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HILBERT FUNCTIONS OF STANDARD k-ALGEBRAS DEFINED BY SKEW-SYMMETRIZABLE MATRICES

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Abstract. Kang and Ko introduced a skew-symmetrizable matrix to describe a structure theorem for complete intersections of grade 4. Let $R = k[w_0, w_1, w_2, \ldots, w_m]$ be the polynomial ring over an algebraically closed field k with indetermiantes w_l and $\deg w_l = 1$, and I_i a homogeneous perfect ideal of grade 3 with type t_i defined by a skew-symmetrizable matrix $G_i (1 \leq t_i \leq 4)$. We show that for $m = 2$ the Hilbert function of the zero dimensional standard k-algebra R/I_i is determined by CI-sequences and a Gorenstein sequence. As an application of this result we show that for $i = 1, 2, 3$ and for $m = 3$ a Gorenstein sequence $h(R/H_i) = (1, 4, h_2, \ldots, h_s)$ is unimodal, where H_i is the sum of homogeneous perfect ideals I_i and J_i which are geometrically linked by a homogeneous regular sequence z in $I_i \cap J_i$.

1. Introduction

Buchsbaum and Eisenbud [2] gave a structure theorem for Gorenstein ideals of grade 3 which says that every Gorenstein ideal I_0 of grade 3 in a noetherian local ring is minimally generated by the maximal order pfaffians of an alternating matrix G_0. Brown [1] gave a structure theorem for a class of perfect ideals I_1 of grade 3 with type 2 and $\lambda(I_1) > 0$, where λ is the numerical invariant introduced by Kustin and Miller [14] to classify classes of Gorenstein ideals of grade 4 by distinguishing free resolutions of different forms. Kang and Ko [11] described a structure theorem for some class of these ideals (This is a special case of Theorem 4.4 [1]): Every perfect ideal I_1 having an odd number of minimal generators for I_1 is generated by the quotients of the maximal order pfaffians of the alternating matrix $\mathcal{A}(G_1)$ induced by a skew-symmetrizable matrix G_1 by an element v_1 (see Example 3.2 and Theorem 3.3). Cho, Kang and Ko [3] and Choi, Kang and Ko [4, 5] constructed some classes of perfect ideals I_i of grade 3 with type t_i defined by a skew-symmetrizable matrix G_i for $i = 1, 2, 3$ (see Definition 3.1 and Examples 3.2 and 3.4). An ideal in these classes is

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generated by the quotients of the maximal order pfaffians of the alternating matrix $\mathcal{A}(G_i)$ induced by a skew-symmetrizable matrix G_i by an element v_i (see Theorems 3.3 and 3.5). We define a sequence $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_s)$ of nonnegative integers with $h_s \neq 0$ to be a *PI-sequence of type* t_i *defined by a skew-symmetrizable matrix* G_i if $\mathbf{h}$ is the Hilbert function of the zero dimensional standard k-algebra $S = R/I_i$, where I_i is a homogeneous perfect ideal of grade 3 with type t_i defined by G_i, and $t_0 = 1, t_1 = t_4 = 2, t_2 = 3$ and $t_3 = 4$. For $i = 0$ Stanley [17] proved that PI-sequence $\mathbf{h} = (1, 3, h_2, \ldots, h_s)$ of type 1 defined by G_0 is unmodal. He used the Buchsbaum and Eisenbud structure theorem for Gorenstein ideals of grade 3 to prove this. For $i = 1, 2, 3$ we characterize these PI-sequences $\mathbf{h} = (1, 3, h_2, \ldots, h_\sigma)$ of type t_i defined by G_i as follows: Let q_i be the degree of the i-th generators for a Gorenstein ideal of grade 3 corresponding to a Gorenstein sequence $\mathbf{g}$ stated below.

(i) $\mathbf{h}$ is a PI-sequence of type 2 defined by G_1 if and only if there exist a Gorenstein sequence $\mathbf{g} = (1, 3, g_2, \ldots, g_\eta)$ and a CI-sequence $\mathbf{c} = (c_0, c_1, c_2, \ldots, c_{\rho_1})$ having the type (q_1, q_2, τ) with $2 \leq c_1 \leq 3$ and $\sigma = \eta + \tau$ such that $h_\sigma = 1$, and $h_i = c_i$ if $0 \leq i \leq \tau - 1$, and $h_i = g_{i-\tau} + c_i$ if $\tau \leq i \leq \sigma$ (Theorem 4.3). The Hilbert function in Example 4.4 is in this class.

(ii) If $\mathbf{h}$ is a PI-sequence of type 3 defined by G_2, then there exist a Gorenstein sequence $\mathbf{g} = (1, 3, g_2, \ldots, g_\eta)$ and two CI-sequences $\mathbf{c} = (c_0, c_1, c_2, \ldots, c_{\rho_1})$ having the type $(\tau, q_2 + \kappa, q_3)$ and $\hat{\mathbf{c}} = (\hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{\hat{\rho}_1})$ having the type (κ, q_1, q_3) with $2 \leq c_1, \hat{c}_1 \leq 3$ and $\sigma = \eta + \tau + \kappa$ such that $h_\sigma = 1$, and $h_i = c_i$ if $0 \leq i \leq \tau - 1$, $h_i = c_i + \hat{c}_{i-\tau}$ if $\tau \leq i < \tau + \kappa$ and $h_i = g_{i-\tau-\kappa} + c_i + \hat{c}_{i-\tau}$ if $\tau + \kappa \leq i \leq \sigma$ (Theorem 4.6). The Hilbert function in Example 4.7 is in this class.

(iii) If $\mathbf{h}$ is a PI-sequence of type 4 defined by G_3, then there exist a Gorenstein sequence $\mathbf{g} = (1, 3, g_2, \ldots, g_\eta)$ and two CI-sequences $\mathbf{c} = (c_0, c_1, c_2, \ldots, c_{\rho_1})$ having the type $(\tau + q_1, \kappa + q_2, \nu + q_3)$ and $\hat{\mathbf{c}} = (\hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{\hat{\rho}_1})$ having the type (q_1, q_2, q_3) with $2 \leq c_1, \hat{c}_1 \leq 3$ and $\sigma = \eta + \tau + \kappa + \nu$ such that $h_\sigma = 1$, and $h_i = c_i$ if $0 \leq i \leq \tau + \kappa + \nu - 1$ and $h_i = g_{i-\tau-\kappa-\nu} + c_i - \hat{c}_{i-\tau-\kappa-\nu}$ if $\tau + \kappa + \nu \leq i \leq \sigma$ (Theorem 4.8). The Hilbert function in Example 4.9 is in this class.

We use Theorems 4.4 [1] or 3.3 to prove (i) and Theorems 3.6 [4] and 3.11 [5] to show (ii) and (iii). We use these results (Theorems 4.3, 4.6 and 4.8), Proposition 5.1 and Lemmas 5.2, 5.3, 5.4 to prove that every Gorenstein sequence $h(R/H_i)$ mentioned in the abstract is unimodal for $i = 1, 2, 3$ (Theorem 5.5). Let $\mathcal{G}_p(4)$ be the set of Gorenstein sequences $h(R/H) = (1, 4, h_2, \ldots, h_s)$, where H is the sum of homogeneous perfect ideals I and J of grade 3 geometrically linked by a homogeneous regular sequence z. There exist many examples of unimodal Gorenstein sequences $h(R/H) = (1, 4, \ldots, h_s)$ in $\mathcal{G}_p(4)$ [7, 8, 13, 15, 16]. We use Proposition 5.10 to show that if a Gorenstein sequence $h(R/H)$ in $\mathcal{G}_p(4)$

falls into one of the following three cases, then $h(R/H)$ is unimodal (Corollary 5.12): Let $\sigma = \sigma(R/(z)), \sigma^* = \sigma(R/I)$ and $\sigma - \sigma^* = \alpha^*$.

