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ALMOST EINSTEIN MANIFOLDS WITH CIRCULANT STRUCTURES

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Abstract. We consider a 3-dimensional Riemannian manifold M with a circulant metric g and a circulant structure q satisfying $q^3 = \mathrm{id}$. The structure q is compatible with g such that an isometry is induced in any tangent space of M. We introduce three classes of such manifolds. Two of them are determined by special properties of the curvature tensor. The third class is composed by manifolds whose structure q is parallel with respect to the Levi-Civita connection of g. We obtain some curvature properties of these manifolds (M, g, q) and give some explicit examples of such manifolds.

1. Introduction

The Riemannian manifolds with additional structures are among the most studied types of manifolds in differential geometry. For example, we will refer to the theory of Riemannian almost product manifolds and to the theory of almost Hermitian manifolds. A. Naveira gave a classification of Riemannian almost product manifolds. It was made by the properties of the tensor ∇P, where ∇ is the Levi-Civita connection determined by the metric and P is the almost product structure ([11]). The class W_0 defined by $\nabla P = 0$ in this classification is common to all classes. Every manifold in this class satisfies the curvature identity $R(x, y, Pz, Pu) = R(x, y, z, u)$, which implies $R(Px, Py, Pz, Pu) = R(x, y, z, u)$. In this vein, almost Hermitian manifolds were classified by A. Gray and L. Hervella ([7]). Due to Gray, in these classes curvature identities are a key to understand their geometry. Substantial results in the geometry of Riemannian manifolds with additional structures are associated with the curvature tensor, the Ricci tensor, the scalar curvatures, the

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Ricci curvature and sectional curvatures of some characteristic 2-planes of the tangent space of the manifolds (for instance [1], [2], [6], [9], [10]).

The main aim of the present paper is to continue the investigations in [5] and [4], and to find more geometric properties of a 3-dimensional Riemannian manifold M with a circulant metric g and an additional circulant structure q with $q^3 = \mathrm{id}$, i.e., a manifold (M, g, q).

The paper is organized as follows. In Section 2, we define a Riemannian manifold (M, g, q) equipped with a circulant metric g and an endomorphism q whose third power is the identity. Moreover, we assume that the local coordinates of q form a circulant matrix. Then q is compatible with g such that an isometry is induced in any tangent space of M. We recall necessary facts about such manifolds and we consider three classes $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$ ($\mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_2$) of manifolds (M, g, q) that are of interest to our further studies. In Section 3, we obtain conditions for the Ricci tensor ρ, which are necessary and sufficient for belonging of (M, g, q) to $\mathcal{L}_2$. In both classes $\mathcal{L}_2$ and $\mathcal{L}_1$, we express the Ricci tensor ρ by the metric g and the structure q and establish that (M, g, q) is an almost Einstein manifold. In Section 4, we are interested in the sectional curvatures of some characteristic 2-planes in $\mathcal{L}_2$, also in $\mathcal{L}_1$. In both classes we find the Ricci curvature in the direction of a non-zero vector x. For a manifold $(M, g, q) \in \mathcal{L}_0$ we obtain a partial differential equation for the scalar curvature and a necessary and sufficient condition for conformal flatness. In Section 5, we construct explicit examples of the considered manifolds (M, g, q).

2. Preliminaries

Let M be a 3-dimensional manifold equipped with a Riemannian metric g. We assume that the metric g at a point $p(X^1, X^2, X^3) \in M$ has the following matrix form

$$(g_{ij}) = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}, \ A > B > 0, \tag{1}$$

where A and B are smooth functions of X^1, X^2, X^3. Then the metric g is positive definite.

Let q be an endomorphism in the tangent space T_pM, whose coordinate matrix with respect to a basis $\{e_1, e_2, e_3\}$ of T_pM is

$$(q_i^j) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2}$$

Obviously

$$q^3 = \mathrm{id},$$

and the structure q is an isometry with respect to g, i.e.,

$$g(qx, qy) = g(x, y). \tag{3}$$

In (3) and further, x, y, z, u will stand for arbitrary elements of the algebra on the smooth vector fields on M or vectors in the tangent space T_pM. The Einstein summation convention is used, the range of the summation indices being always $\{1, 2, 3\}$.

We denote by (M, g, q) the manifold M equipped with the Riemannian metric g and the structure q, defined by (1) and (2).

Let us remark that the matrix (2) generates the commutative algebra `Circ(3)` (over $\mathbb{R}$) of all 3×3 real circulant matrices ([8]).

The Levi-Civita connection on a Riemannian manifold is denoted by ∇. The curvature tensor R of ∇ is defined by

$$R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z.$$

Also, we consider the tensor of type $(0, 4)$ associated with R, defined as follows

$$R(x, y, z, u) = g(R(x, y)z, u).$$

We say that a manifold (M, g, q) is in class $\mathcal{L}_0$ if the structure q is parallel with respect to g, i.e.,

$$\nabla q = 0. \tag{4}$$

We say that a manifold (M, g, q) is in class $\mathcal{L}_1$ if

$$R(x, y, qz, qu) = R(x, y, z, u). \tag{5}$$

We say that a manifold (M, g, q) is in class $\mathcal{L}_2$ if

$$R(qx, qy, qz, qu) = R(x, y, z, u). \tag{6}$$

It is easy to see that $\mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_2$ are valid.

