THE NAVIER-STOKES EQUATIONS WITH INITIAL VALUES IN BESOV SPACES OF TYPE $B_{q,\infty}^{-1+3/q}$

Reinhard Farwig, Yoshikazu Giga, and Pen-Yuan Hsu

Dedicated to the 60th birthday of our colleague Hi Jun Choe

Abstract. We consider weak solutions of the instationary Navier-Stokes system in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ with initial value $u_0 \in L_\sigma^2(\Omega)$. It is known that a weak solution is a local strong solution in the sense of Serrin if u_0 satisfies the optimal initial value condition $u_0 \in B_{q,s_q}^{-1+3/q}$ with Serrin exponents $s_q > 2, q > 3$ such that $\frac{2}{s_q} + \frac{3}{q} = 1$. This result has recently been generalized by the authors to weighted Serrin conditions such that u is contained in the weighted Serrin class $\int_0^T (\tau^\alpha \|u(\tau)\|_q)^s \, d\tau < \infty$ with $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$, $0 < \alpha < \frac{1}{2}$. This regularity is guaranteed if and only if u_0 is contained in the Besov space $B_{q,s}^{-1+3/q}$. In this article we consider the limit case of initial values in the Besov space $B_{q,\infty}^{-1+3/q}$ and in its subspace $\overset{\circ}{B}{}_{q,\infty}^{-1+3/q}$ based on the continuous interpolation functor. Special emphasis is put on questions of uniqueness within the class of weak solutions.

1. Introduction

We consider the Navier-Stokes initial value problem

$$(1.1) \qquad \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f, \ \operatorname{div} u = 0 \ \text{in} \ (0,T) \times \Omega$$
$$u|_{\partial\Omega} = 0, \ u(0) = u_0$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ of class $C^{2,1}$ and a time interval $[0,T)$, $0 < T \le \infty$. For simplicity, the coefficient of viscosity is assumed to be equal to 1.

Let us recall the definition of weak and strong solutions to (1.1) and define special types of strong solutions contained in spaces with weights in time, so-called *strong $L_\alpha^s(L^q)$-solutions*.

Definition 1.1. Let $u_0 \in L^2_\sigma(\Omega)$ be an initial value and let $f = \operatorname{div} F$ with $F = (F_{ij})^3_{i,j=1} \in L^2(0,T;L^2(\Omega))$ be an external force.

(i) A vector field u on $\Omega \times (0,T)$ in the Leray-Hopf class

$$(1.2) \qquad \mathcal{LH}_T = L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_0(\Omega))$$

is called a *weak solution* (in the sense of Leray-Hopf) of the Navier-Stokes system (1.1) with data u_0, f, if the relation

$$(1.3) \quad -\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T}$$

holds for each test function $w \in C^\infty_0([0,T); C^\infty_{0,\sigma}(\Omega))$, and if the energy inequality

$$(1.4) \qquad \frac{1}{2}\|u(t)\|^2_2 + \int^t_0 \|\nabla u\|^2_2 \, \mathrm{d}\tau \le \frac{1}{2}\|u_0\|^2_2 - \int^t_0 (F, \nabla u) \, \mathrm{d}\tau$$

is satisfied for $0 \le t < T$.

(ii) A weak solution u of (1.1) is called a *strong $L^s_\alpha(L^q)$-solution* with exponents $2 < s \le \infty$, $3 < q < \infty$ and weight τ^α in time, where $0 < \alpha < \frac{1}{2}$ and $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$, if additionally the weighted Serrin condition holds:

$$(1.5)$$

$$u \in \begin{cases} L^s_\alpha(0,T;L^q(\Omega)), & \text{i.e.,} \quad \int^T_0 (\tau^\alpha \|u(\tau)\|_q)^s \, \mathrm{d}\tau < \infty, \text{ if } 2 < s < \infty, \\ L^\infty_\alpha(0,T;L^q(\Omega)), & \text{i.e.,} \quad \operatorname{ess\,sup}_{\tau\in(0,T)} \tau^\alpha \|u(\tau)\|_q < \infty, \text{ if } s = \infty. \end{cases}$$

If in (1.5) $\alpha = 0$ and $\frac{2}{s} + \frac{3}{q} = 1$, then u is called a strong solution (in the sense of Serrin).

In this definition we use the usual Lebesgue and Sobolev spaces, $L^q(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $W^{k,q}(\Omega)$ with norm $\|\cdot\|_{W^{k,q}(\Omega)} = \|\cdot\|_{k;q}$, respectively, for $1 < q < \infty$ and $k \in \mathbb{N}$. Let $L^s(L^q) = L^s(0,T;L^q(\Omega))$, $1 < q,s < \infty$, with norm $\|\cdot\|_{L^s(0,T;L^q(\Omega))} = \|\cdot\|_{q,s;T} = \left(\int^T_0 \|\cdot\|^s_q \, \mathrm{d}t\right)^{1/s}$ denote the classical Bochner spaces. If additionally $\alpha \ge 0$ is given, we define the weighted (in time) Bochner spaces $L^s_\alpha(0,T;L^q(\Omega)) = L^s_\alpha(L^q)$ with norm

$$\|\cdot\|_{L^s_\alpha(0,T;L^q(\Omega))} = \|\cdot\|_{L^s_\alpha(L^q)} = \left(\int^T_0 t^{\alpha s} \|\cdot\|^s_q \, \mathrm{d}t\right)^{1/s}.$$

Of course, if $s = \infty$, then

$$L^\infty_\alpha(0,T;L^q(\Omega)) = \big\{ u : (0,T) \to L^q(\Omega) \text{ strongly measurable,}$$

$$(1.6) \qquad \|u\|_{L^\infty_\alpha(0,T;L^q(\Omega))} = \operatorname{ess\,sup}_{\tau\in(0,T)} \tau^\alpha \|u(\tau)\|_q < \infty \big\}.$$

The expression $\langle \cdot, \cdot\rangle_\Omega = \langle \cdot, \cdot\rangle$ denotes the pairing of functions on Ω, and $\langle \cdot, \cdot\rangle_{\Omega,T}$ means the corresponding pairing on $[0,T) \times \Omega$. Furthermore, to deal with solenoidal vector fields we use the smooth function spaces $C^\infty_0(\Omega)$ and $C^\infty_{0,\sigma}(\Omega) = \{v \in C^\infty_0(\Omega) : \operatorname{div} v = 0\}$, and the spaces $L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$,

$W_0^{1,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,q}}$, $W_{0,\sigma}^{1,q}(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{1,q}}$. Throughout this paper, $A = A_2$ denotes the Stokes operator in $L_\sigma^2(\Omega)$. More general, A_q, $1 < q < \infty$, means the Stokes operator in $L_\sigma^q(\Omega)$, and e^{-tA_q}, $t \geq 0$, is the semigroup generated by A_q in $L_\sigma^q(\Omega)$. Note that, with $x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3$, for $F = (F_{ij})_{i,j=1}^3$, $u = (u_1, u_2, u_3)$ we let $\operatorname{div} F = \left(\sum_{i=1}^3 \partial_i F_{ij}\right)_{j=1}^3$, $u \cdot \nabla u = (u \cdot \nabla)u$, so that $u \cdot \nabla u = \operatorname{div}(uu)$ if u is solenoidal; here $uu = (u_i u_j)_{i,j=1}^3$.

We may assume in the following, without loss of generality, that each weak solution $u : [0, T) \to L_\sigma^2(\Omega)$ of (1.1) is weakly continuous, see Sohr [28, V. Theorem 1.3.1]. Therefore, $u(0) = u_0$ is well-defined.

For further properties of weak and strong solutions to (1.1) (in the classical sense, i.e., $\alpha = 0$, $u \in L^s(0, T; L^q(\Omega))$, $\frac{2}{s} + \frac{3}{q} = 1$), we refer to [2, 3, 19, 20, 22, 24, 29]. It is well-known that Serrin's condition (1.5) with $\alpha = 0$ yields the regularity property

$$u \in C^\infty((0, T) \times \overline{\Omega}),$$

provided that $\partial\Omega$ is of class C^∞ and $F \in C^\infty((0, T) \times \overline{\Omega})$. Moreover, we get uniqueness within the class of weak solutions satisfying the energy inequality (1.4), see [28, V. Theorem 1.8.2, Theorem 1.5.1]. In the context of uniqueness a stronger version of the energy inequality (1.4) is helpful: A weak solution satisfies the *strong energy inequality* if

$$(1.7) \qquad \frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^t \|\nabla u\|_2^2 \, d\tau \leq \frac{1}{2}\|u(t_0)\|_2^2 - \int_{t_0}^t (F, \nabla u) \, d\tau$$

holds for a.a. $t_0 \in [0, T)$ including $t_0 = 0$ and for all $t_0 \leq t < T$. It is well-known that for a bounded domain weak solutions constructed by standard approximation procedures (Galerkin approximation, Yosida approximation, difference quotients in time, mollifiers in space and/or time) satisfy (1.7). Finally, if in (1.4) there holds equality, u is said to satisfy the *energy equality*. The condition $u \in L^4(0, T; L^4(\Omega))$ is known to be sufficient to guarantee the energy equality. For conditions weaker than $L^4(L^4)$-integrability for bounded domains we refer to [11].

Since the pioneering work of J. Leray and E. Hopf, see [20, 24], the existence of at least one weak solution u of (1.1) is well-known. However, the existence of a strong solution u could be shown up to now at least in a sufficiently small interval $[0, T)$, $0 < T \leq \infty$, and under additional smoothness conditions on the initial data u_0 and the external force f. The first sufficient condition on the initial data for a bounded domain seems to be due to [22], yielding a solution class of so-called local strong solutions. Since then many results on sufficient initial value conditions for the existence of local strong solutions have been developed, see [2, 12, 15, 16, 19, 21, 23, 26, 28, 29], with weaker and weaker assumptions on u_0, thus making the space of initial values to guarantee the existence of a local strong solution larger and larger.

The optimal condition, i.e., sufficiency and necessity, was found by Sohr, Varnhorn and the first author of this article, see [9, 10], and can be written in terms of (solenoidal) Besov spaces $\mathbb{B}_{q,s_q}^{-1+3/q}(\Omega)$ where $\frac{2}{s_q} + \frac{3}{q} = 1$. This space is defined by real interpolation as

$$\mathbb{B}_{q,s_q}^{-1+\frac{3}{q}} = \big(\mathbb{B}_{q',s_q'}^{\frac{2}{s_q}}\big)' = \big(\mathcal{D}(A_{q'})', L_\sigma^q\big)_{\frac{1}{2}(1+\frac{3}{q}),s_q},$$

where $A_{q'}$ denotes the Stokes operator on the space $L_\sigma^{q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$, of solenoidal vector fields. Note that similar results in the whole space case are well-known.