(p) $\sigma^* \leq [(\sigma - 1)/2]$. A Gorenstein sequence $h(R/H)$ in Example 5.13 belongs to this case.

(q) $[(\sigma-1)/2] < \sigma^*$ and $[(\sigma-1)/2] < \alpha^*$. A Gorenstein sequence $h(R/H)$ in Example 5.14 belongs to this case.

(r) $\alpha^* \leq [(\sigma-1)/2] < \sigma^*$ and $\Delta H(R/I, i) - \Delta H(R/I, \sigma - i) \geq 0$ for $i = \alpha^*, \alpha^* + 1, \ldots, [(\sigma-1)/2]$. A Gorenstein sequence $h(R/H)$ in Example 5.15 belongs to this case.

In Section 2 we review the Hilbert functions of the standard k-algebras. In Section 3 for $i = 1, 2, 3$ we review various properties of perfect ideals I_i of grade 3 defined by a skew-symmetrizable matrix G_i in a noetherian local ring. In Section 4 as we have mentioned above we show that the Hilbert function of a zero dimensional standard k-algebra R/I_i expressed as in terms of a Gorenstein sequence and CI-sequences for $i = 1, 2, 3$. We will see that the numerical invariant λ plays a role of distinguishing between PI-sequences $\mathbf{h} = (1, 3, h_2, \ldots, h_s)$ of type 2 defined by G_1 and by G_4 (see Theorem 4.3 and Example 4.5). In Section 5 we give some lemmas and a proposition for the proof of Theorem 5.5 and Corollary 5.12, and some unimodal Gorenstein sequences in $\mathcal{G}_p(4)$.

2. Preliminaries

Let $S = S_0 + S_1 + S_2 + \cdots$ be a standard k-algebra over a field k. Thus $S_0 = k$, S is generated by the elements of S_1 and S_1 is a finite dimensional k-vector space. The Hilbert function of S is defined by $H(S, t) = \dim_k S_t$ for $t = 0, 1, 2, \ldots$. Thus $H(S, 0) = 1$. Define the *Hilbert series* $H_S(\lambda)$ of S to be the formal power series

$$H_S(\lambda) = \sum_{t=0}^{\infty} H(S, t)\lambda^t \in \mathbb{Z}[[\lambda]].$$

As a consequence of the Hilbert syzygy theorem, we can write $H_S(\lambda)$ in the form

$$H_S(\lambda) = \frac{1 + h_1\lambda + h_2\lambda^2 + \cdots + h_s\lambda^s}{(1+\lambda)^d},$$

where d is the Krull dimension of S. We call $h(S) = (1, h_1, h_2, \ldots, h_s)$ *h-sequence*. We put $\sigma(S) = s$. We say that an ideal I is homogeneous if I is generated by homogeneous elements. We observe that if I is homogeneous, then I inherits a grading $I = I_0 + I_1 + \cdots$ from S given by $I_t = I \cap S_t$. We define

$$H(I, t) = \dim_k I_t \ \text{ and } \ H_I(\lambda) = \sum_{t=0}^{\infty} H(I, t)\lambda^t.$$

Similarly, the quotient ring S/I inherits a grading from S, and $H(S/I,t)$ is always defined with respect to this quotient grading. We note that for any homogeneous ideal I of S,

$$H_S(\lambda) = H_I(\lambda) + H_{S/I}(\lambda). \tag{2.1}$$

The following proposition gives us a characterization of the Hilbert functions of d-dimensional standard complete intersection k-algebras.

Proposition 2.1 ([17])**.** *Let R be the polynomial ring mentioned in the abstract. Let $z_1, z_2, \ldots, z_r$ be a homogeneous regular sequence with $\deg z_i = f_i$. Let S be the complete intersection $S = R/(z_1, z_2, \ldots, z_r)$ with the quotient grading. Then*

$$H_S(\lambda) = \frac{\prod_{j=1}^{r}(1-\lambda^{f_j})}{(1-\lambda)^{m+1}}.$$

The following proposition gives us an information on the Hilbert functions of standard Cohen-Macaulay k-algebras.

Proposition 2.2 ([17])**.** *Let $\mathbf{h} = (h_0, h_1, h_2, \ldots)$ be an infinite sequence of nonnegative integers. The following two conditions are equivalent.*

(1) *There exists a d-dimensional standard Cohen-Macaulay k-algebra S with the Hilbert function $\mathbf{h}$, where d is a positive integer and $S_0 = k$.*
(2) *The power series $(1-\lambda)^d \sum_{i=0}^{\infty} h_i\lambda^i$ is a polynomial in λ, say $p_0 + p_1\lambda + \cdots + p_s\lambda^s$. Moreover, $(p_0, p_1, \ldots, p_s)$ is an O-sequence.*

3. Perfect ideals of grade three defined by skew-symmetrizable matrices

Kang and Ko [10] introduced a skew-symmetrizable matrix to describe a structure theorem for complete intersections of grade 4. We review perfect ideals of grade 3 defined by some skew-symmetrizable matrices. We begin this section with the definition of a skew-symmetrizable matrix.

Definition 3.1. Let R be a commutative ring with identity. An $n \times n$ matrix G over R is said to be *skew-symmetrizable* if there exist nonzero diagonal matrices $D' = \mathrm{diag}\{u_1, u_2, \ldots, u_n\}$ and $D = \mathrm{diag}\{v_1, v_2, \ldots, v_n\}$ with entries in R such that $D'GD$ is an alternating matrix.

Let G be an $n \times n$ skew-symmetrizable matrix with entries in R. Then $D'GD$ is an alternating matrix for some diagonal matrices D' and D. We set $\mathcal{A}(G)$ to be an alternating matrix given by

$$\mathcal{A}(G) = \begin{cases} G & \text{if } G \text{ is alternating,} \\ D'GD & \text{if } G \text{ is not alternating.} \end{cases}$$

We denote $\mathcal{A}(G)_i$ by the pfaffian of the $(n-1) \times (n-1)$ alternating submatrix of $\mathcal{A}(G)$ obtained by deleting the i-th row and column from $\mathcal{A}(G)$. Now we give various homogeneous perfect ideals of grade 3 with type t $(1 \le t \le 4)$

associated with some skew-symmetrizable matrices over a commutative ring R with identity. Let $\tilde{G}_0$ be an $n\times n$ alternating matrix for an odd integer $n>1$. Clearly $\tilde{G}_0$ is skew-symmetrizable. So we have $\mathcal{A}(\tilde{G}_0)=\tilde{G}_0$. Let $v_0=1$ and let $\tilde{x}_i$ be an element by

$$\tilde{x}_i=\mathcal{A}(\tilde{G}_0)_i/v_0 \text{ for } i=1,2,3,\ldots,n.$$

We define $I_0=\tilde{I}_0=\overline{\mathrm{Pf}_{n-1}(\tilde{G}_0)}$ to be the ideal generated by n elements $\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n$. Then it follows from Theorem 2.1 [2] that if $\tilde{I}_0=\overline{\mathrm{Pf}_{n-1}(\tilde{G}_0)}=\mathrm{Pf}_{n-1}(\tilde{G}_0)$ has grade 3, then $\tilde{I}_0$ is a Gorenstein ideal of grade 3. The following example gives us skew-symmetrizable matrices which define classes of perfect ideals I of grade 3 with type 2 and $\lambda(I)>0$ [11].