In [5] it is proved that $(M, g, q) \in \mathcal{L}_0$ if and only if the gradients of the functions A and B satisfy the following equality

$$\mathrm{grad} A = \mathrm{grad} B \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}. \tag{7}$$

Let R_{ijkh} be the components of the curvature tensor R of type $(0, 4)$.

The local form of (5) is $R_{ijlm} q_k^l q_h^m = R_{ijkh}$. Then, using (2), we find

$$R_{1212} = R_{1223} = R_{1231}, \ R_{1313} = R_{1321} = R_{1332}, \ R_{2323} = R_{2331} = R_{2312},$$

which implies

$$R_{1212} = R_{1313} = R_{2323} = -R_{1213} = -R_{1323} = R_{1223}. \tag{8}$$

Vice versa, from (2) and (8) it follows (5).

Hence we arrive at the following:

Proposition 2.1. *The property* (5) *of the manifold* (M, g, q) *is equivalent to the conditions* (8).

In [4] it is proved:

Proposition 2.2. *The property* (6) *of the manifold* (M, g, q) *is equivalent to the conditions*

$$R_{1212} = R_{1313} = R_{2323}, \quad R_{1213} = R_{1323} = -R_{1223}. \tag{9}$$

Definition. A basis of type $\{x, qx, q^2x\}$ of T_pM is called a *q-basis.* In this case we say that *the vector x induces a q-basis of T_pM.* Similarly, a basis $\{x, qx\}$ of a 2-plane $\alpha = \{x, qx\}$ is called a *q-basis.*

In [5], for (M, g, q) it is verified that

(i) A vector $x = (x^1, x^2, x^3)$ induces a q-basis of T_pM if and only if

$$3x^1x^2x^3 \neq (x^1)^3 + (x^2)^3 + (x^3)^3;$$

(ii) If a vector x induces a q-basis of T_pM and $\varphi = \angle(x, qx)$, then

$$\varphi \in \big(0, \frac{2\pi}{3}\big), \qquad \angle(x, qx) = \angle(qx, q^2x) = \angle(x, q^2x) = \varphi;$$

(iii) An orthogonal q-basis of T_pM exists.

3. Almost Einstein manifolds

We consider the associated metric f with g on (M, g, q) determined by

$$f(x, y) = g(x, qy) + g(qx, y). \tag{10}$$

It is an indefinite metric, whose matrix of components is

$$(f_{ij}) = \begin{pmatrix} 2B & A+B & A+B \\ A+B & 2B & A+B \\ A+B & A+B & 2B \end{pmatrix}. \tag{11}$$

The inverse matrices of (g_{ij}) and (f_{ij}) are as follows

$$(g^{ij}) = \frac{1}{D}\begin{pmatrix} A+B & -B & -B \\ -B & A+B & -B \\ -B & -B & A+B \end{pmatrix}, \tag{12}$$

$$(f^{ij}) = \frac{1}{2D}\begin{pmatrix} -A-3B & A+B & A+B \\ A+B & -A-3B & A+B \\ A+B & A+B & -A-3B \end{pmatrix}, \tag{13}$$

where $D = (A - B)(A + 2B)$.

The Ricci tensor ρ and the scalar curvature τ with respect to g are given by the well-known formulas:

$$\rho(y, z) = g^{ij}R(e_i, y, z, e_j), \ \ \tau = g^{ij}\rho(e_i, e_j). \tag{14}$$

Their associated quantities are determined by

$$\rho^*(y, z) = f^{ij}R(e_i, y, z, e_j), \ \ \tau^* = f^{ij}\rho(e_i, e_j). \tag{15}$$

A Riemannian manifold is said to be Einstein if its Ricci tensor ρ is a constant multiple of the metric tensor g, i.e.,

$$\rho(x,y) = \alpha g(x,y). \tag{16}$$

In [13], for locally decomposable Riemannian manifolds is defined a class of almost Einstein manifolds.

For the considered in our paper manifolds, we give the following:

Definition. A Riemannian manifold (M,g,q) is called almost Einstein if the metrics g and f satisfy

$$\rho(x,y) = \alpha g(x,y) + \beta f(x,y), \tag{17}$$

where α and β are smooth functions on M.

3.1. The case $(M,g,q) \in \mathcal{L}_2$

Theorem 3.1. *A manifold (M,g,q) belongs to $\mathcal{L}_2$ if and only if the components of the Ricci tensor ρ are*

$$\rho_{11} = \rho_{22} = \rho_{33}, \quad \rho_{12} = \rho_{13} = \rho_{23}. \tag{18}$$

Proof. Let $(M,g,q) \in \mathcal{L}_2$. Consequently, the components of the curvature tensor R satisfy (9). For brevity, we denote

$$R_1 = R_{1212}, \ R_2 = R_{1213}. \tag{19}$$

Then, having in mind (9), (12) and (14), we get the components of ρ, as follows:

$$\begin{aligned} \rho_{11} = \rho_{22} = \rho_{33} &= \frac{2}{D}\big(-(A+B)R_1 + BR_2\big), \\ \rho_{12} = \rho_{13} = \rho_{23} &= -\frac{1}{D}\big(BR_1 + (A+3B)R_2\big), \end{aligned} \tag{20}$$

i.e., (18).