Recently, this result has been generalized by the authors ([5, 6]) to initial values in

$$\mathbb{B}_{q,s}^{-1+\frac{3}{q}}(\Omega) = \big(\mathcal{D}(A_{q'})', L_\sigma^q\big)_{\frac{1}{2}(1+\frac{3}{q}),s},$$

where $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$, $0 < \alpha < \frac{1}{2}$. An equivalent explicit norm for $u_0 \in \mathbb{B}_{q,s}^{-1+\frac{3}{q}}(\Omega)$ is given by the weighted integral

$$\Big(\int_0^\infty \big(\tau^\alpha \|e^{-\tau A} u_0\|_q\big)^s \mathrm{d}\tau\Big)^{1/s} = \|e^{-\tau A_q} u_0\|_{L_\alpha^s(0,\infty;L^q(\Omega))} < \infty;$$

note that this norm is scaling invariant with respect to the scaling properties of the Navier-Stokes solutions. For further details see [6, §4]. The term $\|A^{-1}u_0\|_q$ which usually appears as an additional term on the left hand side of the norm can be omitted since the semigroup on a bounded domain decays exponentially. Moreover, we note that the interval of integration $(0,\infty)$ may be replaced by any finite interval $(0,\delta)$, yielding a family of equivalent norms. In particular, by choosing $\delta > 0$ small, we can achieve that $\|e^{-\tau A_q} u_0\|_{L_\alpha^s(0,\delta;L^q(\Omega))}$ is as small as we want. Altogether, we get for $s_q < s_1 < s_2 < \infty$ a scale of growing Besov spaces

$$(1.8) \qquad \mathbb{B}_{q,s_q}^{-1+\frac{3}{q}} \subset \mathbb{B}_{q,s_1}^{-1+\frac{3}{q}} \subset \mathbb{B}_{q,s_2}^{-1+\frac{3}{q}} \subset \mathring{\mathbb{B}}_{q,\infty}^{-1+\frac{3}{q}} \subset \mathbb{B}_{q,\infty}^{-1+\frac{3}{q}}.$$

Here $\mathring{\mathbb{B}}_{q,\infty}^{-1+3/q}(\Omega)$ denotes the continuous interpolation space in this scale, also called little Nikol'skii space, see Amann [4, p. 4, p. 8].

The space $\mathring{\mathbb{B}}_{q,\infty}^{-1+3/q}(\Omega)$, $1 < q \neq 3$, was used by Amann [2] to construct a global unique solution in $C\big([0,T); \mathring{\mathbb{B}}_{q,\infty}^{-1+3/q}(\Omega)\big)$ when $\|u_0\|_{\mathbb{B}_{q,\infty}^{-1+3/q}}$ is small; here Ω is a bounded or exterior domain, the whole or half space, including the n-dimensional case with a suitable modification. Recently, Ri et al. [27] showed for all $3 \leq q < \infty$ the existence of a local unique solution $u \in L^\infty\big(0,T; \mathbb{B}_{q,\infty}^0(\Omega)\big)$ for initial values u_0 in $\mathbb{B}_{q,\infty}^0(\Omega)$; if even $u_0 \in \mathring{\mathbb{B}}_{q,\infty}^0(\Omega)$, then $u \in C^0\big([0,T); \mathbb{B}_{q,\infty}^0(\Omega)\big)$. Note that $\mathbb{B}_{3,\infty}^0(\mathbb{R}^3)$ is a scaling invariant space, and that analogous results are obtained for the n-dimensional whole and half space. Similar results to those of this paper and of [6] are discussed by Haak and Kunstmann in [18]; the authors consider the whole space $\mathbb{R}^n$ in different scaling

invariant function spaces, but bounded domains mainly in $L^2(\Omega)$-spaces. In these papers the relation to weak Leray-Hopf solutions is not investigated. For details on the Besov spaces $\mathbb{B}_{q,\infty}^{-1+3/q}(\Omega)$ and $\overset{\circ}{\mathbb{B}}_{q,\infty}^{-1+3/q}(\Omega)$ we refer to Section 5 and particularly to [2, 27].

Whereas the focus of the articles [2, 18, 27] and of numerous articles dealing only with the whole space case are on solutions with values in a given Besov space, our focus is on solutions with initial values in $L_\sigma^2(\Omega)$ intersected with a Besov space such that the solution is also a *weak* one. In this setting the main results of [6] for a bounded smooth domain $\Omega \subset \mathbb{R}^n$ read as follows:

Theorem 1.2 ([6, Theorems 1.2, 1.3]). *Assume $u_0 \in L_\sigma^2(\Omega)$ and $f = \operatorname{div} F$ where $F \in L^2(0, T; L^2(\Omega)) \cap L_{2\alpha}^{s/2}(0, T; L^{q/2}(\Omega))$; here, $2 < s < \infty$, $3 < q < \infty$ and $0 < \alpha < \frac{1}{2}$ satisfying $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$.*

(i) Then there exists a constant $\epsilon_* = \epsilon_*(q, s, \alpha, \Omega) > 0$ with the following property: If

$$(1.9) \qquad \|e^{-\tau A} u_0\|_{L_\alpha^s(0,T;L^q)} + \|F\|_{L_{2\alpha}^{s/2}(0,T;L^{q/2})} \le \varepsilon_*,$$

then the Navier-Stokes system (1.1) has a unique strong $L_\alpha^s(L^q)$-solution with data u_0, f on the interval $[0, T)$.

(ii) The condition $u_0 \in \mathbb{B}_{q,s}^{-1+\frac{3}{q}}(\Omega)$ is sufficient and necessary for the existence of a (unique) local in time strong $L_\alpha^s(L^q)$-solution of the Navier-Stokes system (1.1).

Of course, solutions with initial values in the space $\mathbb{B}_{q,s}^{-1+\frac{3}{q}}(\Omega)$ larger than the optimal space studied in [9, 10] are strong solutions in the sense of Serrin on each interval (δ, T) with $0 < \delta < T$, but not on $(0, T]$. Another disadvantage is related to Serrin's Uniqueness Theorem: It cannot be proved that a weak solution satisfying the energy inequality and a strong $L_\alpha^s(L^q)$-solution with the same data u_0, f coincide. This problem can be solved for so-called *well-chosen* weak solutions constructed by an *admissible* approximation scheme. *E.g.*, weak solutions constructed by a semigroup-Yosida approximation procedure are well-chosen. The same holds under some restrictive conditions for solutions given by Galerkin's method. For details we refer to [6] and in particular to [5].

The aim of this paper is the study of the limit case $s = \infty$, i.e., $u_0 \in \mathbb{B}_{q,\infty}^{-1+\frac{3}{q}}(\Omega)$ working with the largest space in the scale (1.8). The disadvantage of this space is the fact that it is no longer separable and that the norm $\|e^{-\tau A_q} u_0\|_q$ will not converge to 0 as $\tau \to 0$.

Now our first main theorem reads as follows:

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, and let $0 < T \le \infty$, $3 < q < \infty$ and $0 < \alpha < \frac{1}{2}$ with $\frac{3}{q} = 1 - 2\alpha$ be given. Consider the Navier-Stokes equation (1.1) with initial value $u_0 \in L_\sigma^2(\Omega) \cap \mathbb{B}_{q,\infty}^{-1+3/q}(\Omega)$ and an external force $f = \operatorname{div} F$ where $F \in L^2(0, T; L^2(\Omega)) \cap$*

$L^\infty_{2\alpha}(0,T;L^{q/2}(\Omega))$. *Then there exists a constant* $\epsilon_* = \epsilon_*(q,\alpha,\Omega) > 0$ *with the following property: If*

$$(1.10) \qquad \|e^{-\tau A}u_0\|_{L^\infty_\alpha(0,T;L^q)} + \|F\|_{L^\infty_{2\alpha}(0,T;L^{q/2})} \leq \varepsilon_*,$$

then (1.1) *has a strong* $L^\infty_\alpha(L^q)$*-solution with data* u_0, f *on the interval* $[0,T)$.

   This solution is unique in the class of all strong $L^\infty_\alpha(L^q)$*-solutions on* $(0,T)$ *with sufficiently small norm in* $L^\infty_\alpha(L^q)$.

The reader is referred to Theorem 5.1 in Section 5 below for an explanation of the Besov space $\mathbb{B}^{-1+3/q}_{q,\infty}(\Omega)$ which is equipped with norm $\|e^{-\tau A}\cdot\|_{L^\infty_\alpha(0,T;L^q)}$. This space has the disadvantage that in general the term $\tau^\alpha\|e^{-\tau A}u_0\|_q$ does not converge to 0 as $\tau \to 0$. This drawback is removed in the continuous interpolation space $\mathring{\mathbb{B}}^{-1+3/q}_{q,\infty}$ where the property $\lim_{\tau\to0} \tau^\alpha\|e^{-\tau A}u_0\|_q = 0$ is satisfied by definition, see Section 5. By analogy, we define the subspace

$$\mathring{L}^\infty_{2\alpha}\big(0,T;L^{q/2}(\Omega)\big) = \Big\{ F \in L^\infty_{2\alpha}\big(0,T;L^{q/2}(\Omega)\big) : \|F\|_{L^\infty_{2\alpha}(0,t;L^{q/2})} \to 0 \text{ as } t \to 0\Big\}$$

of $L^\infty_{2\alpha}(0,T;L^{q/2}(\Omega))$. In this case, condition (1.10) can be achieved by choosing T sufficiently small, and we get the following variant of Theorem 1.3:

Corollary 1.4. *Under the assumptions of Theorem 1.3 let* $u_0 \in L^2_\sigma(\Omega) \cap \mathring{\mathbb{B}}^{-1+3/q}_{q,\infty}(\Omega)$ *and* $f = \operatorname{div} F$ *where* $F \in L^2(0,T;L^2(\Omega)) \cap \mathring{L}^\infty_{2\alpha}\big(0,T;L^{q/2}(\Omega)\big)$. *Then the Navier-Stokes system* (1.1) *has a unique strong* $L^\infty_\alpha(L^q)$*-solution with data* u_0, f *on some interval* $[0,T'] \subset [0,T)$.