Example 3.2. Let R be a commutative ring with identity and u_1 an element of R. Let n be an odd integer with $n>3$. Let $Y=(y_{ij})$ be an $n\times n$ alternating matrix with $y_{12}=0$ and entries in R. Let A be the submatrix of Y obtained by deleting the first two columns and the last $(n-2)$ rows of Y. We define the $n\times n$ skew-symmetrizable matrix G_1 by

$$G_1=\left[\begin{array}{c|c}\mathbf{0} & u_1A\\ \hline -A^t & Y(1,2)\end{array}\right], \tag{3.1}$$

and $Y(1,2)$ is the $(n-2)\times(n-2)$ alternating submatrix of Y obtained by deleting the first, second rows and columns from Y. The alternating matrix $\mathcal{A}(G_1)$ is obtained by multiplying the first two columns of G_1 by u_1. We note that $\mathcal{A}(G_1)_i$ is divisible by u_1 for every i. Let $v_1=u_1$ and let x_i be an element defined by

$$x_i=\mathcal{A}(G_1)_i/v_1 \text{ for } i=1,2,3,\ldots,n. \tag{3.2}$$

We define $I_1=\overline{\mathrm{Pf}_{n-1}(G_1)}$ to be the ideal generated by n elements $x_1,x_2,\ldots,x_n$. Let $\tilde{G}_1=Y$ be an $n\times n$ alternating matrix obtained from G_1. Let $\tilde{I}_1$ be the ideal generated by the maximal order pfaffians $\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n$ of $\tilde{G}_1$. It follows from Theorem 2.1 [2] that if I_1 has grade 3, then $\tilde{I}_1$ is a Gorenstein ideal of grade 3. We can easily see from (3.2) that $I_1=(\tilde{x}_1,\tilde{x}_2,u_1\tilde{x}_3\ldots,u_1\tilde{x}_n)$.

The following theorem is a special case of Theorem 4.4 [1]. It states that $I_1=\overline{\mathrm{Pf}_{n-1}(G_1)}$ is a perfect ideal of grade 3 satisfying the following properties: (a) I_1 has type 2, (b) the number of generators for I_1 is odd, and (c) $\lambda(I_1)>0$.

Theorem 3.3 ([11])**.** *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let n be an odd integer with $n>3$ and u_1 an element of $\mathfrak{m}$. Let G_1 be the $n\times n$ skew-symmetrizable matrix in* (3.1) *with entries in $\mathfrak{m}$. Let x_i be an element in* (3.2) *for $i=1,2,\ldots,n$.*

(1) *If I_1 is an ideal of grade 3 generated by $x_1,x_2,\ldots,x_n$ and has $\lambda(I_1)>0$, then I_1 is a perfect ideal of type 2.*

(2) *Every perfect ideal I of grade 3 with type 2 and $\lambda(I) > 0$ minimally generated by n elements arises as in the way of* (1).

Next we give two skew-symmetrizable matrices G_2 and G_3 [4, 5] which define perfect ideals of grade 3 with type 3 and with type 4, respectively, linked to an almost complete intersection of grade 3 with even type by a regular sequence.

Example 3.4. Let R be a commutative ring with identity. Let n be an even integer with $n \geq 4$. Let $A = (a_{ij})$ and $Y = (y_{ij})$ be an $n \times 3$ matrix and an $n \times n$ alternating matrix with entries in R, respectively. Let u_1, u_2 and u_3 be three elements of R. Let F be an $3 \times n$ matrix defined by

$$F = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ -a_{12} & -a_{22} & \cdots & -a_{n2} \\ a_{13} & a_{23} & \cdots & a_{n3} \end{bmatrix}.$$

(1) Let G_2 be an $(n+3) \times (n+3)$ skew-symmetrizable matrix by

(3.3)
$$G_2 = \left[\begin{array}{c|c} \mathbf{0} & \overline{F} \\ \hline -F^t & Y \end{array}\right], \text{ where } \overline{F} = \begin{bmatrix} u_2a_{11} & u_2a_{21} & \cdots & u_2a_{n1} \\ -u_1a_{12} & -u_1a_{22} & \cdots & -u_1a_{n2} \\ u_1u_2a_{13} & u_1u_2a_{23} & \cdots & u_1u_2a_{n3} \end{bmatrix}.$$

The alternating matrix $\mathcal{A}(G_2)$ is obtained by multiplying the first column of G_2 by u_2, the second column by u_1, and the third column by u_1u_2. We note that $\mathcal{A}(G_2)_i$ is divisible by u_1u_2 for every i. Let $v_2 = u_1u_2$ and let x_i be an element defined by

(3.4) $$x_i = \mathcal{A}(G_2)_i / v_2 \text{ for } i = 1, 2, 3, \ldots, n+3.$$

We define $I_2 = \overline{\mathrm{Pf}_{n+2}(G_2)}$ to be the ideal generated by $(n+3)$ elements, $x_1, x_2, x_3, \ldots, x_{n+3}$.

(2) Let G_3 be an $(n+3) \times (n+3)$ skew-symmetrizable matrix by

(3.5)
$$G_3 = \left[\begin{array}{c|c} \mathbf{0} & \bar{F} \\ \hline -F^t & Y \end{array}\right], \text{ where } \bar{F} = \begin{bmatrix} u_2u_3a_{11} & u_2u_3a_{21} & \cdots & u_2u_3a_{n1} \\ -u_1u_3a_{12} & -u_1u_3a_{22} & \cdots & -u_1u_3a_{n2} \\ u_1u_2a_{13} & u_1u_2a_{23} & \cdots & u_1u_2a_{n3} \end{bmatrix}.$$

The alternating matrix $\mathcal{A}(G_3)$ induced by G_3 is obtained by multiplying the first column of G_3 by u_2u_3, the second column of it by u_1u_3, and the third column of it by u_1u_2. We note that $\mathcal{A}(G_3)_i$ is divisible by $u_1u_2u_3$ for every i. Let $v_3 = u_1u_2u_3$ and let x_i be an element defined by

(3.6) $$x_i = \mathcal{A}(G_3)_i / v_3 \text{ for } i = 1, 2, 3, \ldots, n+3.$$

We define $I_3 = \overline{\mathrm{Pf}_{n+2}(G_3)}$ to be the ideal generated by $(n+3)$ elements, $x_1, x_2, x_3, \ldots, x_{n+3}$. Let $\tilde{G}_2 = \tilde{G}_3$ be an $(n+3) \times (n+3)$ alternating matrix T

given by

$$T = \left[\begin{array}{c|c} \mathbf{0} & F \\ \hline -F^t & Y \end{array}\right] \tag{3.7}$$

and let T_k be the pfaffian of the $(n+2)\times(n+2)$ alternating submatrix of T obtained by deleting the k-th row and column from T. Let $\tilde{x}_i = T_i$ for $i = 1, 2, 3, \ldots, n+3$. Let $\tilde{I}_2$ and $\tilde{I}_3$ be ideals generated by the $n+3$ elements $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{n+3}$. It is easy to show that if I_2 or I_3 has grade 3, then $\tilde{I}_2$ and $\tilde{I}_3$ are Gorenstein ideals of grade 3. We can also see from (3.4) and (3.6) that $I_2 = (u_1\tilde{x}_1, u_2\tilde{x}_2, \tilde{x}_3, u_1u_2\tilde{x}_4, \ldots, u_1u_2\tilde{x}_{n+3})$ and $I_3 = (u_1\tilde{x}_1, u_2\tilde{x}_2, u_3\tilde{x}_3, u_1u_2u_3\tilde{x}_4, \ldots, u_1u_2u_3\tilde{x}_{n+3})$.