Vice versa, let the components of the Ricci tensor ρ of (M,g,q) satisfy (18). It is known that the curvature tensor R for a 3-dimensional Riemannian manifold is completely determined by the Ricci tensor ρ and the metric g, as follows ([12])

$$\begin{aligned} R(x,y,z,u) = &-g(x,z)\rho(y,u) - g(y,u)\rho(x,z) + g(y,z)\rho(x,u) \\ &+ g(x,u)\rho(y,z) + \frac{\tau}{2}\big(g(x,z)g(y,u) - g(y,z)g(x,u)\big), \end{aligned} \tag{21}$$

which in a local form is

$$R_{ijkl} = -g_{ik}\rho_{jl} - g_{jl}\rho_{ik} + g_{jk}\rho_{il} + g_{il}\rho_{jk} + \frac{\tau}{2}\big(g_{ik}g_{jl} - g_{jk}g_{il}\big).$$

By straightforward computation, for (M,g,q) we get

$$\begin{aligned} R_{1212} &= -A(\rho_{11}+\rho_{22}) + 2B\rho_{12} + \frac{\tau}{2}(A^2-B^2), \\ R_{1313} &= -A(\rho_{11}+\rho_{33}) + 2B\rho_{13} + \frac{\tau}{2}(A^2-B^2), \end{aligned}$$

$$\begin{aligned}
R_{2323} &= -A(\rho_{22}+\rho_{33}) + 2B\rho_{23} + \frac{\tau}{2}(A^2 - B^2),\\
R_{1213} &= -A\rho_{23} + B(\rho_{12}+\rho_{13}-\rho_{11}) + \frac{\tau}{2}B(A-B),\\
R_{1223} &= A\rho_{13} + B(\rho_{22}-\rho_{12}-\rho_{23}) - \frac{\tau}{2}B(A-B),\\
R_{1323} &= -A\rho_{12} + B(\rho_{13}+\rho_{23}-\rho_{33}) + \frac{\tau}{2}B(A-B).
\end{aligned}$$

We substitute (18) in the above equalities and obtain (9), i.e., (M, g, q) is in $\mathcal{L}_2$. □

Theorem 3.2. *A manifold (M, g, q) belongs to $\mathcal{L}_2$ if and only if (M, g, q) is an almost Einstein manifold.*

Proof. Let $(M, g, q) \in \mathcal{L}_2$. From (14) and (15), using (12), (13) and (18), we get the values of the scalar curvatures τ and τ^*, as follows:

$$\tau = \frac{3}{D}\big((A+B)\rho_{11} - 2B\rho_{12}\big), \quad \tau^* = \frac{3}{2D}\big(-(A+3B)\rho_{11} + 2(A+B)\rho_{12}\big). \tag{22}$$

Immediately from (22) we have

$$\rho_{11} = \frac{\tau}{3}(A+B) + \frac{2\tau^*}{3}B, \quad \rho_{12} = \frac{\tau}{6}(A+3B) + \frac{\tau^*}{3}(A+B),$$

and due to (1) and (11) we get

$$\rho_{11} = \frac{\tau}{3}g_{11} + \big(\frac{\tau}{6} + \frac{\tau^*}{3}\big)f_{11}, \quad \rho_{12} = \frac{\tau}{3}g_{12} + \big(\frac{\tau}{6} + \frac{\tau^*}{3}\big)f_{12}.$$

Hence, taking into account (1), (11) and (18), we obtain

$$\rho_{ij} = \frac{\tau}{3}g_{ij} + \big(\frac{\tau}{6} + \frac{\tau^*}{3}\big)f_{ij},$$

i.e.,

$$\rho(x, y) = \frac{\tau}{3}g(x, y) + \big(\frac{\tau}{6} + \frac{\tau^*}{3}\big)f(x, y). \tag{23}$$

Therefore, according to (17), the manifold (M, g, q) is almost Einstein.

Inversely, let (M, g, q) be an almost Einstein manifold. Using (1), (11) and (17) we get (18), i.e., $(M, g, q) \in \mathcal{L}_2$. □

Obviously, from (21) and (23) it follows:

Theorem 3.3. *Let $(M, g, q) \in \mathcal{L}_2$. Then*

$$R = \frac{\tau}{6}\pi_1 + \big(\frac{\tau}{6} + \frac{\tau^*}{3}\big)\pi_2, \tag{24}$$

where

$$\begin{aligned}
\pi_1(x, y, z, u) &= g(y, z)g(x, u) - g(x, z)g(y, u),\\
\pi_2(x, y, z, u) &= g(y, z)f(x, u) + g(x, u)f(y, z)\\
&\quad - g(x, z)f(y, u) - g(y, u)f(x, z).
\end{aligned} \tag{25}$$

Now we suppose that (M, g, q) is an Einstein manifold. From (1) and (16) we obtain (18).