Corollary 1.5. *Suppose that the assumptions of Theorem 1.3 are fulfilled.*
   (i) *The condition*

$$(1.11) \qquad \operatorname*{ess\,sup}_{\tau\in(0,\infty)} \tau^\alpha\|e^{-\tau A}u_0\|_q < \infty$$

is necessary for the existence of a strong $L^\infty_\alpha(L^q)$*-solution* $u \in L^\infty_\alpha(0,T;L^q)$ *of the Navier-Stokes system* (1.1) *with data* u_0, f *in some interval* $[0,T)$, $0 < T \leq \infty$.

   (ii) *If additionally* $F \in \mathring{L}^\infty_{2\alpha}\big(0,T;L^{q/2}(\Omega)\big)$, *then the condition*

$$(1.12) \qquad u_0 \in \mathring{\mathbb{B}}^{-1+\frac{3}{q}}_{q,\infty}(\Omega)$$

is even necessary and sufficient for the existence of a unique strong $L^\infty_\alpha(0,T;L^q)$ *-solution* $u \in \mathring{L}^\infty_\alpha\big(0,T;L^q(\Omega)\big)$ *of the Navier-Stokes system* (1.1).

We note that the solutions constructed in Theorems 1.2 and 1.3, Corollaries 1.4 and 1.5 are continuous in time with values in $\mathbb{B}^{-1+3/q}_{q,s}$ and $\mathring{\mathbb{B}}^{-1+\frac{3}{q}}_{q,\infty}$, i.e.,

$$u \in C\big([0,T];\mathbb{B}^{-1+3/q}_{q,s}(\Omega)\big) \text{ and } u \in C\left([0,T];\mathring{\mathbb{B}}^{-1+3/q}_{q,\infty}(\Omega)\right),$$

respectively. This includes the more classical case $u_0 \in \mathbb{B}^{-1+3/q}_{q,s_q}$ where $\frac{2}{s_q}+\frac{3}{q} = 1$ considered in [9, 10]. For details of the proof we refer to the forthcoming

article [7]. In case that $u_0 \in \mathbb{B}_{q,\infty}^{-1+3/q}(\Omega)$ continuity of $u(t)$ holds on $(0,T]$, but cannot be expected at $t = 0+$.

For the definition of well-chosen weak solutions we need a slight extension of Definition 1.2 in [5] to the case of L^∞-type spaces. Because of the bad approximation properties of the spaces $\mathbb{B}_{q,\infty}^{-1+\frac{3}{q}}(\Omega)$ and $L_{2\alpha}^\infty\big(0,T;L^{q/2}(\Omega)\big)$ it is convenient to work in this definition immediately with the smaller spaces $\overset{\circ}{\mathbb{B}}_{q,\infty}^{-1+3/q}$ and $\overset{\circ}{L}_{2\alpha}^\infty\big(0,T;L^{q/2}(\Omega)\big)$. Of course, a version with the spaces $\mathbb{B}_{q,\infty}^{1+3/q}(\Omega)$ and $L_{2\alpha}^\infty\big(0,T;L^{q/2}(\Omega)\big)$ is possible, but looks awkward.

Definition 1.6. A *well-chosen weak solution* v is a weak solution of the Navier-Stokes system (1.1) with $v(0) = u_0 \in L_\sigma^2(\Omega)$ and force $f = \mathrm{div}\, F$, $F \in L^2\big(0,T;L^2(\Omega)\big)$, satisfying the strong energy inequality (1.7), defined by a concrete so-called *admissible* approximation procedure, and compatible with the notion of $L_\alpha^\infty(L^q)$-solutions in the following sense:

(1) The initial value $u_0 \in \overset{\circ}{\mathbb{B}}_{q,\infty}^{-1+\frac{3}{q}}$ is approximated by a sequence $(u_{0n}) \subset L_\sigma^2(\Omega) \cap \mathbb{B}_{q,\infty}^{-1+\frac{3}{q}}(\Omega)$ converging to u_0 in $L_\sigma^2(\Omega) \cap \mathbb{B}_{q,\infty}^{-1+\frac{3}{q}}(\Omega)$ as $n \to \infty$.

(2) The force $F \in L^2\big(0,T;L^2(\Omega)\big) \cap \overset{\circ}{L}_{2\alpha}^\infty\big(0,T;L^{q/2}(\Omega)\big)$ is approximated by a sequence $(F_n) \subset L^2\big(0,T;L^2(\Omega)\big) \cap L_{2\alpha}^\infty\big(0,T;L^{q/2}(\Omega)\big)$ such that $F_n \to F$ in both spaces.

(3) The approximation method yields approximate weak solutions (u_n), uniformly bounded in $\mathcal{LH}_T$, and containing a subsequence (u_{n_k}) such that $u_{n_k} \rightharpoonup v$ in Leray-Hopf's class $\mathcal{LH}_T$, i.e., $u_{n_k} \rightharpoonup v$ in $L^2\big(0,T; H^1(\Omega)\big)$ and $u_{n_k} \overset{*}{\rightharpoonup} v$ in $L^\infty\big(0,T;L_\sigma^2(\Omega)\big)$ as $k \to \infty$.

(4) (u_n) is uniformly bounded in $L_\alpha^\infty(0,T';L^q)$ for some $T' \in (0,T]$.

Remark 1.7. (1) The crucial part of Definition 1.6 for an admissible approximation procedure is assumption (4) on (u_n).

(2) The strong convergence $u_{0n} \to u_0$ in $L^2(\Omega)$ in Definition 1.6(1) can be replaced by the corresponding weak convergence. By analogy, the strong convergence $F_n \to F$ in $L^2\big(0,T;L^2(\Omega)\big)$ may be replaced by a weak one due to Definition 1.6(2).

(3) Although the assumptions on F, F_n do not imply that $\|F_n\|_{L_{2\alpha}^\infty(0,t;L^{q/2})}$ converges to 0 as $t \to 0$ uniformly in $n \in \mathbb{N}$, the following smallness condition is satisfied due to Definition 1.6(2): For any $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ and $T' \in (0,T]$ such that

$$(1.13) \qquad \|F_n\|_{L_{2\alpha}^\infty(0,T';L^{q/2})} \leq \varepsilon \ \text{ for all } n \geq N_\varepsilon.$$

By analogy, for any $\varepsilon > 0$ there exist an $N_\varepsilon \in \mathbb{N}$ and $T' \in (0,T]$ such that

$$(1.14) \qquad \sup_{(0,T')} \tau^\alpha \|e^{-\tau A_q} u_{0n}\|_q \leq \varepsilon \ \text{ for all } n \geq N_\varepsilon.$$

Now our main theorem on uniqueness reads as follows.

Theorem 1.8. *Under the assumptions of Theorem 1.3 let $u_0 \in L^2_\sigma(\Omega) \cap \overset{\circ}{\mathbb{B}}{}^{-1+3/q}_{q,\infty}(\Omega)$ and $f = \operatorname{div} F$ with $F \in L^2\big(0,T;L^2(\Omega)\big) \cap \overset{\circ}{L}{}^\infty_{2\alpha}\big(0,T;L^{q/2}(\Omega)\big)$ be given. Furthermore, let $u \in \overset{\circ}{L}{}^\infty_\alpha\big(0,T;L^q(\Omega)\big)$ be the unique strong $L^\infty_\alpha(L^q)$-solution of* (1.1) *with data u_0, F.*

(i) *The solution u is unique within the class of all well-chosen weak solutions of* (1.1) *in the sense of Definition 1.6.*

(ii) *Assume that each subsequence of (u_n) converging weakly in $\mathcal{LH}_T$ converges weakly to any weak solution of* (1.1). *Then the whole sequence (u_n) converges to u. Moreover, for any sequence of initial values (u_{0n}) and external forces (F_n) approximating u_0 and F in the sense of Definition 1.6*(1), (2), *respectively, and generating approximate solutions (u_n) with a subsequence weakly convergent in $\mathcal{LH}_T$ to any weak solution of* (1.1), *the whole sequence (u_n) converges weakly in $\mathcal{LH}_T$ to u.*

The crucial point is to show that an approximation procedure for the construction of weak solutions is admissible in the sense of Definition 1.6.

Theorem 1.9. *Let $3 < q < \infty$, $0 < \alpha < \frac{1}{2}$ and $\frac{3}{q} = 1 - 2\alpha$. Then the Yosida approximation scheme and, if $3 < q \le 4$, the Galerkin approximation scheme are admissible. To be more precise, in this context these methods are defined as follows*:

(i) (*The Yosida approximation scheme*) *Let $J_n = (I + \frac{1}{n}A^{1/2})^{-1}$ denote the Yosida operator, let $u_{0n} = J_n u_0$, and assume that $F_n \to F$ in $L^2\big(0,T;L^2(\Omega)\big) \cap \overset{\circ}{L}{}^\infty_{2\alpha}(0,T^*;L^{q/2}(\Omega))$ for some $0 < T^* \le T$. Then the approximate solution u_n is defined as the solution of the approximate Navier-Stokes system*

$$\partial_t u_n - \Delta u_n + (J_n u_n) \cdot \nabla u_n + \nabla p_n = \operatorname{div} F_n, \quad \operatorname{div} u_n = 0,$$

(1.15)

$$u_n|_{\partial\Omega} = 0, \quad u_n(0) = u_{0n}.$$

(ii) (*The Galerkin approximation scheme*) *Let Π_n denote the L^2_σ-projection onto the space of the first n eigenfunctions of the Stokes operator A_2, and suppose that $u_{0n} \in \Pi_n L^2_\sigma(\Omega)$ as well as $F_n \in L^2(0,T;L^2(\Omega))$ satisfy the assumptions of Definition 1.6*(1), (2). *Then let u_n denote the Galerkin approximation of* (1.1) *with data u_{0n}, F_n.*

(iii) *In both cases* (i) *and* (ii) *the assumption in Theorem 1.8*(ii) *is satisfied. Hence the whole sequence given by these admissible approximation schemes converges to the well-chosen weak solution, irrespective of the sequences (u_{0n}) and (F_n).*

2. Preliminaries

For the reader's convenience, we first explain some well-known properties of the Stokes operator. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{2,1}$, let $[0,T), 0 < T \le \infty$, be a time interval, and let $1 < q < \infty$. Then $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ denotes the Helmholtz projection, and the Stokes operator $A_q = -P_q\Delta :$

$\mathcal{D}(A_q) \to L_\sigma^q(\Omega)$ is defined with domain $\mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega)$ and range $R(A_q) = L_\sigma^q(\Omega)$. Since $P_q v = P_\gamma v$ for $v \in L^q(\Omega) \cap L^\gamma(\Omega)$ and $A_q v = A_\gamma v$ for $v \in \mathcal{D}(A_q) \cap \mathcal{D}(A_\gamma)$, $1 < \gamma < \infty$, we sometimes write $A_q = A$ to simplify the notation if there is no misunderstanding. Furthermore, let $A_q^\alpha : \mathcal{D}(A_q^\alpha) \to L_\sigma^q(\Omega)$, $-1 \le \alpha \le 1$, denote the fractional powers of A_q. It holds $\mathcal{D}(A_q) \subseteq \mathcal{D}(A_q^\alpha) \subseteq L_\sigma^q(\Omega)$, $R(A_q^\alpha) = L_\sigma^q(\Omega)$ if $0 \le \alpha \le 1$. We note that $(A_q^\alpha)^{-1} = (A_q^{-\alpha})$ and $(A_q)' = A_{q'}$ where $\frac{1}{q} + \frac{1}{q'} = 1$.