The following theorem says that I_2 and I_3 are perfect ideals of grade 3 with type 3 and with type 4, respectively, linked to an almost complete intersection of grade 3 with even type by a regular sequence.

Theorem 3.5 ([4, 5])**.** *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let n be an even integer with $n \geq 4$. Let G_2 and G_3 be skew-symmetrizable matrices in Example* 3.4 *with entries in $\mathfrak{m}$.*

(1) *Let x_i be an element in* (3.4) *and I_2 an ideal generated by $(n+3)$ elements $x_1, x_2, \ldots, x_{n+3}$. If $x = x_1, x_2, x_3$ is a regular sequence in I_2, then*
   (a) $(x) : I_2$ *is an almost complete intersection of grade* 3 *with type n, and*
   (b) I_2 *is a perfect ideal of grade* 3 *with type* 3.
(2) *Let x_i be an element in* (3.6) *and I_3 an ideal generated by $(n+3)$ elements $x_1, x_2, \ldots, x_{n+3}$. If $x = x_1, x_2, x_3$ is a regular sequence in I_3, then*
   (a) $(x) : I_3$ *is an almost complete intersection of grade* 3 *with type n, and*
   (b) I_3 *is a perfect ideal of grade* 3 *with type* 4.

Proof. See the proof of (1) of Theorem 3.6 [4] for the proof of (1) of Theorem 3.5. The proof of (2) is similar to that of (1) [5]. □

The following example gives us a skew-symmetrizable matrix G_4 which defines a class of perfect ideals I of grade 3 with type 2 [3].

Example 3.6. Let R be a commutative ring with identity. Let n be an odd integer with $n > 1$ and u_4 a regular element of R. Let $A = (a_{ij}), C = (c_{ij})$ and $Y = (y_{ij})$ be an $n \times 4$ matrix, a 4×4 alternating matrix, and an $n \times n$ alternating matrix, respectively. We define G_4 to be an $(n+4) \times (n+4)$

skew-symmetrizable matrix as follows:

$$G_4 = \left[\begin{array}{c|c} C & u_4A^t \\ \hline -A & Y \end{array}\right]. \tag{3.8}$$

Let $v_4 = u_4^2$ and let x_i be an element defined by

$$x_i = \mathcal{A}(G_4)_i/v_4 \text{ for } i = 1, 2, 3, \ldots, n + 4. \tag{3.9}$$

We define $I_4 = \overline{\mathrm{Pf}_{n+3}(G_4)}$ to be the ideal generated by $n+4$ elements $x_1, x_2, \ldots, x_{n+4}$.

Theorem 3.17 [3] says that if I_4 has grade 3, then I_4 is a perfect ideal of grade 3 with type 2. The minimal free resolution of R/I_4 described in [3]. I_4 contains a class of perfect ideals I of grade 3 with type 2 and $\lambda(I) = 0$ (see Example 3.18 [3]).

We close this section with the following remark.

Remark 3.7. (1) Theorems 3.3 and 3.5 are true for the polynomial ring R mentioned in the abstract and the homogeneous perfect ideal I_i of grade 3 for $i = 1, 2, 3$.

(2) A perfect ideal I_i of grade 3 mentioned in this section is algebraically linked to an almost complete intersection of grade 3 by a regular sequence for $i = 1, 2, 3$. A structure theorem for such a perfect ideal I_i appears in [9].

4. Hilbert functions of the standard *k*-algebras defined by skew-symmetrizable matrices

In this section we characterize the Hilbert function of the standard k-algebra $S = R/I_i$, where R is the polynomial ring mentioned in the abstract and I_i is a homogeneous perfect ideal of grade 3 in R generated by the quotients of the maximal order pfaffians of the alternating matrix $\mathcal{A}(G_i)$ by an element v_i. We say that a sequence $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_s)$ of nonnegative integers with $h_s \neq 0$ is a Gorenstein sequence if there exists a zero dimensional standard Gorenstein k-algebra S with the Hilbert function $\mathbf{h}$. Stanley characterized a Gorenstein sequence $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_s)$ with $h_1 \leq 3$.

Theorem 4.1 ([17]). *Let* $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_s)$ *be a sequence of nonnegative integers with* $h_1 \leq 3$ *and* $h_s \neq 0$. *Then* $\mathbf{h}$ *is a Gorenstein sequence if and only if*

(1) $h_i = h_{s-i}$ *for each* $i (0 \leq i \leq s)$, *and*
(2) $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_t - h_{t-1})$ *is an O-sequence for* $t = [\frac{s}{2}]$.

Here is an example.

Example 4.2. $\mathbf{h} = (1, 3, 6, 8, 6, 3, 1)$ is a Gorenstein sequence. To see this, by Theorem 4.1 it is sufficient to show that $(1, 2, 3, 2)$ is an O-sequence. Let $m = 1$. Let $K = (w_0^3, w_0^2w_1^2, w_1^3)$ be the ideal generated by $w_0^3, w_0^2w_1^2$ and w_1^3.

Then K is a perfect ideal of grade 2 and the Hilbert series of R/K is $H_{R/K}(\lambda) = 1 + 2\lambda + 3\lambda^2 + 2\lambda^3$. Hence it follows from Theorem 2.2 [17] that $(1, 2, 3, 2)$ is an O-sequence.

Next we turn to the Hilbert function of the standard k-algebra $S = R/I_1$, where I_1 is a homogeneous perfect ideal of grade 3 with type 2 defined by a skew-symmetrizable matrix G_1 in (3.1). The minimal free resolution of R/I_1 is

$$(4.1) \qquad \mathbb{G} : 0 \longrightarrow \bigoplus_{i=1}^{2} R(-\bar{s}_i) \xrightarrow{f_3} \bigoplus_{i=1}^{n+1} R(-\bar{p}_i) \xrightarrow{f_2} \bigoplus_{i=1}^{n} R(-\bar{q}_i) \xrightarrow{f_1} R,$$

where for each i, f_i is a homogeneous map of degree 0 given by

$$\begin{aligned} V &= \begin{bmatrix} -x_2 & x_1 & 0 & \cdots 0 \end{bmatrix}^t, \\ f_1 &= \begin{bmatrix} x_1 & x_2 & \cdots x_n \end{bmatrix}, \\ f_2 &= \begin{bmatrix} G_1 & V \end{bmatrix}, \\ f_3 &= \begin{bmatrix} Y_1 & Y_2 & Y_3 & \cdots & Y_n & 0 \\ 0 & 0 & Y_{123} & \cdots & Y_{12n} & -u_1 \end{bmatrix}^t, \end{aligned}$$

and the shifted degrees are

$$\begin{aligned} &\bar{q}_i = \deg x_i \ \text{ for } \ i = 1, 2, \ldots, n, \\ &\bar{p}_1 = \deg y_{l1} + \bar{q}_l \ \text{ for some } \ l(3 \le l \le n), \\ &\bar{p}_2 = \deg y_{l2} + \bar{q}_l \ \text{ for some } \ l(3 \le l \le n), \\ &\bar{p}_i = \deg u_1 + \deg y_{li} + \bar{q}_l \ \text{ or } \ \bar{p}_i = \deg y_{ci} + \bar{q}_c \ \text{ for } \ i = 3, 4, \ldots, n, \\ &\quad l = 1 \ \text{ or } \ l = 2, \ \text{ and } \ c \text{ is an integer with } \ 3 \le c \le n, \\ &\bar{p}_i = \bar{q}_1 + \bar{q}_2 \ \text{ for } \ i = n + 1, \\ &\bar{s}_1 = \deg Y_j + \bar{p}_j \ \text{ for some } \ j(1 \le j \le n), \\ &\bar{s}_2 = \deg Y_{12j} + \bar{p}_j \ \text{ or } \ \bar{s}_2 = \deg u_1 + \bar{p}_{n+1} \ \text{ for some } \ j(3 \le j \le n). \end{aligned}$$