Consequently, we establish the following:

Proposition 3.4. *Every Einstein manifold* (M, g, q) *belongs to* $\mathcal{L}_2$.

Let $(M, g, q) \in \mathcal{L}_2$. By using (9), (13) and (15), we calculate the components of the Ricci tensor ρ^*:

$$\begin{aligned} \rho^*_{11} = \rho^*_{22} = \rho^*_{33} &= \frac{1}{D}\big((A+3B)R_1 - (A+B)R_2\big), \\ \rho^*_{12} = \rho^*_{13} = \rho^*_{23} &= \frac{1}{2D}\big((A+B)R_1 + (3A+5B)R_2\big). \end{aligned} \tag{26}$$

For brevity, we denote

$$\rho_1 = \rho_{11},\ \ \rho_2 = \rho_{12},\ \ \rho^*_1 = \rho^*_{11},\ \ \rho^*_2 = \rho^*_{12}. \tag{27}$$

Let (M, g, q) be an Einstein manifold. Then $(M, g, q) \in \mathcal{L}_2$ and its Ricci tensor ρ satisfies (23). We compare (16) and (23), and bearing in mind (1) and (11), we get the following system for $3\alpha - \tau$ and $\tau^* + \frac{\tau}{2}$:

$$B\big(3\alpha - \tau\big) - (A+B)\big(\tau^* + \frac{\tau}{2}\big) = 0, \quad A\big(3\alpha - \tau\big) - 2B\big(\tau^* + \frac{\tau}{2}\big) = 0.$$

The determinant of the system is $D = (A-B)(A+2B) \neq 0$. Hence its only solution is $3\alpha - \tau = 0$ and

$$\tau^* + \frac{\tau}{2} = 0. \tag{28}$$

In this case (22) and (27) imply

$$B\rho_1 = A\rho_2. \tag{29}$$

From (20), (27) and (29) we get

$$(A+B)R_2 = BR_1. \tag{30}$$

We substitute the latter equality in (20) and also in (26). Thus, having in mind (27), we obtain

$$\rho^*_1 = -\frac{A+B}{2A}\rho_1, \quad \rho^*_2 = -\frac{A+3B}{4B}\rho_2. \tag{31}$$

Inversely, let (M, g, q) be in $\mathcal{L}_2$ and the components of the Ricci tensors ρ and ρ^* satisfy (31). Then, from (20), (26) and (27), we obtain (30). We substitute (30) in (20) and find (29). Further, from (22) and (29) we have (28). Hence (23) implies $\rho(x, y) = \frac{\tau}{3} g(x, y)$, i.e., (M, g, q) is an Einstein manifold.

Therefore, we arrive at the following:

Theorem 3.5. *Let* $(M, g, q) \in \mathcal{L}_2$. *Then the following propositions are equivalent*:

(i) (M, g, q) *is an Einstein manifold*;
(ii) *The scalar curvatures* τ *and* τ^* *satisfy* (28);
(iii) *The components of the Ricci tensor* ρ *satisfy* (29);

(iv) *The components of the curvature tensor* R *satisfy* (30);
(v) *The components of the Ricci tensors* ρ *and* ρ^* *satisfy* (31).

3.2. The case $(M, g, q) \in \mathcal{L}_1$

Theorem 3.6. *Let* $(M, g, q) \in \mathcal{L}_2$. *Then the following propositions are equivalent*:

(i) $(M, g, q) \in \mathcal{L}_1$;
(ii) *The components of the Ricci tensor* ρ *satisfy* $\rho_1 = -2\rho_2$;
(iii) *The scalar curvatures* τ *and* τ^* *satisfy* $\tau^* = -\tau$;
(iv) *The components of the Ricci tensors* ρ *and* ρ^* *satisfy* $\rho_1^* = -\rho_1$, $\rho_2^* = -\rho_2$.

Proof. Let $(M, g, q) \in \mathcal{L}_1$. Thus (8) and (19) imply $R_1 = -R_2$. Then, from (20), (26) and (27), we obtain the components of the Ricci tensors ρ and ρ^*, as follows:

$$\begin{aligned} \rho_1 &= -\frac{2}{A-B}R_1, \quad \rho_2 = \frac{1}{A-B}R_1, \\ \rho_1^* &= \frac{2}{A-B}R_1, \quad \rho_2^* = -\frac{1}{A-B}R_1. \end{aligned} \tag{32}$$

Substituting $\rho_1 = -2\rho_2$ into (22), we find the values of the scalar curvatures

$$\tau^* = -\tau, \qquad \tau = \frac{3}{A-B}\rho_1. \tag{33}$$

Vice versa, if $(M, g, q) \in \mathcal{L}_2$ and

a) $\tau^* = -\tau$, then (22) and (27) imply $\rho_1 = -2\rho_2$. From the latter equality and (20) it follows $R_1 = -R_2$, so $(M, g, q) \in \mathcal{L}_1$.

b) $\rho_1^* = -\rho_1$, $\rho_2^* = -\rho_2$, then bearing in mind (20), (26) and (27) we get $R_1 = -R_2$, i.e., $(M, g, q) \in \mathcal{L}_1$. □

According to (23) and the first equality of (33) we have the following:

Theorem 3.7. *Let* $(M, g, q) \in \mathcal{L}_1$. *Then*

$$\rho(x, y) = \frac{\tau}{6}\big(2g(x, y) - f(x, y)\big). \tag{34}$$

Immediately from (1), (11) and (34) we obtain:

Corollary 3.8. *The Ricci tensor* ρ *of a manifold* $(M, g, q) \in \mathcal{L}_1$ *is degenerate.*

Remark 3.9. A manifold $(M, g, q) \in \mathcal{L}_1$ doesn't admit Einstein metric.

