Generally, Theorem 3.5 ensures that if f satisfies the assumptions $(\mathcal{M})$, $(\mathcal{F})$ and (19), then for every parameter λ belonging to the real interval $\Lambda_\Omega := (0, \lambda^\star)$, where

$$\lambda^\star := \alpha p m_0 \sup_{\gamma>0} \frac{\gamma^{\alpha p}}{\Pi(f,\gamma,\sigma)},$$

the problem (1) possesses at least one non-zero solution $u_{0,\lambda} \in \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$. Moreover $\|u_{0,\lambda}\|_{ap} \to 0$ as $\lambda \to 0^+$.

Remark 3.9. We note that, in particular, if f is $(\alpha p - 1)$-sublinear at infinity, Theorem 3.5 ensures that problem (1) admits at least one non-zero weak solution for every positive parameter λ. Moreover, in our case, the obtained solution is non-zero, while the classical direct method approach, that can be accept in this context, ensures the existence of at least one solution that may be zero.

Remark 3.10. A careful analysis of the proof of Theorem 3.5 ensures that the result still remains true if condition (19) is replaced by the more general assumption

$$\limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^{\alpha p}} = +\infty.$$

Moreover, the previous asymptotic condition at zero can be replaced by the following form

$$\limsup_{\xi\to 0^+} \frac{f(\xi)}{\xi^{\alpha p-1}} = +\infty. \tag{21}$$

Therefore, it is natural to obtain the following result.

Theorem 3.11. *Let $M(t) \geq t^{\beta-1}$ for all $t \in \mathbb{R}^+$ where $1 < \beta < \min\left\{\frac{N}{N-p}, \frac{N}{N-p(a+1)}\right\}$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\lim_{t\to 0^+} \frac{f(t)}{t^{\beta p-1}} = +\infty \quad \text{and} \quad \lim_{t\to +\infty} \frac{f(t)}{t^{\beta p-1}} = 0.$$

Then there exists $\lambda^\star > 0$ such that for every $\lambda \in (0, \lambda^\star)$, problem

$$\begin{cases} -M\left(\int_\Omega |x|^{-ap}|\nabla u(x)|^p \mathrm{d}x\right) \operatorname{div}\left(|x|^{-ap}|\nabla u(x)|^{p-2}\nabla u\right) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

possesses at least one non-zero weak solution $u_{0,\lambda} \in \mathrm{C}_0^\infty(\Omega)$. Moreover, we have

$$\left(\int_\Omega |x|^{-ap}|\nabla u_{0,\lambda}|^p \mathrm{d}x\right)^{\frac{1}{p}} \to 0$$

as $\lambda \to 0^+$ and the mapping

$$\lambda \mapsto \frac{1}{p}\widehat{M}\left(\int_\Omega |x|^{-ap}|\nabla u_{0,\lambda}|^p \mathrm{d}x\right) - \int_\Omega \left(\int_0^{u_{0,\lambda}} f(t)\mathrm{d}t\right) \mathrm{d}x$$

is negative and strictly decreasing in $(0, \lambda^\star)$.

As it follows, we show how the former analysis can be used to pass from the existence of at least one nontrivial solution to that of at least two nontrivial solutions. This objective will emerge by using the specific nature of the initially found solution, namely a local minimum. The information is then useful in guaranteeing the existence of a second solution as a critical point of mountain pass type. Accordingly, we start with the following theorem, where the celebrated Ambrosetti-Rabinowitz condition is necessary.

Theorem 3.12. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) \neq 0$ and the assumptions $(\mathcal{M})$, $(\mathcal{F})$ and (21) hold. Furthermore, assume that*

(AR) *there are constants $\nu > \frac{m_1 \alpha p}{m_0}$ and $\varrho > 0$ such that, for all $|\xi| \geq \varrho$, one has*

$$0 < \nu F(\xi) \leq \xi f(\xi). \tag{22}$$

Then for each $\lambda \in \Lambda_\Omega$, problem (1) admits at least two non-trivial weak solutions in the space $\mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$.