We say that a sequence $\mathbf{c} = (c_0, c_1, c_2, \ldots, c_\rho)$ of nonnegative integers with $c_\rho \neq 0$ is a CI-sequence having the type $(d_0, d_1, d_2, \ldots, d_m)$ if $\mathbf{c}$ is the Hilbert function of a zero dimensional standard complete intersection k-algebra $S = R/I$, where I is a homogeneous complete intersection generated by a homogeneous regular sequence $z = z_0, z_1, z_2, \ldots, z_m$ with $\deg z_i = d_i$. It follows from Proposition 2.1 that $\rho = \sum_{i=0}^{m}(d_i - 1)$. We define a sequence $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_\sigma)$ of nonnegative integers with $h_\sigma \neq 0$ to be a *Brown sequence* if there exists a zero dimensional standard k-algebra $S = R/I$ with the Hilbert function $\mathbf{h}$, where I is a homogeneous perfect ideal of type 2 with $\lambda(I) > 0$. Now we characterize a class of Brown sequences $\mathbf{h}$ with $h_1 = 3$ by using Theorems 3.3 or 4.4 [1]. We say that $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_\sigma)$ is a Brown sequence with $h_1 = 3$ defined by a skew-symmetrizable matrix G_1 in (3.1) if $\mathbf{h}$ is the Hilbert function of the zero dimensional standard k-algebra $S = R/I_1$, where $I_1 = \overline{\mathrm{Pf}_{n-1}(G_1)}$ is a homogeneous perfect ideal of grade 3 with type 2 and $\lambda(I_1) > 0$.

Theorem 4.3. *With the notation above,* $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_\sigma)$ *is a Brown sequence with* $h_1 = 3$ *defined by a skew-symmetrizable matrix* G_1 *in* (3.1) *if and only if there exist a Gorenstein sequence* $\mathbf{g} = (g_0, g_1, g_2, \ldots, g_\eta)$ *and a CI-sequence* $\mathbf{c} = (c_0, c_1, c_2, \ldots, c_{\rho_1})$ *with* $g_1 = 3$ *and* $2 \le c_1 \le 3$ *satisfying three following properties*:

(1) $\mathbf{c}$ *has the type* (q_1, q_2, τ), *where* q_i *is the degree of the* i*-th generator for the Gorenstein ideal* $\tilde{I}_1$ *of grade* 3 *corresponding to* $\mathbf{g}$ *for* $i = 1, 2, 3, \ldots, n$,

(2) $\sigma = \eta + \tau$,

(3) $h_\sigma = 1$ *and*

$$h_i = \begin{cases} c_i & \text{if } 0 \le i \le \tau - 1 \\ g_{i-\tau} + c_i & \text{if } \tau \le i \le \sigma, \end{cases}$$

where we set $g_i = 0$ *if* $\eta < i \le \sigma$ *and* $c_i = 0$ *if* $\rho_1 < i \le \sigma$.

Proof. Let $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_\sigma)$ be a Brown sequence with $h_1 = 3$ defined by a skew-symmetrizable matrix G_1 in (3.1). Then there exists a homogeneous perfect ideal I_1 of grade 3 with type 2 and $\lambda(I_1) > 0$ in the polynomial ring $R = k[w_0, w_1, w_2]$ over the algebraically closed field k with $\deg w_i = 1$ such that $\mathbf{h}$ is the Hilbert function of $S = R/I_1$. First we show the existence of a Gorenstein sequence $\mathbf{g}$. By Theorems 3.3 or 4.4 [1] there exists an $n \times n$ skew-symmetrizable matrix G_1 in (3.1) such that $I_1 = \overline{\mathrm{Pf}_{n-1}(G_1)}$. Let $Y = (y_{ij})$ be an $n \times n$ alternating matrix and let $\tilde{I}_1 = \mathrm{Pf}_{n-1}(Y)$ be the ideal in Example 3.2. Then the grade of $\tilde{I}_1$ is less than or equal to 3. Since $I_1 \subseteq \tilde{I}_1$ and I_1 has grade 3, $\tilde{I}_1$ has grade 3. Hence by Theorem 2.1 [2] $S = R/\tilde{I}_1$ is a zero dimensional standard Gorenstein k-algebra. The minimal free resolution of $R/\tilde{I}_1$ is given in [2]. Hence there exists a Gorenstein sequence $\mathbf{g} = (g_0, g_1, g_2, \ldots, g_\eta)$, where

$$g_i = H(R/\tilde{I}_1, i) \quad \text{for } i = 0, 1, 2, \ldots, \eta.$$

Now we prove (1). The minimal free resolution of R/I_1 is given in (4.1). We note that

$$h_i = H(R/I_1, i) \quad \text{for } i = 0, 1, 2, \ldots, \sigma.$$

Let $\tau = \deg u_1$. Since R/I_1 and $R/\tilde{I}_1$ are zero dimensional, it follows from the consequence of the Hilbert syzygy theorem that

$$\sum_{i=0}^{\eta} H(R/\tilde{I}_1, i)\lambda^i = \frac{\tilde{g}(\lambda)}{(1-\lambda)^3} \quad \text{and} \quad \sum_{i=0}^{\sigma} H(R/I_1, i)\lambda^i = \frac{\tilde{h}(\lambda)}{(1-\lambda)^3}, \tag{4.2}$$

where

$$\tilde{g}(\lambda) = 1 - \sum_{i=1}^{n} \lambda^{\frac{1}{2}(s-r_i)} + \sum_{i=1}^{n} \lambda^{\frac{1}{2}(s+r_i)} - \lambda^s, \tag{4.3}$$

$$\tag{4.4} \begin{aligned}\tilde{h}(\lambda) = 1 - \sum_{i=1}^{2} \lambda^{\frac{1}{2}(s-r_i)} - \sum_{i=3}^{n} \lambda^{\frac{1}{2}(s-r_i+2\tau)} + \sum_{i=1}^{n} \lambda^{\frac{1}{2}(s+r_i+2\tau)} \\ + \lambda^{s-\frac{1}{2}(r_1+r_2)} - \lambda^{s+\tau} - \lambda^{s+\tau-\frac{1}{2}(r_1+r_2)},\end{aligned}$$

and r_i and s are integers given in [2] (see 466 page). Let

$$\sum_{i=0}^{\sigma} e_i \lambda^i = \tilde{h}(\lambda) - \tilde{g}(\lambda)$$

be the difference of two polynomials $\tilde{h}(\lambda)$ and $\tilde{g}(\lambda)$. Then we have