Theorem 3.10. *Let* $(M, g, q) \in \mathcal{L}_1$. *Then*

$$R = \frac{\tau}{6}(\pi_1 - \pi_2), \tag{35}$$

where π_1 *and* π_2 *are determined by* (25).

Proof. For $(M, g, q) \in \mathcal{L}_1$ the conditions of Theorem 3.3 are valid. Hence we apply the first equality of (33) in (24) and we get (35). □

4. Some curvature properties

The sectional curvature of a non-degenerate 2-plane $\{x, y\}$ spanned by the vectors $x, y \in T_pM$ is the value ([13])

$$\mu(x, y) = \frac{R(x, y, x, y)}{g(x, x)g(y, y) - g^2(x, y)}. \tag{36}$$

Let x induce a q-basis of T_pM for (M, q, g) and $\sigma = \{x, qx\}$ be a 2-plane. Evidently, if $y \in \sigma$ and $y \neq x$, then $qy \notin \sigma$. Consequently, σ has only two q-bases: $\{x, qx\}$ and $\{-x, -qx\}$. That's why the sectional curvature $\mu(x, qx)$ depends only on $\varphi = \angle(x, qx)$.

In [4] it is proved that, if $(M, g, q) \in \mathcal{L}_2$ and x induces a q-basis, then $\mu(x, qx) = \mu(x, q^2x) = \mu(qx, q^2x)$.

Now we recall that the Ricci curvature in the direction of a non-zero vector x is the value

$$r(x) = \frac{\rho(x, x)}{g(x, x)}. \tag{37}$$

4.1. The case $(M, q, g) \in \mathcal{L}_2$

Theorem 4.1. *Let $(M, g, q) \in \mathcal{L}_2$ and a vector x induce a q-basis. Then*

$$\mu(x, qx) = \mu(x, q^2x) = \mu(qx, q^2x) = -\frac{\tau(1 + 3\cos\varphi) + 4\tau^* \cos\varphi}{6(1 + \cos\varphi)}\ , \tag{38}$$

where $\varphi = \angle(x, qx)$.

Proof. Let a vector x induce a q-basis. In [5], for (M, g, q) it is verified that $g(x, qx) = g(x, q^2x) = g(qx, q^2x) = g(x, x)\cos\varphi$, where $\varphi \in (0, \frac{2\pi}{3})$. From (3) and (10) we find $f(x, x) = 2g(x, qx)$, $f(x, qx) = g(x, x) + g(x, q^2x)$ and $f(x, q^2x) = g(x, x) + g(x, qx)$. Then (24), (25) and (36) imply (38). □

Corollary 4.2. *Let $(M, g, q) \in \mathcal{L}_2$ and a vector x induce an orthonormal q-basis. Then*

$$\mu(x, qx) = \mu(x, q^2x) = \mu(qx, q^2x) = -\frac{\tau}{6}.$$

Theorem 4.3. *Let $(M, g, q) \in \mathcal{L}_2$ and a vector x induce a q-basis. Then*

$$r(x) = r(qx) = r(q^2x) = \frac{\tau}{3}(1 + \cos\varphi) + \frac{2\tau^*}{3}\cos\varphi, \tag{39}$$

where $\varphi = \angle(x, qx)$.

Proof. Let $(M, g, q) \in \mathcal{L}_2$. According to Theorem 3.2, the Ricci tensor ρ is given by (23). Then, using (3) and (10), we find

$$\rho(x, x) = \rho(qx, qx) = \rho(q^2x, q^2x) = \frac{\tau}{3}g(x, x) + \big(\frac{\tau}{6} + \frac{\tau^*}{3}\big)f(x, x). \tag{40}$$

Let a vector x induce a q-basis. From (10), (37) and (40) it follows (39). □

4.2. The case $(M,g,q) \in \mathcal{L}_1$

If $(M,g,q) \in \mathcal{L}_1$, then (38) is valid. According to Theorem 3.6, we have $\tau^* = -\tau$. In this case we get the following:

Theorem 4.4. *Let $(M,g,q) \in \mathcal{L}_1$ and a vector x induce a q-basis. Then*

$$\mu(x,qx) = \mu(x,q^2x) = \mu(qx,q^2x) = -\frac{\tau}{6}\tan^2\frac{\varphi}{2}\ , \tag{41}$$

where $\varphi = \angle(x,qx)$.

Now we substitute $\tau^* = -\tau$ in (39) and obtain:

Theorem 4.5. *Let $(M,g,q) \in \mathcal{L}_1$ and a vector x induce a q-basis. Then*

$$r(x) = r(qx) = r(q^2x) = \frac{\tau}{3}(1-\cos\varphi), \tag{42}$$

where $\varphi = \angle(x,qx)$.