Proof. Fix $\lambda \in \Lambda_\Omega$. Owing to the assumptions $(\mathcal{M})$, $(\mathcal{F})$ and (21), Theorem 3.5 ensures that problem (1) admits at least one weak non-zero solution u_1 in $\mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ which is a local minimum of the functional J_λ as defined in the proof of Theorem 3.1. Now, we prove the existence of the second local minimum distinct from the first one. To this goal, we verify the hypotheses of the mountain-pass theorem for the functional J_λ. Clearly, the functional J_λ is of class C^1 and $J_\lambda(0) = 0$. The first part of proof guarantees that $u_1 \in \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ is a nontrivial local minimum for J_λ in $\mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$. We can assume that u_1 is a strict local minimum for J_λ in $\mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$. Therefore, there is $\rho > 0$ such that $\inf_{\|u-u_1\|=\rho} I_\lambda(u) > I_\lambda(u_1)$, so condition [39, (I_1), Theorem 2.2] is verified. By integrating the condition (22), there exist constants $a_1, a_2 > 0$ such that

$$F(x) \geq a_1|x|^\nu - a_2$$

for all $t \in \mathbb{R}$. Now, choosing any $u \in \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap}) \setminus \{0\}$, one has

$$\begin{aligned}
J_\lambda(\tau u) &= (\Phi - \lambda\Psi)(\tau u) \\
&\leq \frac{m_1}{\alpha p}\|\tau u\|^{\alpha p} - \lambda \int_\Omega |x|^{-p(a+1)+c} F(\tau u(x))\mathrm{d}x \\
&\leq \frac{m_1 \alpha \tau^p}{\alpha p}\|u\|^{\alpha p} - \lambda\tau^\nu a_1 \int_\Omega |x|^{-p(a+1)+c}|u(x)|^\nu \mathrm{d}x \\
&\quad + \lambda a_2 \mu\big(\Omega, p(a+1) - c\big) \\
&\to -\infty, \qquad \tau \to +\infty.
\end{aligned}$$

Thus condition [39, (I_2), Theorem 2.2] is satisfied. Therefore the functional J_λ satisfies the geometry of mountain pass. Moreover, J_λ satisfies the (PS)-condition. Indeed, assume that $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{J_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $J'_\lambda(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that

$$|J_\lambda(u_n)| \leq c_0, \ |J'_\lambda(u_n)| \leq c_0 \ \text{ for all}\, n \in \mathbb{N}.$$

Therefore, we infer to deduce from the definition of J'_λ and the assumption (AR) that

$$\begin{aligned}
c_0 + c_1\|u_n\| &\geq \nu J_\lambda(u_n) - J'_\lambda(u_n)(u_n) \\
&\geq \big(\frac{\nu m_0}{\alpha p} - m_1\big)\|u_n\|^{\alpha p} \\
&\quad - \lambda \int_\Omega |x|^{-p(a+1)+c} \left(\nu F(t, u_n(t)) - f(t, u_n(t))(u_n(t))\right) \mathrm{d}t \\
&\geq \big(\frac{\nu m_0}{\alpha p} - m_1\big)\|u_n\|^{\alpha p}
\end{aligned}$$

for some $c_1 > 0$. Since $\nu > \frac{m_1 \alpha p}{m_0}$, this implies that (u_n) is bounded. Now, as the same argument in [20, Lemma 2.4], we can prove $\{u_n\}$ converges strongly to u in $\mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$. Consequently, J_λ satisfies (PS)-condition. Hence, by the classical theorem of Ambrosetti and Rabinowitz [39, Theorem 2.2] we establish a critical point u_2 of J_λ such that $J_\lambda(u_2) > J_\lambda(u_1)$. Since $f(0) \neq 0$, u_1 and u_2 are two distinct non-trivial weak solutions of (1) and the proof is completed. □

Remark 3.13. The non-triviality of the second weak solution ensured by Theorem 3.12 can be achieved also in the case $f(0) = 0$ requiring the extra conditions at zero

$$\limsup_{\xi\to 0^+} \frac{F(\xi)}{|\xi|^{\alpha p}} = +\infty, \tag{23}$$

and

$$\liminf_{\xi\to 0^+} \frac{F(\xi)}{|\xi|^{\alpha p}} > -\infty. \tag{24}$$

Indeed, let $0 < \bar{\lambda} < \lambda^*$ where

$$\lambda^* = \alpha p m_0 \sup_{\gamma>0} \frac{\gamma^{\alpha p}}{\Pi(f,\gamma,\sigma)}.$$

Then there exists $\bar{\gamma} > 0$ such that

$$\frac{\bar{\lambda}}{\alpha p m_0} < \frac{\bar{\gamma}^{\alpha p}}{\Pi(f,\bar{\gamma},\sigma)}.$$