Now we recall the embedding estimate

$$(2.1) \quad \|v\|_q \le c\|A_\gamma^\alpha v\|_\gamma , \ v \in D(A_\gamma^\alpha), \ 1 < \gamma \le q < \infty, \ 2\alpha + \frac{3}{q} = \frac{3}{\gamma}, \ 0 \le \alpha \le 1,$$

and the estimate

$$(2.2) \quad \|A_q^\alpha e^{-tA_q} v\|_q \le c t^{-\alpha} e^{-\delta t} \|v\|_q , \ v \in L_\sigma^q(\Omega), \ 0 \le \alpha \le 1, \ t > 0,$$

with constants $c = c(\Omega, q) > 0$, $\delta = \delta(\Omega, q) > 0$, see [1, 8, 13, 14, 17, 29]. Using (2.1), (2.2) with $1 < \gamma \le q < \infty$, $2\beta + \frac{3}{q} = \frac{3}{\gamma}$ and constants c, $\delta > 0$ not depending on t, we obtain for $v \in L_\sigma^\gamma(\Omega)$ that $A^{-\beta} v \in L_\sigma^q(\Omega)$ and that

$$(2.3) \quad \|e^{-tA} v\|_q \le c t^{-\beta} e^{-\delta t} \|v\|_\gamma, \ t > 0.$$

Consequently, $\|e^{-tA} u_0\|_q$ with $u_0 \in L_\sigma^2(\Omega)$ is well-defined at least for $t > 0$, and there holds $\text{ess sup}_{(\delta,\infty)}(\tau^\alpha \|e^{-\tau A} u_0\|_q) < \infty$ for any $\delta > 0$ and $\alpha > 0$. In particular, the conditions $\sup_{(0,\infty)}(\tau^\alpha \|e^{-\tau A} u_0\|_q) < \infty$ and $\sup_{(0,\delta)}(\tau^\alpha \|e^{-\tau A} u_0\|_q) < \infty$ are equivalent for any $\delta > 0$.

Further note that $\mathcal{D}(A_q^{1/2}) = W_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega)$ and that the norms

$$\|A_q^{\frac{1}{2}} v\|_q \approx \|\nabla v\|_q , \ v \in \mathcal{D}(A_q^{\frac{1}{2}}),$$

are equivalent. In particular, if $q = 2$, then $\|A_2^{1/2} v\|_2 = \|\nabla v\|_2$ for $v \in \mathcal{D}(A^{1/2})$.

Consider any $g = \operatorname{div} G$ with $G = (G_{ij})_{i,j=1}^3 \in L^q(\Omega)$. Then a duality argument, see [16, Lemma 2.1], [28, Lemma 2.6.1], shows that $A_q^{-1/2} P_q \operatorname{div} G \in L_\sigma^q(\Omega)$ is well-defined by the identity $\langle A_q^{-\frac{1}{2}} P_q \operatorname{div} G, \varphi \rangle = -\langle G, \nabla A_{q'}^{-\frac{1}{2}} \varphi \rangle$ for $\varphi \in L_\sigma^{q'}(\Omega)$, and that

$$(2.4) \quad \|A_q^{-\frac{1}{2}} P_q \operatorname{div} G\|_q \le c\|G\|_q$$

holds with $c = c(\Omega, q) > 0$.

The Yosida approximation operator J_n defined by $J_n = (I + \frac{1}{n} A^{1/2})^{-1}$, $n \in \mathbb{N}$, has on each space $L_\sigma^q(\Omega)$, $1 < q < \infty$, the following fundamental properties:

$$(2.5) \quad \|J_n\|_{\mathcal{L}(L_\sigma^q)} + \left\|\frac{1}{n} A^{\frac{1}{2}} J_n\right\|_{\mathcal{L}(L_\sigma^q)} \le C_q < \infty \text{ for all } n \in \mathbb{N},$$

$$(2.6) \quad J_n u \to u \text{ in } L_\sigma^q(\Omega) \text{ for each } u \in L_\sigma^q(\Omega) \text{ as } n \to \infty.$$

Finally, we recall a weighted version of the Hardy-Littlewood-Sobolev inequality on weighted L^s-spaces on $\mathbb{R}$, cf. [30, 31],

$$L_\alpha^s(\mathbb{R}) = \Big\{ u : \|u\|_{L_\alpha^s} = \Big(\int_\mathbb{R} (|\tau|^\alpha |u(\tau)|)^s \, \mathrm{d}\tau \Big)^{1/s} < \infty \Big\}, \quad \alpha \in \mathbb{R}, \ s \geq 1.$$

Lemma 2.1. *Let* $0 < \lambda < 1$, $1 < s_1 \leq s_2 < \infty$, $-\frac{1}{s_1} < \alpha_1 < 1 - \frac{1}{s_1}$, $-\frac{1}{s_2} < \alpha_2 < 1 - \frac{1}{s_2}$ *and* $\frac{1}{s_1} + (\lambda + \alpha_1 - \alpha_2) = 1 + \frac{1}{s_2}$, $\alpha_2 \leq \alpha_1$. *Then the integral operator*

$$I_\lambda f(t) = \int_\mathbb{R} |t - \tau|^{-\lambda} f(\tau) \, \mathrm{d}\tau$$

is bounded as operator $I_\lambda : L_{\alpha_1}^{s_1}(\mathbb{R}) \to L_{\alpha_2}^{s_2}(\mathbb{R})$.

3. Proof of Theorem 1.3, Corollaries 1.4 and 1.5

Proof of Theorem 1.3. Let E_{f,u_0} denote the solution of the instationary Stokes problem on $\Omega \times (0, T)$ with data f, u_0:

$$\partial_t v - \Delta v + \nabla p = f, \ \operatorname{div} v = 0$$
$$v|_{\partial\Omega} = 0, \ v(0) = u_0,$$

i.e.,

$$E_{f,u_0}(t) = e^{-tA} u_0 + \int_0^t A^{1/2} e^{-(t-\tau)A} A^{-1/2} P \operatorname{div} F(\tau) \, \mathrm{d}\tau$$

(3.1) $$=: E_{0,u_0}(t) + E_{f,0}(t).$$

Evidently, the assumptions $u_0 \in L_\sigma^2(\Omega)$ and $F \in L^2(0, T; L^2(\Omega))$ imply that $E_{f,u_0} \in C^0([0, T]; L^2) \cap L^2(0, T; H^1)$, satisfying the energy equality. Moreover,

$$\|\nabla E_{f,u_0}\|_{L^2(0,T;L^2)} \leq c(\|u_0\|_2 + \|F\|_{L^2(0,T;L^2(\Omega))}).$$

By assumption (1.10) there holds $E_{0,u_0} \in L_\alpha^\infty(L^q)$. Finally, using the estimates (2.1) and (2.2) with $2\beta + \frac{3}{q} = \frac{3}{q/2}$ for $q > 3$, i.e., $\beta = \frac{3}{2q} = \frac{1}{2} - \alpha$,

$$\|E_{f,0}(t)\|_q \leq c \int_0^t \|A^{\frac{1}{2}+\beta} e^{-(t-\tau)A} (A^{-\frac{1}{2}} P \operatorname{div}) F(\tau)\|_{\frac{q}{2}} \, \mathrm{d}\tau$$

$$\leq c \int_0^t (t-\tau)^{-1+\alpha} \tau^{-2\alpha} \tau^{2\alpha} \|F(\tau)\|_{\frac{q}{2}} \, \mathrm{d}\tau$$

(3.2) $$\leq c \operatorname*{ess\,sup}_{(0,t)} \|\tau^{2\alpha} F(\tau)\|_{\frac{q}{2}} \int_0^t (t-\tau)^{-1+\alpha} \tau^{-2\alpha} \, \mathrm{d}\tau$$

$$= ct^{-\alpha} \operatorname*{ess\,sup}_{(0,t)} \|\tau^{2\alpha} F(\tau)\|_{\frac{q}{2}}.$$

Hence we proved that $\|E_{f,0}\|_{L_\alpha^\infty(0,t;L^q)} \leq c\|F\|_{L_{2\alpha}^\infty(0,t;L^{q/2})}$ and even
(3.3)
$$\|E_{0,u_0}\|_{L_\alpha^\infty(0,t;L^q)} + \|E_{f,0}\|_{L_\alpha^\infty(0,t;L^q)} \leq c(\|e^{-\tau A} u_0\|_{L_\alpha^\infty(0,t;L^q)} + \|F\|_{L_{2\alpha}^\infty(0,t;L^{q/2})})$$

with a constant $c > 0$ independent of $t > 0$.