4.3. The case $(M,g,q) \in \mathcal{L}_0$

Proposition 4.6. *Let $(M,g,q) \in \mathcal{L}_0$. Then for the scalar curvature τ we have*

$$\tau_1 + \tau_2 + \tau_3 = 0, \tag{43}$$

where $\tau_i = \frac{\partial\tau}{\partial X^i}$.

Proof. It is known that in a Riemannian manifold for the scalar curvature τ and the Ricci tensor ρ it is valid

$$\nabla_i\rho^i_k = \frac{1}{2}\nabla_k\tau, \quad \rho^i_k = \rho_{ak}g^{ai}. \tag{44}$$

For $(M,g,q) \in \mathcal{L}_0$ the Ricci tensor has the expression (34), which in a local form is

$$\rho_{ij} = \frac{\tau}{6}\big(2g_{ij} - f_{ij}\big). \tag{45}$$

From (4) and (10) we have

$$\nabla f = 0. \tag{46}$$

Using (44)–(46), we obtain

$$\tau_k = \frac{\tau_a}{3}\big(2\delta^a_k - f_{kb}g^{ab}\big),$$

where δ^a_k are the Kronecker symbols. Then, from (11) and (12), we get (43). □

Theorem 4.7. *A manifold $(M,g,q) \in \mathcal{L}_0$ is conformally flat if and only if the scalar curvature τ is a constant.*

Proof. It is known that the Cotton tensor C for a Riemannian manifold is defined in the following way ([3]):

$$C_{ijk} = \nabla_k\rho_{ij} - \nabla_j\rho_{ik} + \frac{1}{4}\big(\nabla_k\tau g_{ij} - \nabla_j\tau g_{ik}\big). \tag{47}$$

In 3-dimensional manifolds the Cotton tensor is prominent as the substitute for the Weyl tensor. It is conformally invariant and its vanishing is equivalent to conformal flatness.

For $(M,g,q)\in\mathcal{L}_0$, from (45)–(47), we get

$$C_{ijk}=\frac{1}{6}\big(\tau_k g_{ij}-\tau_j g_{ik}+2\tau_j f_{ik}-2\tau_k f_{ij}\big),\quad \tau_i=\frac{\partial\tau}{\partial X^i}. \tag{48}$$

Let τ be a constant. Hence, according to (48), we have $C_{ijk}=0$. Inversely, let $C_{ijk}=0$. Then from (48) and taking into account (1) and (11), we obtain the following system for $\tau_1,\ \tau_2,\ \tau_3$:

$$(2A+B)\tau_i+(A-4B)\tau_j=0,\quad i\neq j.$$

Since $A>B>0$, then the only solution is $\tau_1=\tau_2=\tau_3=0$. $\square$

5. Examples of manifolds (M,g,q)

Now we give examples in order to demonstrate explicitly that all classes presented are nonempty indeed.

Let (M,g,q) be determined by (1) and (2). We denote

$$A_i=\frac{\partial A}{\partial X^i}\ ,\ B_i=\frac{\partial B}{\partial X^i}\ ,\ A_{ij}=\frac{\partial^2 A}{\partial X^i\partial X^j}\ ,\ B_{ij}=\frac{\partial^2 B}{\partial X^i\partial X^j}.$$

We will use the following:

Theorem 5.1 ([4])**.** *The nonzero components of the curvature tensor R of type $(0,4)$ of the manifold (M,g,q) are*

$$\begin{aligned}
R_{1212}=&\ \frac{1}{2}(2B_{21}-A_{11}-A_{22})\\
&+\frac{A+B}{4D}\Big(2A_3B_2-A_3^2+(B_1-B_2-B_3)(B_1+B_2-B_3)\Big)\\
&-\frac{2B}{4D}\Big((A_1-B_2)(B_1+B_2-B_3)-A_1A_3+A_3B_2\Big),\\
R_{1313}=&\ \frac{1}{2}(2B_{31}-A_{11}-A_{33})\\
&+\frac{A+B}{4D}\Big(2A_2B_3-A_2^2+(-B_1+B_2+B_3)(-B_1+B_2-B_3)\Big)\\
&-\frac{2B}{4D}\Big((A_1-B_3)(B_1-B_2+B_3)-A_1A_2+A_2B_3\Big),\\
R_{2323}=&\ \frac{1}{2}(2B_{23}-A_{22}-A_{33})\\
&+\frac{A+B}{4D}\Big(2B_3A_1-A_1^2+(B_1-B_2+B_3)(B_1-B_2-B_3)\Big)\\
&-\frac{2B}{4D}\Big((A_2-B_3)(B_2-B_1+B_3)-A_1A_2+A_1B_3\Big),\\
R_{1213}=&\ \frac{1}{2}(B_{21}+B_{31}-B_{11}-A_{23})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{A+B}{4D}\Big(A_1(B_2-B_3+B_1)+2B_3(B_3-B_2-B_1)+A_2A_3)\Big)\\
&-\frac{B}{4D}\Big(A_1^2+A_2^2+A_3^2+2A_1(A_2-B_3)-2A_2B_3\\
&-2A_3(B_1-B_3)+(B_1-B_2-B_3)(B_1+B_2-B_3)\Big),
\end{aligned}
$$