Let Φ and Ψ be as given in (10) and (11), respectively. Due to Theorem 3.12, for every $\lambda \in (0, \bar{\lambda})$ there exists a critical point of $J_\lambda = \Phi - \lambda\Psi$ such that $u_\lambda \in \Phi^{-1}(-\infty, r_\lambda)$ where $r_\lambda := \frac{m_0}{\alpha p}\bar{\gamma}^{\alpha p}$. In particular, u_λ is a global minimum of the restriction of J_λ to $\Phi^{-1}(-\infty, r_\lambda)$. We will prove that the function u_λ cannot be trivial. Let us show that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{25}$$

Owing to the assumptions (23) and (24), we can consider a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants ι, κ (with $\iota > 0$) such that

$$\lim_{n\to+\infty} \frac{F(\xi_n)}{|\xi_n|^{\alpha p}} = +\infty,$$

and

$$F(\xi) \geq \kappa |\xi|^{\alpha p}$$

for every $\xi \in [0, \iota]$. We consider a set $\mathcal{G} \subset B$ of positive measure and a function $v \in \mathrm{W}_0^{1,p}(\Omega, |x|^{-ap})$ such that

- ($k1$) $v(t) \in [0, 1]$ for every $t \in \Omega$;
- ($k2$) $v(t) = 1$ for every $t \in \mathcal{G}$.

Hence, fix $N > 0$ and consider a real positive number η with

$$N < \frac{2\alpha p\eta \text{ meas}(\mathcal{G}) + 2\alpha p\kappa \int_{\Omega\setminus\mathcal{G}} |v(t)|^2 \mathrm{d}t}{m_0\|v\|_{a,p}^{\alpha p}}.$$

Then there is $n_0 \in \mathbb{N}$ such that $\xi_n < \iota$ and

$$F(\xi_n) \geq \eta|\xi_n|^{\alpha p}$$

for every $n > n_0$. Now, for every $n > n_0$, by the properties of the function v (that is $0 \leq \xi_n v(t) < \iota$ for n large enough), one has

$$\begin{aligned}\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} &= \frac{\text{meas}(\mathcal{G})F(\xi_n) + \int_{\Omega\setminus\mathcal{G}} G(\xi_n v(t))\mathrm{d}t}{\Phi(\xi_n v)}\\ &> \frac{2\alpha p\eta \text{ meas}(\mathcal{G}) + 2\alpha p\kappa \int_{\Omega\setminus\mathcal{G}} |v(t)|^2 \mathrm{d}t}{m_0\|v\|_{a,p}^{\alpha p}} > N.\end{aligned}$$

Since N could be arbitrarily large, we get

$$\lim_{n\to\infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty,$$

from which (25) clearly follows. So, there exists a sequence $\{\omega_n\} \subset X$ strongly converging to zero such that, for n large enough, $\omega_n \in \Phi^{-1}(-\infty, r_\lambda)$ and

$$J_\lambda(\omega_n) = \Phi(\omega_n) - \lambda\Psi(\omega_n) < 0.$$

Since u_λ is a global minimum of the restriction of J_λ to $\Phi^{-1}(-\infty, r_\lambda)$, we obtain

$$J_\lambda(u_\lambda) < 0,$$

so that u_λ is not trivial.

In what follows, we present one application of Theorem 3.12 as follows.

Example 3.14. Let $N = 2$, $p = \frac{3}{2}$, $a = \frac{1}{4}$, $c = \frac{7}{8}$, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2;\ x_1^2 + x_2^2 \leq 4\} \subset \mathbb{R}^2$, $x_0 = (0, 1) \in \Omega$, $R_0 = \frac{1}{2}$, $M(t) = tK(t)$ for all $t \in \mathbb{R}^+$ where

$$K(t) = \begin{cases} 1 + t - [t], & [t] \text{ is even,}\\ 1 + |t - [t+1]|, & [t] \text{ is odd}\end{cases}$$

and $[t]$ is the integer part of t,

$$f(t) = 1 + t^6 \quad \text{for all}\, t \in \mathbb{R}.$$

We observe that $\min\left\{\frac{N}{N-p}, \frac{N-p(a+1)+c}{N-p(a+1)}\right\} = \min\{4, 8\} = 4$, thus M satisfies the condition $(\mathcal{M})$ by $m_0 = 1$, $m_1 = 2$ and $\alpha = 2$. Also, M and f are two continuous functions, $f(0) = 1 \neq 0$,

$$\lim_{\xi\to 0^+} \frac{f(\xi)}{\xi^{\alpha p-1}} = \lim_{\xi\to 0^+} \frac{1 + \xi^6}{\xi^2} = +\infty.$$