$$\begin{aligned}\sum_{i=0}^{\sigma} e_i \lambda^i = & - \sum_{i=3}^{n} \lambda^{\frac{1}{2}(s-r_i+2\tau)} + \sum_{i=1}^{n} \lambda^{\frac{1}{2}(s+r_i+2\tau)} + \lambda^{s-\frac{1}{2}(r_1+r_2)} - \lambda^{s+\tau} \\ & - \lambda^{s+\tau-\frac{1}{2}(r_1+r_2)} + \sum_{i=3}^{n} \lambda^{\frac{1}{2}(s-r_i)} - \sum_{i=1}^{n} \lambda^{\frac{1}{2}(s+r_i)} + \lambda^s \\ & = (1-\lambda^\tau)\left(\sum_{i=3}^{n} \lambda^{\frac{1}{2}(s-r_i)} - \sum_{i=1}^{n} \lambda^{\frac{1}{2}(s+r_i)} + \lambda^s + \lambda^{s-\frac{1}{2}(r_1+r_2)}\right) \qquad (4.5) \\ & = (1-\lambda^\tau)\left(-\tilde{g}(\lambda) + 1 - \sum_{i=1}^{2} \lambda^{\frac{1}{2}(s-r_i)} + \lambda^{s-\frac{1}{2}(r_1+r_2)}\right) \\ & = (1-\lambda^\tau)\left(-\tilde{g}(\lambda) + (1-\lambda^{\frac{1}{2}(s-r_1)})(1-\lambda^{\frac{1}{2}(s-r_2)})\right).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{i=0}^{\sigma} e_i \lambda^i = & - (1-\lambda^\tau)\tilde{g}(\lambda) + (1-\lambda)^3 \prod_{j=1}^{2} (1+\lambda+\lambda^2+\cdots+\lambda^{q_j-1}) \\ & \times (1+\lambda+\lambda^2+\cdots+\lambda^{\tau-1}),\end{aligned}$$

where $q_i = \frac{1}{2}(s-r_i)$ for $i = 1, 2$. Let $\rho_1 = q_1 + q_2 + \tau - 3$. Let $c(\lambda)$ be the polynomial defined by

$$c(\lambda) = \sum_{i=0}^{\rho_1} c_i \lambda^i = \prod_{i=1}^{2} (1+\lambda+\lambda^2+\cdots+\lambda^{q_i-1})(1+\lambda+\lambda^2+\cdots+\lambda^{\tau-1}).$$

Then $c(\lambda)$ is the Hilbert series of the zero dimensional standard complete intersection k-algebra $R/(w_0^{q_1}, w_1^{q_2}, w_2^{\tau})$. Hence (1) is proved. It follows from (4.2), (4.3) and (4.4) that the degrees of two polynomials $\tilde{g}(\lambda)$ and $\tilde{h}(\lambda)$ are

$$s = \eta + 3 \quad \text{and} \quad \sigma + 3 = s + \tau,$$

respectively. This proves (2). Finally we prove (3). We set

$$g_i = 0 \quad \text{for } i = \eta+1, \eta+2, \ldots, \sigma.$$

It follows from (4.2) and (4.5) that

$$\sum_{i=0}^{\sigma}(h_i - g_i)\lambda^i = \frac{\sum_{i=0}^{\sigma} e_i\lambda^i}{(1-\lambda)^3} = -(1-\lambda^\tau)\sum_{i=0}^{\eta} g_i\lambda^i + c(\lambda). \tag{4.6}$$

Hence

$$h_i = \begin{cases} c_i & \text{if } 0 \le i \le \tau - 1 \\ g_{i-\tau} + c_i & \text{if } \tau \le i \le \sigma. \end{cases}$$

Since $\rho_1 = s - \frac{1}{2}(r_1 + r_2) + \tau - 3$ and $\sigma = s + \tau - 3$, it follows that $\rho_1 < \sigma = \eta + \tau$. Hence it follows from (4.6) that $h_\sigma = g_\eta = 1$. This proves (3). Conversely, we assume that the three properties (1), (2) and (3) are true. Since $h_1 = 3$ and $\mathbf{g}$ is a Gorenstein sequence, by Theorem 2.1 [2], there exists a homogeneous Gorenstein ideal K of grade 3 such that

$$\sum_{i=0}^{\eta} g_i\lambda^i = \sum_{i=0}^{\eta} H(R/K, i)\lambda^i = \frac{\tilde{g}(\lambda)}{(1-\lambda)^3}, \tag{4.7}$$

where $\tilde{g}(\lambda)$ is the polynomial in (4.3). Furthermore, we can see from [2] (see page 466) that $q_i = \frac{1}{2}(s - r_i)$ for $i = 1, 2$. Then $\rho_1 = q_1 + q_2 + \tau - 3$. By (1) we have

$$c(\lambda) = \sum_{i=0}^{\rho_1} c_i\lambda^i = \frac{\prod_{i=1}^{2}(1-\lambda^{q_i})(1-\lambda^\tau)}{(1-\lambda)^3}.$$

Since $\sigma = \eta + \tau$, (3) implies that

$$\sum_{i=0}^{\sigma} h_i\lambda^i = \sum_{i=0}^{\eta+\tau} h_i\lambda^i = \sum_{i=0}^{\tau-1} c_i\lambda^i + \sum_{i=\tau}^{\eta+\tau} g_{i-\tau}\lambda^i + \sum_{i=\tau}^{\rho_1} c_i\lambda^i.$$

We want to show that

$$\sum_{i=0}^{\sigma} h_i\lambda^i = \frac{\tilde{h}(\lambda)}{(1-\lambda)^3},$$

where $\tilde{h}(\lambda)$ is the polynomial in (4.4). Since $g_i = 0$ for $\eta < i \le \sigma$, it follows from (4.7) that

$$\sum_{i=\tau}^{\eta+\tau} g_{i-\tau}\lambda^i = \lambda^\tau \sum_{i=0}^{\eta} g_i\lambda^i = \frac{\lambda^\tau\tilde{g}(\lambda)}{(1-\lambda)^3}.$$

A direct computation shows that

$$\sum_{i=0}^{\sigma} h_i\lambda^i = \sum_{i=0}^{\rho_1} c_i\lambda^i + \sum_{t=\tau}^{\eta+\tau} g_{i-\tau}\lambda^i = \frac{\lambda^\tau\tilde{g}(\lambda)}{(1-\lambda)^3} + \frac{\prod_{i=1}^{2}(1-\lambda^{q_i})(1-\lambda^\tau)}{(1-\lambda)^3}$$
$$= \frac{\tilde{h}(\lambda)}{(1-\lambda)^3}.$$

The last identity follows from (4.5). It follows from (4.2) and Theorems 3.3 or 4.4 [1] that there exists a zero dimensional standard k-algebra $S = R/I_1$ with the Hilbert function $\mathbf{h} = (h_0, h_1, h_2, \dots, h_\sigma)$, where I_1 is a homogeneous perfect ideal of grade 3 with type 2 and $\lambda(I_1) > 0$ defined by a skew-symmetrizable matrix G_1 in (3.1). This completes the proof. □

We give an example which demonstrates Theorem 4.3.