$$
\begin{aligned}
R_{1223}=&\frac{1}{2}(B_{22}-B_{12}-B_{23}+A_{13})\\
&+\frac{A+B}{4D}\Big(A_2(B_2+B_3-B_1)-(2B_3-A_1)(2B_2-A_3)\Big)\\
&-\frac{B}{4D}\Big(A_2^2-A_1^2+A_3^2+2A_1(B_2+B_3)+2A_2(B_2-B_3)\\
&+2A_3(B_3-B_1)-4B_2B_3+(B_1+B_2-B_3)(B_1-B_2-B_3)\Big),
\end{aligned}
$$

$$
\begin{aligned}
R_{1323}=&\frac{1}{2}(B_{23}-B_{33}+B_{13}-A_{12})\\
&+\frac{A+B}{4D}\Big((2B_2-A_1)(2B_3-A_2)-A_3(-B_1+B_2+B_3\Big)\\
&-\frac{B}{4D}\Big(A_1^2-A_2^2-A_3^2-2A_1(B_2+B_3)+2A_2(B_1-B_2)\\
&+2A_3(B_2-B_3)+4B_2B_3+(-B_1+B_2+B_3)(B_1-B_2+B_3)\Big),
\end{aligned}
$$

where $D=(A-B)(A+2B)$. The rest of the nonzero components are obtained by the properties

$$R_{ijkh}=R_{khij},\ R_{ijkh}=-R_{jikh}=-R_{ijhk}.$$

5.1. An example in $\mathcal{L}_2$.

We consider a manifold (M,g,q) with

$$A=a\exp(X^1+X^2+X^3),\ B=b\exp(X^1+X^2+X^3),\ a,b\in\mathbb{R}, \tag{49}$$

where $a>b>0$.

Obviously $A>B>0$. We substitute the functions (49) and their derivatives in the equalities of Theorem 5.1 and calculate the components of the curvature tensor R:

$$
\begin{aligned}
&R_{1212}=R_{1313}=R_{2323}=\frac{(b-a)(5a+7b)\exp(X^1+X^2+X^3)}{4(a+2b)}\ ,\\
&R_{1213}=R_{1323}=-R_{1223}=\frac{(b-a)(a+5b)\exp(X^1+X^2+X^3)}{4(a+2b)}.
\end{aligned}
\tag{50}
$$

We check directly that the conditions (9) are valid, but the conditions (8) are not valid.

Therefore, we establish the following:

Theorem 5.2. *The manifold (M,g,q) with (49) belongs to $\mathcal{L}_2$ but doesn't belong to $\mathcal{L}_1$.*

Substituting (50) into (20) and (26), and due to (19) and (27), we obtain the components of the Ricci tensors ρ and ρ^*:

$$\rho_1 = \frac{5a+b}{2(a+2b)}\ , \quad \rho_2 = \frac{a+11b}{4(a+2b)}, \quad \rho_1^* = -1, \quad \rho_2^* = -1. \tag{51}$$

Hence, from (22) and (27) we calculate the values of the scalar curvatures τ and τ^*:

$$\begin{aligned} \tau &= \frac{15}{2(a+2b)\exp\left(X^1+X^2+X^3\right)}\ , \\ \tau^* &= -\frac{3}{(a+2b)\exp\left(X^1+X^2+X^3\right)}. \end{aligned} \tag{52}$$

Let a vector x induce a q-basis and $\varphi = \angle(x, qx)$. Then, from (38) and (52), we get

$$\mu(x, qx) = -\frac{5+7\cos\varphi}{4(a+2b)(1+\cos\varphi)\exp\left(X^1+X^2+X^3\right)}. \tag{53}$$

Using (39) and (52) we find

$$r(x) = \frac{5+\cos\varphi}{2(a+2b)\exp\left(X^1+X^2+X^3\right)}. \tag{54}$$

Therefore, we arrive at the following:

Proposition 5.3. *For the manifold* (M, g, q) *with* (49)*, the following assertions are valid*:

(i) *The components of the Ricci tensors* ρ *and* ρ^* *are* (51);
(ii) *The scalar curvatures* τ *and* τ^* *are* (52);
(iii) *The sectional curvatures of the* 2*-planes* $\{x, qx\}$, $\{x, q^2x\}$, $\{qx, q^2x\}$ *are* (53);
(iv) *The Ricci curvature* $r(x)$ *in the direction of a non-zero vector* x *is* (54).

Corollary 5.4. *For the manifold* (M, g, q) *with* (49)*, the Ricci curvature satisfies the inequalities*:

$$\frac{9}{4(a+2b)\exp\left(X^1+X^2+X^3\right)} < r(x) < \frac{3}{(a+2b)\exp\left(X^1+X^2+X^3\right)}.$$

The proof follows immediately from the conditions $\varphi \in \left(0, \frac{2\pi}{3}\right)$ and (54).