We then set $\tilde{u} = u - E_{f,u_0}$ which solves the (Navier-)Stokes system

$$\partial_t \tilde{u} - \Delta \tilde{u} + u \cdot \nabla u + \nabla p = 0, \ \text{div}\, \tilde{u} = 0$$
$$\tilde{u}|_{\partial\Omega} = 0, \ \tilde{u}(0) = 0.$$

At least formally, we can write

$$(3.4) \qquad \tilde{u}(t) = -\int_0^t A^{1/2} e^{-(t-\tau)A} (A^{-1/2} P \, \text{div})(u \otimes u)(\tau) \, \mathrm{d}\tau$$

so that we define the nonlinear operator

$$(3.5) \ \ \mathcal{F}(\tilde{u})(t) = -\int_0^t A^{1/2} e^{-(t-\tau)A} (A^{-1/2} P \, \text{div})(u \otimes u)(\tau) \, \mathrm{d}\tau, \ u = \tilde{u} + E_{f,u_0}.$$

With $\beta = \frac{3}{2q} = \frac{1}{2} - \alpha$ we get as in (3.1), (3.2)

$$(3.6) \qquad \|\mathcal{F}(\tilde{u})(t)\|_q \le c \int_0^t (t-\tau)^{-1+\alpha} \tau^{-2\alpha} (\tau^\alpha \|u\|_q)^2 \, \mathrm{d}\tau.$$

We proceed as in (3.2), (3.3) and conclude that

$$(3.7) \qquad \qquad \|\mathcal{F}\tilde{u}\|_{L_\alpha^\infty(0,t;L^q)} \le c\|u\|_{L_\alpha^\infty(0,t;L^q)}^2.$$

Since $u = \tilde{u} + E_{f,u_0}$ we get from (3.3), (3.7) that for any $T > 0$

(3.8)
$$\|\mathcal{F}\tilde{u}\|_{L_\alpha^\infty(0,T;L^q)} \le c_0 \big(\|\tilde{u}\|_{L_\alpha^\infty(0,T;L^q)} + \|F\|_{L_{2\alpha}^\infty(0,T;L^{q/2})} + \|e^{-\tau A} u_0\|_{L_\alpha^\infty(0,T;L^q)}\big)^2,$$

where $c_0 = c_0(\Omega, q) > 0$ is independent of T and the data. With the abbreviation

$$b = b(T) := \|F\|_{L_{2\alpha}^\infty(0,T;L^{q/2})} + \|e^{-\tau A} u_0\|_{L_\alpha^\infty(0,T;L^q)}$$

we obtain from (3.8) the estimate

$$(3.9) \qquad \|\mathcal{F}\tilde{u}\|_{L_\alpha^\infty(0,T;L^q)} + b \le c_0(\|\tilde{u}\|_{L_\alpha^\infty(0,T;L^q)} + b)^2 + b.$$

Assume the smallness condition

$$(3.10) \qquad 4bc_0 = 4c_0\big(\|F\|_{L_{2\alpha}^\infty(0,T;L^{q/2})} + \|e^{-\tau A} u_0\|_{L_\alpha^\infty(0,T;L^q)}\big) < 1.$$

Obviously, the quadratic equation $r = c_0 r^2 + b$ has a minimal positive root given by $r_1 = 2b(1 + \sqrt{1 - 4bc_0})^{-1} \in (b, 2b)$. Note that $r_1 = r_1(b, c_0)$ is increasing in b as well as in c_0 and that $r_1 - b = c_0 r_1^2 \in (c_0 b^2, 4c_0 b^2)$. We conclude that $\mathcal{F}$ maps the non-empty closed ball $\mathcal{B} = \{v \in L_\alpha^\infty(0,T;L^q) : \|v\|_{L_\alpha^\infty(0,T;L^q)} \le r_1 - b\}$ into itself. Moreover, it is straightforward to modify the above estimates to show that for $\tilde{u}, \hat{u} \in \mathcal{B}$

$$\|\mathcal{F}\tilde{u} - \mathcal{F}\hat{u}\|_{L_\alpha^\infty(0,T;L^q)} \le 4bc_0 \|\tilde{u} - \hat{u}\|_{L_\alpha^\infty(0,T;L^q)}.$$

Since $4bc_0 < 1$, Banach's Fixed Point Theorem proves the existence of a unique fixed point $\tilde{u} \in L_\alpha^\infty(0,T;L^q)$ of $\mathcal{F}$ in $\mathcal{B}$; this fixed point $\tilde{u}$ solves (3.4). Hence the mild solution $u = \tilde{u} + E_{f,u_0}$ is contained in $L_\alpha^\infty(0,T;L^q)$.

Now we will prove that this mild solution u is indeed a weak solution. To this aim, we need the following lemmata.

Lemma 3.1. *The mild solution u constructed in the above procedure satisfies $\nabla u \in L^2(0, T; L^2(\Omega))$.*

Proof. We use a modification of the proof described in [10]. Since for the moment differentiability properties of the mild solution u are yet unknown, we apply the Yosida operator $J_n = (I + \frac{1}{n} A^{\frac{1}{2}})^{-1}, n \in \mathbb{N}$, to (3.4) and write $J_n P \operatorname{div} u \otimes u$ in the form $J_n P \operatorname{div}(u \otimes (\tilde{u} + E_{f,u_0}))$, $\tilde{u} = (I + \frac{1}{n} A^{\frac{1}{2}})\tilde{u}_n$, where $\tilde{u}_n = J_n \tilde{u}$. Then we have

$$J_n P \operatorname{div} u \otimes u$$
$$= J_n P(u \cdot \nabla E_{f,u_0}) + J_n P(u \cdot \nabla \tilde{u}_n) + \frac{1}{n} J_n P \operatorname{div}(u \otimes A^{\frac{1}{2}} \tilde{u}_n)$$
$$= J_n P(u \cdot \nabla E_{f,u_0}) + J_n P(u \cdot \nabla \tilde{u}_n) + \frac{1}{n} A^{\frac{1}{2}} J_n (A^{-\frac{1}{2}} P \operatorname{div})(u \otimes A^{\frac{1}{2}} \tilde{u}_n).$$

By (2.5), (2.6) and Hölder's inequality with $\frac{1}{\gamma} = \frac{1}{2} + \frac{1}{q}$ we obtain the estimate

$$\|J_n P \operatorname{div}(u \otimes u)\|_\gamma \le c \|u\|_q \big(\|\nabla E_{f,u_0}\|_2 + \|\nabla \tilde{u}_n\|_2 + \|A^{\frac{1}{2}} \tilde{u}_n\|_2 \big)$$
$$= c \|u\|_q \big(\|\nabla E_{f,u_0}\|_2 + 2\|A^{\frac{1}{2}} \tilde{u}_n\|_2 \big).$$

Applying $A^{\frac{1}{2}} J_n$ to (3.4) it holds the identity

$$A^{\frac{1}{2}} \tilde{u}_n(t) = - \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} J_n P \operatorname{div}(u \otimes u)(\tau) \, \mathrm{d}\tau,$$

and by (2.1) with $2\beta + \frac{3}{2} = \frac{3}{\gamma}$, i.e., $\beta = \frac{3}{2q} = \frac{1}{2} - \alpha$, the estimate

$$\|A^{\frac{1}{2}} \tilde{u}_n(t)\|_2 \le c \int_0^t \|A^{\frac{1}{2}+\beta} e^{-(t-\tau)A}\| \|J_n P \operatorname{div}(u \otimes u)(\tau)\|_\gamma \, \mathrm{d}\tau$$
$$\le c \int_0^t (t-\tau)^{-1+\alpha} \|u(\tau)\|_q \big(\|\nabla E_{f,u_0}(\tau)\|_2 + 2\|A^{\frac{1}{2}} \tilde{u}_n(\tau)\|_2 \big) \, \mathrm{d}\tau.$$

Now let $s_1 = 2, \alpha_1 = \alpha, s_2 = 2, \alpha_2 = 0, \lambda = 1 - \alpha$, so that $\frac{1}{2} + (1 - \alpha + \alpha - 0) = 1 + \frac{1}{2}$. By Lemma 2.1 we have for any $0 < T_1 \le T$

$$\|A^{\frac{1}{2}} \tilde{u}_n(t)\|_{L^2(0,T_1;L^2)}$$
$$\le c \Big(\int_0^{T_1} \big(\tau^\alpha \|u\|_q (\|\nabla E_{f,u_0}\|_2 + \|A^{\frac{1}{2}} \tilde{u}_n\|_2) \big)^2 \mathrm{d}\tau \Big)^{1/2}$$
$$\le c_1 \|u\|_{L^\infty_\alpha(0,T_1;L^q)} \big(\|\nabla E_{f,u_0}\|_{L^2(0,T_1;L^2)} + \|A^{\frac{1}{2}} \tilde{u}_n\|_{L^2(0,T_1;L^2)} \big).$$

Next, to achieve for $u = E_{f,u_0} + \tilde{u}$ the smallness condition

$$(3.11) \qquad\qquad c_1 \|u\|_{L^\infty_\alpha(0,T_1;L^q)} \le \frac{1}{2}$$

note that, with a constant $c_2 = c_2(\Omega, q) > 0$,

$$c_1 \|u\|_{L^\infty_\alpha(0,T_1;L^q)} \le c_2 \big(\|e^{-\tau A} u_0\|_{L^\infty_\alpha(0,T_1;L^q)} + \|F\|_{L^\infty_{2\alpha}(0,T_1;L^{q/2})} + r_1 - b \big)$$
$$\le c_2(b + r_1 - b) \le 2 c_2 b.$$

Thus, in order to satisfy (3.11) by means of the condition

(3.12) $$4c_* b < 1, \quad c_* := \max(c_0, c_2),$$

it suffices to replace in (3.10) c_0 by c_2. Then the absorption argument yields the estimate

$$\|A^{\frac{1}{2}} \tilde{u}_n\|_{L^2(0,T_1;L^2)} \le 2c_1 \|u\|_{L^\infty_\alpha(0,T_1;L^q)} \|\nabla E_{f,u_0}\|_{L^2(0,T_1;L^2)} < \infty$$

independent of $n \in \mathbb{N}$. Consequently, using reflexivity arguments, $A^{\frac{1}{2}} \tilde{u}, \nabla \tilde{u} \in L^2(0,T_1;L^2)$ and $\nabla u \in L^2(0,T_1;L^2)$. $\qquad \square$

Lemma 3.2. *Under the assumptions of Lemma 3.1 we have the following results*:

(i) $u \in L^{s_2}(0,T;L^{q_2})$ *for all* $\frac{2}{s_2} + \frac{3}{q_2} = \frac{3}{2}$, $2 \le s_2 \le \infty$, $2 \le q_2 \le 6$.

(ii) $\|\tilde{u}(t)\|_2 \to 0$ *and* $u(t) \to u_0$ *in* $L^2(\Omega)$ *as* $t \to 0+$.

(iii) $u \in L^4_{\alpha/(2+8\alpha)}(0,T;L^4(\Omega))$.

(iv) u *satisfies the energy equality on* $[0,T]$.

Proof. (i) From Lemma 3.1 we know that $\nabla u \in L^2(0,T;L^2)$. Moreover, by (2.3) with $\beta = \frac{3}{2q} = \frac{1}{2} - \alpha$, Hölder's inequality implies that

$$\|\tilde{u}(t)\|_2 \le c \int_0^t (t-\tau)^{-\frac{1}{2}+\alpha} \tau^{-\alpha} (\tau^\alpha \|u\|_q) \|\nabla u\|_2 \, d\tau$$

(3.13) $$\le C \|u\|_{L^\infty_\alpha(0,t;L^q)} \|\nabla u\|_{L^2(0,t;L^2)};$$

we note that here $\alpha > 0$ is necessary. Hence

$$\|\tilde{u}\|_{L^\infty(0,t;L^2)} \le C \|u\|_{L^\infty_\alpha(0,t;L^q)} \|\nabla u\|_{L^2(0,t;L^2)}.$$

From the properties $u \in L^\infty(L^2)$ and $\nabla u \in L^2(L^2)$ it follows immediately that $u \in L^{s_2}(L^{q_2})$ as required in (i).