Example 4.4. $\mathbf{h} = (1, 3, 6, 8, 4, 1)$ is a Brown sequence defined by a skew-symmetrizable matrix G_1 in (3.1) given as follows

$$G_1 = \begin{bmatrix} 0 & 0 & w_1w_2 & w_2^2 & -w_0w_2 & 0 & 0 \\ 0 & 0 & w_0w_2 & 0 & 0 & -w_0w_2 & w_2^2 \\ -w_1 & -w_0 & 0 & w_1 & w_0 & w_2 & 0 \\ -w_2 & 0 & -w_1 & 0 & w_0 & w_1 & 0 \\ w_0 & 0 & -w_0 & -w_0 & 0 & w_2 & w_1 \\ 0 & w_0 & -w_2 & -w_1 & -w_2 & 0 & w_0 \\ 0 & -w_2 & 0 & 0 & -w_1 & -w_0 & 0 \end{bmatrix},$$

where $v_1 = u_1 = w_2$. So $\tau = \deg u_1 = \deg w_2 = 1$. Let $m = 2$. Let $I_1 = (x_1, x_2, \dots, x_7)$ be the ideal in Example 3.2. Then a direct computation by CoCoA 4.7.5, Algebra system shows that I_1 is a perfect ideal of grade 3 with type 2. Since $x = x_1, x_2, x_3$ is a regular sequence such that $(x) : I_1$ is an almost complete intersection of grade 3, Proposition 2.5 [1] gives us that $\lambda(I) > 0$. $\mathbf{h} = (h_0, h_1, h_2, h_3, h_4, h_5) = (1, 3, 6, 8, 4, 1)$ is the Hilbert function of R/I_1. Hence $\sigma = 5$. Let Y be an 7×7 alternating matrix in Example 3.2. We can get Y from G_1. Let $\tilde{I}_1$ be the ideal generated by the maximal order pfaffians of Y. Since I_1 has grade 3 and $I_1 \subset \tilde{I}_1$, $\tilde{I}_1$ is a Gorenstein ideal of grade 3. The Hilbert function of $R/\tilde{I}_1$ is $\mathbf{g} = (g_0, g_1, g_2, g_3, g_4) = (1, 3, 6, 3, 1)$. Hence $\eta = 4$ and $\sigma = 5 = 4 + 1 = \eta + \tau$. We know that $q_1 = 3$ and $q_2 = 3$. Since $\mathbf{c} = (c_0, c_1, c_2, c_3, c_4) = (1, 2, 3, 2, 1)$ is a CI-sequence having type $(3, 3, 1)$, it follows that

$$h_i = c_i \quad \text{for } i = 0, \quad \text{and} \quad h_i = g_{i-1} + c_i \quad \text{for } i = 1, 2, 3, 4, 5,$$

where we set $g_i = 0$ and $c_i = 0$ for $i = 5$.

For $i = 0, 1, 2, 3, 4$ we let G_i be a skew-symmetrizable matrix in Section 3 (for $i = 0$ we set $G_0 = \tilde{G}_0$). We say that a sequence $\mathbf{h} = (h_0, h_1, h_2, \dots, h_\sigma)$ of nonnegative integers with $h_s \neq 0$ is a *PI-sequence of type t_i defined by a skew-symmetrizable matrix G_i* if $\mathbf{h}$ is the Hilbert function of the zero dimensional standard k-algebra $S = R/I_i$, where I_i is a homogeneous perfect ideal of grade 3 with type t_i defined by G_i. For example, Theorems 2.1 [2] and 4.2 [17] say that every Gorenstein sequence $\mathbf{h}$ with $h_1 = 3$ is a PI-sequence of type 1 defined by G_0. Moreover, Theorems 3.3 and 3.5 say that there exist many PI-sequences of type t_i defined by G_i, where $t_i = 2, 3, 4$ for $i = 1, 2, 3$. The following example gives us a PI-sequence $\mathbf{h}$ of type 2 with $h_1 = 3$ which does not belong to a class of Brown sequences with $h_1 = 3$ defined by G_1 in (3.1).

Example 4.5. Let $\mathbf{h} = (1, 3, 6, 10, 8, 4, 1)$ be a sequence of positive integers. Let $m = 2$. First we show that $\mathbf{h}$ is not the Hilbert function of R/I_1, where I_1

is a homogeneous perfect ideal of grade 3 mentioned in Theorem 3.3. Suppose that $\mathbf{h}$ is the Hilbert function of R/I_1. Let $\mathbf{g}$ be a Gorenstein sequence in Theorem 4.3 and let $\tilde{I}_1$ be a Gorenstein ideal of grade 3 corresponding to $\mathbf{g}$. Let q_1 and q_2 be the integers mentioned in (1) of Theorem 4.3. Since $h_3 = 10$ is equal to the number of monomials of degree 3 in R, it follows from (2.1) that the degrees of generators for I_1 are greater than or equal to 4. Hence q_1 and q_2 are greater than or equal to 4. Let σ and ρ_1 be the integers mentioned in Theorem 4.3. Then $\sigma = 6$ and $\rho_1 < \sigma = 6$. However, this is contrary to the fact that $\rho_1 = q_1 + q_2 + \tau - 3$, $q_1 + q_2 - 2 \geq 6$ and $\tau - 1 \geq 0$. Hence $\mathbf{h}$ is not the Hilbert function of R/I_1. Now we show that $\mathbf{h}$ is the Hilbert function of the zero dimensional standard k-algebra R/I_4, where $I_4 = \overline{\mathrm{Pf}_6(G_4)}$ is a homogeneous perfect ideal of grade 3 defined as follows: Let G_4 be an 7×7 skew-symmetrizable matrix in (3.8) given by

$$G_4 = \begin{bmatrix} 0 & 0 & 0 & w_1 & w_2^2 & 0 & 0 \\ 0 & 0 & w_0 & w_2 & 0 & w_0w_2 & 0 \\ 0 & -w_0 & 0 & 0 & 0 & w_1w_2 & 0 \\ -w_1 & -w_2 & 0 & 0 & 0 & 0 & w_0w_2 \\ -w_2 & 0 & 0 & 0 & 0 & w_0^2 & w_1^2 \\ 0 & -w_0 & -w_1 & 0 & -w_0^2 & 0 & w_2^2 \\ 0 & 0 & 0 & -w_0 & -w_1^2 & -w_2^2 & 0 \end{bmatrix}.$$

Then $I_4 = (x_1, x_2, \ldots, x_7)$ is a perfect ideal of grade 3 with type 2, where $v_4 = w_2^2$ and $x_i = \mathcal{A}(G_4)/v_4$ for $i = 1, 2, \ldots, 7$ (see (3.9)). Theorem 3.17 [3] says that I_4 has type 2. The Hilbert function of R/I_4 is $\mathbf{h} = (1, 3, 6, 10, 8, 4, 1)$. Hence $\mathbf{h}$ is a PI-sequence of type 2 defined by G_4. Since $L = (x) : I_4$ is a perfect ideal of grade 3 minimally generated by five elements for any regular sequence $x = x_i, x_j, x_k$ in I_4, Proposition 2.5 [1] says that $\lambda(I_4) = 0$.

Now we characterize the Hilbert function of a zero dimensional standard k-algebra R/I_2, where I_2 is a homogeneous perfect ideal of grade 3 with type 3 defined by a skew-symmetrizable matrix G_2 in (3.3). (1) of Theorem 3.5 says that I_2 is a perfect ideal of grade 3 with type 3 linked to a homogeneous almost complete intersection of grade 3 with even type by a regular sequence.