5.2. An example in $\mathcal{L}_1$

Let (M, g, q) be a manifold with

$$A = a(X^1+X^2+X^3),\ B = b(X^1+X^2+X^3),\ a, b \in \mathbb{R}, \tag{55}$$

where

$$X^1+X^2+X^3 > 0, \qquad a > b > 0.$$

Evidently $A > B > 0$. According to Theorem 5.1 and equalities (55) we obtain

$$\begin{aligned} &R_{1212} = R_{1313} = R_{2323} \\ &\qquad = R_{1223} = -R_{1213} = -R_{1323} = \frac{-(a-b)^2}{4(a+2b)(X^1+X^2+X^3)}. \end{aligned} \tag{56}$$

We check directly that the conditions (8) are valid, but the conditions (7) for the functions (55) are not valid.

Thus, we have the following:

Theorem 5.5. *The manifold* (M, g, q) *with* (55) *belongs to* $\mathcal{L}_1$ *but doesn't belong to* $\mathcal{L}_0$.

From (19), (32), (33) and (56), we obtain the components of the Ricci tensor ρ and the value of the scalar curvature τ, as follows:

$$\rho_1 = \frac{(a-b)}{2(a+2b)(X^1+X^2+X^3)^2}\ ,\ \rho_2 = -\frac{1}{2}\rho_1, \tag{57}$$

$$\tau = \frac{3}{2(a+2b)(X^1+X^2+X^3)^3}. \tag{58}$$

Let a vector x induce a q-basis and $\varphi = \angle(x, qx)$. From (41) and (58) we get

$$\mu(x, qx) = -\frac{\tan^2\frac{\varphi}{2}}{4(X^1+X^2+X^3)^3(a+2b)}. \tag{59}$$

Due to (42) and (58) we find

$$r(x) = \frac{1-\cos\varphi}{2(a+2b)(X^1+X^2+X^3)^3}. \tag{60}$$

Therefore, we establish the following:

Proposition 5.6. *For the manifold* (M, g, q) *with* (55), *the following assertions are valid:*

(i) *The components of the Ricci tensor* ρ *are* (57);
(ii) *The scalar curvature* τ *is* (58);
(iii) *The sectional curvatures of the* 2*-planes* $\{x, qx\}$, $\{x, q^2x\}$, $\{qx, q^2x\}$ *are* (59);
(iv) *The Ricci curvature* $r(x)$ *in the direction of a non-zero vector* x *is* (60).

Corollary 5.7. *For the manifold* (M, g, q) *with* (55), *the Ricci curvature satisfies the inequalities*:

$$0 < r(x) < \frac{3}{4(a+2b)(X^1+X^2+X^3)^3}.$$

Since φ is in the range $\left(0, \frac{2\pi}{3}\right)$ and due to (60) the proof follows.

5.3. An example in $\mathcal{L}_0$

Let (M, g, q) be a manifold with

$$A = (X^1)^2 + (X^2)^2 + (X^3)^2 \ , \ B = X^1X^2 + X^1X^3 + X^2X^3, \tag{61}$$

where

$$X^1X^2 + X^1X^3 + X^2X^3 > 0.$$

We verify that $A > B > 0$ and (7) are valid.

Consequently, we have the following:

Theorem 5.8. *The manifold* (M, g, q) *with* (61) *belongs to* $\mathcal{L}_0$.

According to Theorem 5.1 and (61) we find

$$R_{1212} = R_{1313} = R_{2323} = -R_{1213} = -R_{1323} = R_{1223} = -1.$$

Hence, using (19), (32) and (33), we get the components of the Ricci tensor

$$\rho_1 = \frac{2}{A - B} \ , \quad \rho_2 = -\frac{1}{2}\rho_1, \tag{62}$$

and the value of the scalar curvature

$$\tau = \frac{6}{(A - B)^2}. \tag{63}$$

Let a vector x induce a q-basis and $\varphi = \angle(x, qx)$. From (41) and (63) we obtain

$$\mu(x, qx) = -\frac{\tan^2 \frac{\varphi}{2}}{(A - B)^2}. \tag{64}$$

Due to (42) and (63) we find

$$r(x) = \frac{2(1 - \cos \varphi)}{(A - B)^2}. \tag{65}$$

Therefore, we establish the following:

Proposition 5.9. *For the manifold* (M, g, q) *with* (61)*, the following assertions are valid*:

(i) *The components of the Ricci tensor* ρ *are* (62);
(ii) *The scalar curvature* τ *is* (63);
(iii) *The sectional curvatures of the* 2*-planes* $\{x, qx\}$*,* $\{x, q^2x\}$*,* $\{qx, q^2x\}$ *are* (64);
(iv) *The Ricci curvature* $r(x)$ *in the direction of a non-zero vector* x *is* (65).

Corollary 5.10. *For the manifold* (M, g, q) *with* (61)*, the Ricci curvature satisfies the inequalities*:

$$0 < r(x) < \frac{3}{(A - B)^2}.$$

6. Conclusion

In Section 3, it is verified that every Einstein manifold (M, g, q) belongs to $\mathcal{L}_2$. Also are obtained necessary and sufficient conditions for (M, g, q) to be an Einstein manifold.

Further, it remains the problem of obtaining of explicit examples of 3-dimensional Einstein manifolds (M, g, q).

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