(ii) From (i), to be more precise from (3.13), we conclude that $\|\tilde{u}(t)\|_2 \to 0$ as $t \to 0$. Since also $e^{-tA} u_0 \to u_0$ in $L^2(\Omega)$ and $E_{f,0}(t) \to 0$ in $L^2(\Omega)$ as $t \to 0$, (ii) is proven.

(iii) Given q, α and $\beta = \frac{1}{2+8\alpha}$ we define q_1, s_1 by $\frac{1}{4} = \frac{\beta}{q} + \frac{1-\beta}{q_1}$ and $\frac{2}{s_1} + \frac{3}{q_1} = \frac{3}{2}$, *i.e.*, $s_1 = 4(1-\beta)$. From Hölder's inequality we know that $\|u\|_4 \le \|u\|_q^\beta \|u\|_{q_1}^{1-\beta}$. Hence

$$\int_0^T \tau^{4\alpha\beta} \|u\|_4^4 \, d\tau \le \int_0^T (\tau^\alpha \|u\|_q)^{4\beta} \|u\|_{q_1}^{4(1-\beta)} \, d\tau$$

$$\le \|u\|_{L^\infty_\alpha(L^q)}^{4\beta} \|u\|_{L^{4(1-\beta)}(L^{q_1})}^{4(1-\beta)} < \infty.$$

(iv) From (iii) we know that $u \in L^4(\epsilon,T;L^4)$ for all $0 < \epsilon < T$. So, by [28, IV. Theorem 2.3.1, Lemma 2.4.2] and for a.a. $\epsilon \in (0,T)$, u is the unique weak solution in $L^4(\epsilon,T;L^4)$ on the interval (ϵ,T) of the linear Stokes problem

$$\partial_t u - \Delta u + \nabla p = \operatorname{div} \tilde{F}, \quad \operatorname{div} u = 0$$

$$u|_{\partial\Omega} = 0, \quad u|_{t=\epsilon} = u(\epsilon)$$

with external force div $\tilde{F}$, $\tilde{F} = F - u \otimes u \in L^2(\epsilon, T; L^2)$ and initial value $u(\epsilon) \in L^4(\Omega) \subset L^2(\Omega)$. Therefore, u satisfies the energy equality on (ϵ, T), i.e.,

$$\frac{1}{2}\|u(t)\|_2^2 + \int_\epsilon^t \|\nabla u\|_2^2 \, \mathrm{d}\tau = \frac{1}{2}\|u(\epsilon)\|_2^2 - \int_\epsilon^t (F, \nabla u) \, \mathrm{d}\tau$$

for all $t \in (\epsilon, T)$ and a.a. $\epsilon \in (0, T)$. Letting $\epsilon \to 0$ we conclude in view of (ii) that u satisfies the energy equality even on $[0, T)$. □

Concerning uniqueness we note that $r_1 - b = c_* r_1^2 \in (c_* b^2, 4c_* b^2)$. Hence u coincides with any other strong $L_\alpha^\infty(0, T; L^q(\Omega))$-solution v satisfying $\tilde{v} = v - E_{f, u_0} \in \mathcal{B}$; this can be achieved when $\|\tilde{v}\|_{L_\alpha^\infty(0, T'; L^q)} \leq c_* b^2$.

By Lemma 3.2 we obtain that u is a weak solution of (1.1); this completes the proof of Theorem 1.3. □

Proof of Corollary 1.4. The proof follows the lines of the proof of Theorem 1.3. However, by the assumptions on u_0 and F conditions (3.10) and (3.12) can be achieved by choosing $T' > 0$ sufficiently small. Given any other strong $L_\alpha^\infty(0, T; L^q)$-solution v we find $T' \leq T$ such that $\|\tilde{v}\|_{L_\alpha^\infty(0, T'; L^q)} < c_* b^2$. Then v coincides with u on $(0, T')$. □

Proof of Corollary 1.5. (i) Assume that $u \in L_\alpha^\infty(0, T; L^q)$, $0 < T \leq \infty$, is a strong $L_\alpha^\infty(L^q)$ solution of (1.1). Recall that $E_{0, u_0} = u - \tilde{u} - E_{f, 0}$ where by (3.7) $\tilde{u} = \mathcal{F}\tilde{u} \in L_\alpha^\infty(L^q)$, and by (3.2) $E_{f, 0} \in L_\alpha^\infty(L^q)$. Hence $E_{0, u_0} \in L_\alpha^\infty(L^q)$ as well, and (1.11) is satisfied.

(ii) By assumption on F and in view of the estimate (3.2), we conclude that $E_{f, 0} \in \overset{\circ}{L}_\alpha^\infty(0, T; L^q)$. Given that $u \in \overset{\circ}{L}_\alpha^\infty(0, T; L^q)$, also $\tilde{u} = \mathcal{F}\tilde{u} \in \overset{\circ}{L}_\alpha^\infty(0, T; L^q)$ due to (3.7). Hence we get that $u_0 \in \overset{\circ}{\mathbb{B}}_{q, \infty}^{-1+3/q}$. □

4. Proof of Theorems 1.8 and 1.9

Proof of Theorem 1.8. By Definition 1.6(3) there exists a sequence of approximate weak solutions (u_n) bounded in $\mathcal{LH}_T$ such that a subsequence (u_{n_k}) converges to a weak solution $v \in \mathcal{LH}_T$ of (1.1) satisfying the strong energy equality (1.7).

Since (u_n) is uniformly bounded in $L_\alpha^\infty(0, T'; L^q)$ with T' from Definition 1.6(4) we find a subsequence $(u_{n'_k})$ of (u_{n_k}) converging weakly-* in $L_\alpha^\infty(0, T'; L^q)$ to an element $v' \in L_\alpha^\infty(0, T'; L^q)$. Now, since $u_{n_k} \rightharpoonup v$ in $\mathcal{LH}_{T'}$, we may conclude that $v = v'$ on $(0, T')$; in particular, v' is a weak and even a strong $L_\alpha^\infty(L^q)$-solution of (1.1) on $(0, T')$. Since strong $L_\alpha^\infty(L^q)$-solutions are unique by Theorem 1.3 on some interval $(0, T'')$, $v = v' = u$ on $(0, T'') \subset (0, T')$. This uniqueness also implies that any other subsequence (u_{m_k}) of (u_n) converging weakly in $\mathcal{LH}_{T'}$ to any weak solution actually converges weakly to v' on $(0, T'')$ as $k \to \infty$. Hence the whole sequence (u_n) converges weakly to v on $(0, T'')$. Moreover, again due to uniqueness, this result will hold for any sequence (u_{0n}) and (F_n) with convergence properties as in Definition 1.6.

If $T'' < T$, then we find due to (SEI) applied to v some $0 < T^* \leq T''$ such that the weak solution v satisfies the energy estimate on $[T^*, T)$ with initial time T^*. Since $\frac{3}{q} = 1 - 2\alpha < 1$, there exists $2 < s < \infty$ with $\frac{2}{s} + \frac{2}{q} = 1$ such that $u \in L^s(T^*, T; L^q(\Omega))$ is a "classical" strong solution, and Serrin's Uniqueness Theorem implies that $u = v$ even on $[0, T)$. $\square$

Proof of Theorem 1.9. (i) Given u_0, u_{0n} and F, F_n as in Definition 1.6 classical L^2-methods, see [28, Ch. V.2], prove the existence of a unique approximate solution $u_n \in \mathcal{LH}_T$ of (1.15) and the convergence of a subsequence of (u_n) to a weak solution $u \in \mathcal{LH}_T$ of (1.1). Indeed, each u_n satisfies the energy equality, and consequently the energy estimate

$$\|u_n(t)\|_2^2 + \int_0^t \|\nabla u_n\|_2^2 \, \mathrm{d}\tau \leq \|u_{0n}\|_2^2 + \int_0^t \|F_n\|_2^2 \, \mathrm{d}\tau,$$

with a right-hand side uniformly bounded with respect to $n \in \mathbb{N}$ and $0 < t < T$ due to the convergence properties of u_{0n}, F_n in Definition 1.6, see also Remark 1.7(2). Finally, $(\partial_t u_n)$ is uniformly bounded in $L^{4/3}(0, T; H_{0,\sigma}^1(\Omega)')$, see [28, Lemma V. 2.6.1, Theorem V. 1.6.2]. Hence, by the Aubin-Lions-Simon compactness theorem for Bochner spaces, there exists a subsequence (u_{n_k}) of (u_n) and $v \in \mathcal{LH}_T$ such that

$$(4.1) \qquad u_{n_k} \rightharpoonup v \ \text{ in } \ \mathcal{LH}_T, \quad u_{n_k} \to v \ \text{ in } \ L^2\big(0, T; L_\sigma^2(\Omega)\big)$$

as $k \to \infty$. Furthermore,

$$(4.2) \qquad u_{n_k}(t) \to v(t) \ \text{ in } L_\sigma^2(\Omega) \text{ for a.a. } t \in (0, T)$$

as $k \to \infty$; this step needs the extraction of a further subsequence, as the case may be. Now (4.1) allows us to pass to the limit in (1.15) and to show that v is a weak solution of (1.1) in the sense of Leray-Hopf. In particular, v satisfies the energy inequality (1.4), and due to (4.2) even the strong energy inequality (1.7).