Moreover, taking into account that

$$\lim_{|\xi|\to+\infty} \frac{\xi f(\xi)}{F(\xi)} = \lim_{|\xi|\to+\infty} \frac{\xi + \xi^7}{\xi + \frac{1}{7}\xi^7} = 7 > 6 = \frac{m_1\alpha p}{m_0},$$

by choosing $\nu = 7 > 6 = \frac{m_1 \alpha p}{m_0}$, there exists $\varrho > 1$ such that the assumption (AR) in Theorem 3.12 is fulfilled for all $|\xi| > \varrho$. Hence, by choosing $\sigma = \frac{1}{2}$ and applying Theorem 3.12 and Remark 3.2, for every $\lambda > 0$, the problem

$$\begin{cases} -M\left(\int_\Omega \frac{\sqrt{|\nabla u(x)|^3}}{\sqrt[8]{|x|^3}} \mathrm{d}x\right) \operatorname{div}\left(\frac{\nabla u}{\sqrt[8]{|x|^3}\sqrt{|\nabla u|}}\right) = \lambda \frac{1+u^6}{|x|} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

possesses at least two nontrivial weak solutions.

Finally, as a consequence of Theorems 3.3 and 3.4, we can obtain the following existence result of three solutions.

Theorem 3.15. *Suppose that the assumptions* $(\mathcal{M})$ *and* $(\mathcal{F})$ *are satisfied and* $f(0) \neq 0$. *Moreover, assume that there exist four positive constants* γ, $\sigma \in (0,1)$, $\bar{\gamma}$ *and* $\bar{\sigma} \in (0,1)$ *with*

$$\begin{aligned} \bar{\gamma} &< \frac{\sqrt[\alpha p]{m_1}\sqrt[p]{\mu(B_{R_0}(x_0)\backslash B_{\bar{\sigma}R_0}(x_0), ap)}|t_0|}{\sqrt[\alpha p]{m_0}R_0(1-\bar{\sigma})} \\ &\leq \frac{\sqrt[\alpha p]{m_1}\sqrt[p]{\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap)}|t_0|}{\sqrt[\alpha p]{m_0}R_0(1-\sigma)} < \gamma \end{aligned}$$

such that (16) *and* (17) *hold, and*

$$\begin{aligned} &\frac{\Pi(f,\gamma,\sigma)}{\alpha p m_0 \gamma^{\alpha p}} \\ &< \frac{\alpha p\left(\Pi(f,\bar{\gamma},\bar{\sigma}) - \Upsilon(F,\bar{\sigma})\right)}{m_0\bar{\gamma}^{\alpha p} - m_1|t_0|^{\alpha p}R_0^{-\alpha p}(1-\sigma)^{-\alpha p}\left(\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap)\right)^\alpha} \end{aligned} \tag{26}$$

is satisfied. Then for each

$$\lambda \in \Lambda = \left(\max\left\{\tilde{\lambda}, \frac{\alpha p m_1|t_0|^{\alpha p}\left(\mu(B_{R_0}(x_0)\backslash B_{\sigma R_0}(x_0), ap)\right)^\alpha}{R_0^{\alpha p}(1-\sigma)^{\alpha p}\Upsilon(F,\sigma)}\right\}, \frac{\alpha p m_0 \gamma^{\alpha p}}{\Pi(f,\gamma,\sigma)}\right),$$

problem (1) *possesses at least three weak solutions* $u_{0,\lambda}$, $\bar{u}_{0,\lambda}$ *and* $\tilde{u}_{0,\lambda}$ *such that*

$$\|u_{0,\lambda}\|_{a,p} < \gamma \quad \text{and} \quad \|\bar{u}_{0,\lambda}\|_{a,p} > \sqrt[\alpha p]{\frac{m_0}{m_1}}\bar{\gamma},$$

Proof. First, in view of (26), we have $\Lambda \neq \emptyset$. Next, fix $\lambda \in \Lambda$. Employing Theorem 3.3, there is a positive weak solution $u_{0,\lambda}$ such that

$$\|u_{0,\lambda}\|_{a,p} < \gamma,$$

which is a local minimum for the associated functional J_λ, while Theorem 3.4 ensures a weak solution $\bar{u}_{0,\lambda}$ such that

$$\|\bar{u}_{0,\lambda}\|_{a,p} > \sqrt[\alpha p]{\frac{m_0}{m_1}}\bar{\gamma}$$

which is a local minimum for J_λ. Arguing as in the proof of Theorem 3.1, we observe that the functional J_λ is coercive, then it satisfies the (PS)-condition.

Hence, the conclusion follows from the mountain pass theorem as given by Pucci and Serrin (see [38]). □

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