In the second step of the proof we improve the previous results by exploiting the properties of u_0 in $\overset{\circ}{\mathbb{B}}_{q,\infty}^{-1+3/q}$ and of F in $\overset{\circ}{L}_{2\alpha}^\infty\big(0, T; L^{q/2}\big)$, see Definition 1.6. Since (u_{0n}) converges strongly to u_0 in $\mathbb{B}_{q,\infty}^{-1+3/q}$ we find some $T' \in (0, T]$ and $N_* \in \mathbb{N}$ such that

$$\sup_{\tau \in (0, T')} \|\tau^\alpha e^{-\tau A} u_{0n}\|_q \leq \frac{\varepsilon_*}{2}$$

for all $n \geq N_*$ where $\varepsilon_* > 0$ is the absolute constant from (1.10), see also (1.14). Furthermore, since $F_n \to F$ in $L_{2\alpha}^\infty\big(0, T; L^{q/2}(\Omega)\big)$ we may also assume that

$$\|F_n\|_{L_{2\alpha}^\infty(0, T'; L^{q/2})} \leq \frac{\varepsilon_*}{2}$$

for all $n \geq N_*$, see (1.13). We follow the construction of strong $L_\alpha^\infty(L^q)$-solutions in the proof of Theorem 1.3, decompose the solution u_n of (1.15) into

$u_n = \tilde{u}_n + E_n$ where E_n solves the linear nonhomogeneous Stokes problem with data u_{0n}, F_n, i.e.,

$$(4.3) \qquad E_n(t) = e^{-tA}u_{0n} + \int_0^t A^{1/2}e^{-(t-\tau)}\big(A^{-1/2}P\ \mathrm{div}\big)F_n(\tau)\,\mathrm{d}\tau.$$

Then $\tilde{u}_n = u_n - E_n$ has an integral representation based on the variation of constants formula and can be considered as solution of the fixed point problem $\tilde{u}_n = \mathcal{F}_n\tilde{u}_n$ in $L_\alpha^\infty\big(0, T'; L^q(\Omega)\big)$ where

$$\mathcal{F}_n\tilde{u}_n(t) = -\int_0^t A^{1/2}e^{-(t-\tau)A}\big(A^{-1/2}P\ \mathrm{div}\big)\big(J_n(\tilde{u}_n + E_n) \otimes (\tilde{u}_n + E_n)\big)(\tau)\,\mathrm{d}\tau;$$

note that $\mathcal{F}_n$ differs from $\mathcal{F}$ in (3.5) only by the additional term J_n. Due to fundamental properties of the Yosida operators J_n the fixed point of $\mathcal{F}_n$ can be constructed by Banach's Fixed Point Theorem in the same way as in the proof of Theorem 1.3. By the assumptions on u_n, F_n and (3.8), (3.9), (3.10), $\tilde{u}_n, u_n$ satisfy the estimate

$$(4.4) \qquad \|\tilde{u}_n\|_{L_\alpha^\infty(0,T';L^q)}, \|u_n\|_{L_\alpha^\infty(0,T';L^q)} \le C\varepsilon_*$$

with a constant $C > 0$ independent of $n \ge N_*$. Thus $(\tilde{u}_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ are bounded in $L_\alpha^\infty(0,T'; L^q)$.

(ii) It is well known that the Stokes operator A_2 on the bounded $C^{2,1}$-domain $\Omega \subset \mathbb{R}^3$ admits an orthonormal basis of eigenfunctions $\psi_k \in \mathcal{D}(A_2)$ with corresponding eigenvalues λ_k monotonically increasing to ∞ as $k \to \infty$. For $n \in \mathbb{N}$ let

$$\Pi_n : L_\sigma^2(\Omega) \to V_n := \mathrm{span}\{\psi_1, \ldots, \psi_n\} \subset L_\sigma^2(\Omega)$$

denote the corresponding orthogonal projection. Obviously, $\|\Pi_n\|_{\mathcal{L}(L_\sigma^2(\Omega))} = 1$ for all $n \in \mathbb{N}$. In the Galerkin method we are looking for a solution $u_n : [0, T) \to V_n$ of the ordinary differential $n \times n$-system

$$(4.5) \qquad \begin{aligned} (\partial_t u_n, \psi_k) + (\nabla u_n, \nabla\psi_k) - (u_n \otimes u_n, \nabla\psi_k) &= -(F_n, \nabla\psi_k) \\ u_n(0) &= u_{0n} \in V_n \end{aligned}$$

on $(0, T)$ for each $k = 1, \ldots, n$. By the L^2-assumptions on u_{0n} and F_n we know that there exists a sequence of unique solutions (u_n) to (4.5) bounded in $\mathcal{LH}_T$. Moreover, $(\partial_t u_n)$ is uniformly bounded in $L^{4/3}\big(0, T; H_{0,\sigma}^1(\Omega)'\big)$. As in the first part of the proof we find a subsequence (u_{n_k}) of (u_n) and a vector field v satisfying (4.1) and (4.2). In particular, $v \in \mathcal{LH}_T$ is a weak solution to (1.1) satisfying (1.7).

The crucial question is whether u_n is also a strong $L_\alpha^\infty(0, T'; L^q)$-solution, uniformly bounded in n. To address this problem we consider arbitrary linear combinations of $(4.5)_1$ to see that for all $w \in \mathcal{D}(A^{1/2})$

$$(4.6) \qquad \begin{aligned} (\partial_t u_n, \Pi_n w) + (\nabla u_n, \nabla\Pi_n w) - (u_n \otimes u_n, \nabla\Pi_n w) &= -(F_n, \nabla\Pi_n w) \\ u_n(0) &= u_{0n} \in V_n. \end{aligned}$$

Since $\Pi_n = P\Pi_n$, $P^* = P$, A commutes with Π_n, and $(\nabla u_n, \nabla \Pi_n w) = (Au_n, w)$, we may omit the test function $w \in H^1_{0,\sigma}(\Omega)$ and rewrite (4.6) in the form

$$(4.7) \quad \partial_t u_n + Au_n + \Pi_n P \operatorname{div}(u_n \otimes u_n) = \Pi_n P \operatorname{div} F_n, \quad u_n(0) = u_{0n} \in V_n.$$

Thus $u_n(t)$ can be considered as a solution in $W^{1,4/3}(0,T)$ (with respect to time) of an abstract Cauchy problem and as a mild solution with integral representation

$$(4.8) \quad u_n(t) = e^{-tA}u_{0n} - \int_0^t A^{1/2} e^{-(t-\tau)A} \left(A^{-1/2}\Pi_n P \operatorname{div} \right)(u_n \otimes u_n - F_n)(\tau)\, d\tau.$$

Although $\|\Pi_n\|_{\mathcal{L}(L^2_\sigma(\Omega))} = 1$ and $A^{-1/2}P\operatorname{div} \in \mathcal{L}(L^q(\Omega))$ for each $1 < q < \infty$, similar estimates will neither hold for Π_n on $L^q_\sigma(\Omega)$ nor for the operator $A^{-1/2}\Pi_n P\operatorname{div}$ on $L^q(\Omega)$ uniformly in $n \in \mathbb{N}$. Actually, an estimate of the type $\|\Pi_n\|_{\mathcal{L}(L^q_\sigma(\Omega))} \le c(q)$ uniformly in $n \in N$ is questionable when $q \ne 2$.

Using (2.1), (2.3) and exploiting the uniform boundedness and commutator properties of Π_n on $L^2_\sigma(\Omega)$ we get that

$$(4.9) \quad \begin{aligned} &\|A^{1/2}e^{-(t-\tau)A}(A^{-1/2}\Pi_n P\operatorname{div})(u_n \otimes u_n - F_n)\|_q \\ &\le c\|A^{\frac{3}{4}+\alpha}e^{-(t-\tau)A}(A^{-1/2}\Pi_n P\operatorname{div})(u_n \otimes u_n - F_n)\|_2 \\ &\le c\|A^{\frac{3}{4}+\alpha}e^{-(t-\tau)A}(A^{-1/2}P\operatorname{div})(u_n \otimes u_n - F_n)\|_2 \\ &\le c(t-\tau)^{-1+\alpha}\|u_n \otimes u_n - F_n\|_{q/2}. \end{aligned}$$

This estimate directly implies that with a constant $c > 0$ independent of $n \in \mathbb{N}$ and T

$$\|u_n - e^{-tA}u_{0n}\|_{L^\infty_\alpha(0,T;L^q)} \le c\left(\|u_n\|^2_{L^\infty_\alpha(0,T;L^q)} + \|F_n\|_{L^\infty_{2\alpha}(0,T;L^{q/2})}\right).$$

Then by standard arguments we find $T' \in (0,T)$ independent of $n \in \mathbb{N}$ such that $(u_n) \subset L^\infty_\alpha(0,T';L^q)$ is uniformly bounded.

Now we complete the proof as in the previous case. $\qquad\square$

5. Interpretation in terms of Besov spaces

For $1 < q < \infty$, $1 \le r \le \infty$ and $t \in \mathbb{R}$ let $B^t_{q,r}(\mathbb{R}^3)$ denote the usual Besov spaces, see [32, 2.3.1], and define for the bounded domain $\Omega \subset \mathbb{R}^3$ the space $B^t_{q,r}(\Omega)$ by restriction of elements in $B^t_{q,r}(\mathbb{R}^3)$ in the sense of distributions to Ω; the norm of $u \in B^t_{q,r}(\Omega)$ is given by the infimum of norms of all $v \in B^t_{q,r}(\mathbb{R}^3)$ such that $v_{|\Omega} = u$. Concerning Besov spaces on Ω with vanishing trace - if possible - the definition is modified as follows: Considering only vector fields rather than scalar-valued functions and the range $t \in [-2, 2]$ we follow Amann

[4], [2] and define

$$(5.1) \qquad \mathbf{B}_{q,r}^t(\Omega) = \begin{cases} \{u \in B_{q,r}^t(\Omega)^3; \ \gamma u = 0\} & 1/q < t \le 2 \\ \{u \in B_{q,r}^{1/q}(\mathbb{R}^3)^3; \ \mathrm{supp}(u) \subset \overline{\Omega}\} & 1/q = t \\ B_{q,r}^t(\Omega)^3 & 0 \le t < 1/q \\ \left(\mathbf{B}_{q',r'}^{-t}(\Omega)\right)' \ (1 < r \le \infty) & -2 \le t < 0, \end{cases}$$

where γ denotes the trace operator defined by $\gamma u = u_{|\partial\Omega}$ for continuous functions. For spaces of solenoidal vector fields on Ω let

$$(5.2) \qquad \mathbb{B}_{q,r}^t(\Omega) = \begin{cases} \mathbf{B}_{q,r}^t(\Omega) \cap L_\sigma^q(\Omega) & 0 < t \le 2 \\ \mathrm{cl}\big(C_{0,\sigma}^\infty(\Omega)\big) \text{ in } \mathbf{B}_{q,r}^0(\Omega) & t = 0 \\ \left(\mathbb{B}_{q',r'}^{-t}(\Omega)\right)' \ (1 < r \le \infty) & -2 \le t < 0 \end{cases} ;$$

here "cl" denotes the closure. Note that $u \in \mathbb{B}_{q,r}^t(\Omega)$ with $\frac{1}{q} < t \le 2$ vanishes on $\partial\Omega$ ($\gamma u = 0$) by (5.1), but that only the normal component of u vanishes on $\partial\Omega$ when $0 < t \le \frac{1}{q}$ since $u \in L_\sigma^q(\Omega)$.

Moreover, we need the spaces

$$\mathring{\mathbb{B}}_{q,\infty}^t(\Omega) := \mathrm{cl}\big(\mathbf{H}_q^t(\Omega) \cap L_\sigma^q(\Omega)\big) \text{ in } \mathbb{B}_{q,\infty}^t(\Omega), \quad -2 \le t \le 2,$$

where $\mathbf{H}_q^t(\Omega)$ is a Bessel potential space defined by restriction of the usual Bessel potential space $H_q^t(\mathbb{R}^3)^3$ to vector fields on Ω (and vanishing on $\partial\Omega$ as in (5.1)), cf. [4, pp. 3–4]. For $0 < |t| < 2$ these spaces are also called *little Nikol'skii space* and denoted by $\mathrm{n}_{q,0,\sigma}^t(\Omega)$. Then, using the notation $(\cdot, \cdot)_{\theta,r}$, $1 \le r < \infty$, of real interpolation, and $(\cdot, \cdot)_{\theta,\infty}^0$ for the continuous interpolation functor, Theorem 3.4 in [2] states that

$$(5.3) \qquad (L_\sigma^q(\Omega), \mathcal{D}(A_q))_{\theta,r} = \mathbb{B}_{q,r}^{2\theta}(\Omega), \quad 0 < \theta < 1,$$

$$(5.4) \qquad (L_\sigma^q(\Omega), \mathcal{D}(A_q))_{\theta,\infty}^0 = \mathring{\mathbb{B}}_{q,\infty}^{2\theta}(\Omega), \quad 0 < \theta < 1.$$

Note that $\mathcal{D}(A_q)$ is equipped with its graph norm, and that for a bounded domain this graph norm can be simplified to $\|A_q \cdot\|_q$. As is well-known ([25, Proposition 6.2, Exercise 6.1.1(1)]), equivalent norms on the spaces $(L_\sigma^q(\Omega), \mathcal{D}(A_q))_{\theta,r}$, $1 \le r < \infty$, are given by

$$\|u\|_{\mathbb{B}_{q,r}^{2\theta}} \sim \left(\int_0^T \left(\tau^{1-\theta} \|A_q e^{-\tau A_q} u\|_q \right)^r \frac{\mathrm{d}\tau}{\tau} \right)^{1/r},$$

where $T \in (0, \infty)$ can be chosen arbitrarily and an additional term $\|u\|_q$ on the right-hand side can be omitted since Ω is bounded. For $r = \infty$ we have

$$(5.5) \qquad \|u\|_{\mathbb{B}_{q,r}^{2\theta}} \sim \sup_{(0,T)} \tau^{1-\theta} \|A_q e^{-\tau A_q} u\|_q.$$

In the case (5.4) $\overset{\circ}{\mathbb{B}}{}^{2\theta}_{q,\infty}(\Omega)$ may be equipped with the equivalent norm given in (5.5), but elements $u \in \overset{\circ}{\mathbb{B}}{}^{2\theta}_{q,\infty}(\Omega)$ enjoy the further property that

$$(5.6) \qquad \lim_{\tau \to 0} \tau^{1-\theta} \|A_q e^{-\tau A_q} u\|_q = 0.$$

The conclusion is that the spaces $(L^q_\sigma(\Omega), \mathcal{D}(A_q))_{\theta,\infty}$ and $(L^q_\sigma(\Omega), \mathcal{D}(A_q))^0_{\theta,\infty}$ are special Besov spaces characterized by (5.5) and (5.5)-(5.6), respectively.

The aim is to find similar characterizations of those spaces and norms used in Sections 1-4. Theorem 3.4 in [2] applies to negative exponents of regularity as well. E.g., for $-1 < \theta < 0$ and $1 < r < \infty$, we have by (5.2)

$$\big(L^q_\sigma(\Omega), \mathcal{D}(A_{q'})'\big)_{-\theta,r} = \big(\big(L^{q'}_\sigma(\Omega), \mathcal{D}(A_{q'})\big)_{-\theta,r'}\big)' = \big(\mathbb{B}^{-2\theta}_{q',r'}(\Omega)\big)' = \mathbb{B}^{2\theta}_{q,r}(\Omega).$$

To deal with the cases $r = 1$ and $r = \infty$ note that A_q is an isomorphism from $\mathcal{D}(A_q)$ to $L^q_\sigma(\Omega)$ and also from $L^q_\sigma(\Omega)$ to $\mathcal{D}(A_{q'})'$. Hence, for $1 \le r \le \infty$ and $-1 < \theta < 0$

$$(5.7) \qquad \big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)_{1+\theta,r} = A\big(\big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1+\theta,r}\big),$$

with a similar result for the continuous interpolation functor $(\cdot, \cdot)^0_{\theta,\infty}$. Then we get the following characterizations of real interpolation spaces of $\mathcal{D}(A_{q'})'$ and $L^q_\sigma(\Omega)$ (here $-1 < \theta < 0$):

$$(5.8) \qquad \big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)_{1+\theta,r} = \mathbb{B}^{2\theta}_{q,r}(\Omega), \quad 1 \le r < \infty,$$

$$(5.9) \qquad \big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)_{1+\theta,\infty} = \mathbb{B}^{2\theta}_{q,\infty}(\Omega) \cong \mathbf{B}^{2\theta}_{q,\infty}(\Omega)/\big(\mathbb{B}^{-2\theta}_{q',1}(\Omega)\big)^\perp,$$

$$\big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)^0_{1+\theta,\infty} = \overset{\circ}{\mathbb{B}}{}^{2\theta}_{q,\infty}(\Omega) = \mathrm{cl}\big(\mathbf{H}^2_q(\Omega)\big) \text{ in } \big(\mathbb{B}^{-2\theta}_{q',1}(\Omega)\big)'$$

$$(5.10) \qquad\qquad\qquad = \mathrm{cl}\big(L^q_\sigma(\Omega)\big) \text{ in } \mathbb{B}^{2\theta}_{q,\infty}(\Omega).$$

Actually, (5.8) for $r = 1$ and (5.10) follow from [2, Theorem 3.4], [4, p. 4], for $-1 < \theta < 0$. Moreover, to prove (5.9) we use the duality theorem of real interpolation to get that identity $\big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)_{1+\theta,\infty} = \big(\big(L^{q'}_\sigma(\Omega), \mathcal{D}(A_{q'})\big)_{-\theta,1}\big)'$ $= \big(\mathbb{B}^{-2\theta}_{q',1}(\Omega)\big)'$ and the definition $\mathbb{B}^{2\theta}_{q,\infty}(\Omega) = \big(\mathbb{B}^{-2\theta}_{q',1}(\Omega)\big)'$ in (5.2); this space is called Nikol'skii space $N^{2\theta}_{q,0,\sigma}(\Omega)$ in [2]. The second part of (5.9) is found in the proof of [2, Remark 3.6] and a consequence of the isomorphism

$$\mathbb{B}^{2\theta}_{q,\infty}(\Omega) = \big(\mathbb{B}^{-2\theta}_{q',1}(\Omega)\big)' \cong \mathbf{B}^{2\theta}_{q,\infty}(\Omega)/(\mathbb{B}^{-2\theta}_{q',1}(\Omega))^\perp;$$

here $\mathbf{B}^{2\theta}_{q,\infty}(\Omega) = \big(\mathbf{B}^{-2\theta}_{q',1}(\Omega)\big)'$ by [32, Theorems 4.3.2 and 4.8.1], since in our application $-2\theta = 2\alpha > \frac{1}{q'} - 1$ and $-2\theta - \frac{1}{q'} \notin \mathbb{Z}$. Recall that the characterization (5.7) also holds when $r = \infty$.

Thus for any $1 \leq r \leq \infty$ and $-1 < \theta < 0$, by (5.7), (5.8), (5.9) and (5.3), $\left(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\right)_{1+\theta,r} = A(\mathbb{B}^{2+2\theta}_{q,r}(\Omega)) = \mathbb{B}^{2\theta}_{q,r}(\Omega)$ and has the equivalent norm

(5.11)

$$\|u\|_{A(\mathbb{B}^{2+2\theta}_{q,r})} \sim \Big(\int_0^T \big(\tau^{-\theta}\|e^{-\tau A_q}u\|_q\big)^r \, \frac{d\tau}{\tau}\Big)^{1/r}, \quad -1 < \theta < 0, \, 1 \leq r < \infty,$$

(5.12)

$$\|u\|_{A(\mathbb{B}^{2+2\theta}_{q,\infty})} \sim \sup_{\tau \in (0,T)} \tau^{-\theta}\|e^{-\tau A_q}u\|_q, \quad -1 < \theta < 0.$$

This result was used in [10] when $\frac{2}{r} + \frac{3}{q} = 1$, $\theta = 0$, $2 < r < \infty$.

Let us summarize these results for the case $\theta = -\alpha = \frac{1}{2}\left(\frac{3}{q} - 1\right)$ needed in this paper.

Theorem 5.1. *Let* $3 < q < \infty$, $0 < \alpha < \frac{1}{2}$ *and* $\frac{3}{q} = 1 - 2\alpha$ *such that* $\frac{1}{2}(\frac{3}{q} + 1) = 1 - \alpha$. *Choose any* $T \in (0, \infty)$.

   (i) *The real interpolation space* $\left(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\right)_{1-\alpha,\infty}$ *coincides with the space of Besov-type* $\mathbb{B}^{-1+3/q}_{q,\infty}(\Omega)$ *and has the equivalent norm*

$$\sup_{(0,T)} \tau^\alpha \|e^{-\tau A_q}u\|_q.$$

   (ii) *The interpolation space* $\left(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\right)^0_{1-\alpha,\infty}$ *equals the Besov space* $\overset{\circ}{\mathbb{B}}{}^{-1+3/q}_{q,\infty}(\Omega)$, *equipped with the equivalent norm* $\sup_{(0,T)} \tau^\alpha\|e^{-\tau A_q}u\|_q$, *such that additionally the property* $\lim_{\tau \to 0} \tau^\alpha\|e^{-\tau A_q}u\|_q = 0$ *holds for* $u \in \overset{\circ}{\mathbb{B}}{}^{-1+3/q}_{q,\infty}(\Omega)$.

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Reinhard Farwig
Fachbereich Mathematik
Technische Universität Darmstadt
64289 Darmstadt, Germany
E-mail address: farwig@mathematik.tu-darmstadt.de

Yoshikazu Giga
Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914, Japan
E-mail address: labgiga@ms.u-tokyo.ac.jp

Pen-Yuan Hsu
Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914, Japan
E-mail address: pyhsu@ms.u-tokyo.ac.jp