Theorem 4.6. *Let* $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_\sigma)$ *be a sequence of nonnegative integers with* $h_1 = 3$ *and* $h_\sigma \neq 0$. *If* $\mathbf{h}$ *is the Hilbert function of the zero dimensional standard* k*-algebra* R/I_2, *where* I_2 *is a homogeneous perfect ideal of grade* 3 *with type* 3 *defined by a skew-symmetizable matrix* G_2 *in* (3.3), *then there exist a Gorenstein sequence* $\mathbf{g} = (g_0, g_1, g_2, \ldots, g_\eta)$ *with* $g_1 = 3$ *and two CI-sequences* $\mathbf{c} = (c_0, c_1, c_2, \ldots, c_{\rho_1})$ *and* $\hat{\mathbf{c}} = (\hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{\hat{\rho}_1})$ *with* $2 \leq c_1, \hat{c}_1 \leq 3$ *satisfying the following three properties*:

(1) $\mathbf{c}$ *and* $\hat{\mathbf{c}}$ *have the types* $(\tau, q_2 + \kappa, q_3)$ *and* (κ, q_1, q_3), *respectively, where* q_i *is the degree of the* i*-th generator for the Gorenstein ideal* $\tilde{I}_2$ *of grade* 3 *corresponding to* $\mathbf{g}$ *for* $i = 1, 2, 3, \ldots, n + 3$, *respectively,*

(2) $\sigma = \eta + \tau + \kappa$,

(3) $h_\sigma = 1$ *and*

$$h_i = \begin{cases} c_i & \text{if } 0 \le i \le \tau - 1 \\ c_i + \hat{c}_{i-\tau} & \text{if } \tau \le i < \tau + \kappa \\ g_{i-\tau-\kappa} + c_i + \hat{c}_{i-\tau} & \text{if } \tau + \kappa \le i < \sigma, \end{cases}$$

where we set $g_i = 0$ *if* $\eta < i \le \sigma$, $c_i = 0$ *if* $\rho_1 < i \le \sigma$ *and* $\hat{c}_i = 0$ *if* $\hat{\rho}_1 < i \le \sigma$.

Proof. The proof is similar to that of *if* part of Theorem 4.3. Let $\tilde{I}_2 = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{n+3})$ be a homogeneous Gorenstein ideal of grade 3 in Example 3.4 and $\mathbf{g}$ the Hilbert function of a zero dimensional standard Gorenstein k-algebra $R/\tilde{I}_2$. The Hilbert series of $R/\tilde{I}_2$ is

$$H_{R/\tilde{I}_2}(\lambda) = \sum_{i=0}^{\eta} g_i \lambda^i = \frac{\tilde{g}(\lambda)}{(1-\lambda)^3},$$

where $\tilde{g}(\lambda)$ is mentioned in (4.3) and we replace n with $n+3$. Let I_2 be a homogeneous perfect ideal of grade 3 with type 3 defined by a skew-symmetrizable matrix G_2 in (3.3). The minimal free resolution of R/I_2 described in [4] as follows:

(4.9)

$$\mathbb{F}_{hom} : 0 \longrightarrow \bigoplus_{i=1}^{3} R(-\bar{s}_i) \xrightarrow{f_3} \bigoplus_{i=1}^{n+5} R(-\bar{p}_i) \xrightarrow{f_2} \bigoplus_{i=1}^{n+3} R(-\bar{q}_i) \xrightarrow{f_1} R,$$

where

$$f_1 = \begin{bmatrix} x_1 & x_2 & x_3 \cdots & x_{n+3} \end{bmatrix}, \ f_2 = \begin{bmatrix} \mathbf{0} & \bar{F} & B \\ -F^t & Y & \mathbf{0} \end{bmatrix}, \ f_3 = \begin{bmatrix} C \\ Q \\ N \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & x_3 \\ x_3 & 0 \\ -x_2 & -x_1 \end{bmatrix}, C = \begin{bmatrix} 0 & -\mathrm{Pf}(Y) & T_1 \\ \mathrm{Pf}(Y) & 0 & T_2 \\ 0 & 0 & T_3 \end{bmatrix}, Q = \begin{bmatrix} -q_{21} & q_{11} & T_4 \\ -q_{22} & q_{12} & T_5 \\ \vdots & \vdots & \vdots \\ -q_{2n} & q_{1n} & T_{n+3} \end{bmatrix},$$

$$N = \begin{bmatrix} 0 & u_1 & 0 \\ u_2 & 0 & 0 \end{bmatrix}, \quad q_{ij} = (-1)^{i+1} \sum_{1 \le k \le r} Y_{jk} a_{ki} \text{ for } i = 1, 2,$$

and the shifted degrees are

$\bar{q}_i = \deg x_i$ for $i = 1, 2, \dots, n+3$,

$\bar{p}_i = \deg a_{ji} + \bar{q}_{j+3}$ for $i = 1, 2, 3$, and for some $j (1 \le j \le n)$,

$\bar{p}_i = d + \deg a_{i-3,l} + \bar{q}_l$ for $i = 4, 5, \dots, n+3$ and for some $l (1 \le l \le 3)$, or

$\bar{p}_i = \deg a_{m,i-3} + \bar{q}_{m+3}$ for $i = 4, 5, \dots, n+3$ and for some $m (1 \le m \le n)$,

$\bar{p}_i = \bar{q}_3 + \bar{q}_2$ for $i = n+4$ and $\bar{p}_i = \bar{q}_1 + \bar{q}_3$ for $i = n+5$,

$\bar{s}_1 = \deg \mathrm{Pf}(Y) + \bar{p}_2$ or $\bar{s}_1 = \deg q_{2l} + \bar{p}_{l+3}$ for some $l(1 \le l \le n)$ or
$\bar{s}_1 = \deg u_2 + \bar{p}_{n+5}$,
$\bar{s}_2 = \deg \mathrm{Pf}(Y) + \bar{p}_1$ or $\bar{s}_2 = \deg q_{1l} + \bar{p}_{l+3}$ for some $l(1 \le l \le n)$ or
$\bar{s}_2 = \deg u_1 + \bar{p}_{n+4}$,
$\bar{s}_3 = \deg T_m + \bar{p}_m$ for some $m(1 \le m \le n+3)$,

$$d = \begin{cases} \deg u_2 & \text{if } l = 1 \\ \deg u_1 & \text{if } l = 2 \\ \deg u_1 + \deg u_2 & \text{if } l = 3. \end{cases}$$

Let $\tau = \deg u_1$ and $\kappa = \deg u_2$. Since R/I_2 is zero dimensional, the Hilbert series of R/I_2 is

$$\sum_{i=0}^{\sigma} H(R/I_2, i)\lambda^i = \sum_{i=0}^{\sigma} h_i \lambda^i = \frac{\tilde{h}(\lambda)}{(1-\lambda)^3},$$

where

$$\begin{aligned} \tilde{h}(\lambda) = 1 &- \lambda^{\frac{1}{2}(s-r_1)+\tau} - \lambda^{\frac{1}{2}(s-r_2)+\kappa} - \lambda^{\frac{1}{2}(s-r_3)} \\ &- \sum_{i=4}^{n+3} \lambda^{\frac{1}{2}(s-r_i)+\tau+\kappa} + \sum_{i=1}^{n+3} \lambda^{\frac{1}{2}(s+r_i)+\tau+\kappa} + \lambda^{s-\frac{1}{2}(r_2+r_3)+\kappa} \\ &+ \lambda^{s-\frac{1}{2}(r_1+r_3)+\tau} - \lambda^{s-\frac{1}{2}(r_2+r_3)+\tau+\kappa} - \lambda^{s-\frac{1}{2}(r_1+r_3)+\tau+\kappa} - \lambda^{s+\tau+\kappa}. \end{aligned}$$

The difference of two polynomials $\tilde{h}(\lambda)$ and $\tilde{g}(\lambda)$ is

$$\begin{aligned} &\tilde{h}(\lambda) - \tilde{g}(\lambda) \\ = &- (1-\lambda^{\tau+\kappa})\tilde{g}(\lambda) - \lambda^{\tau+\kappa} + \sum_{i=1}^{3} \lambda^{\frac{1}{2}(s-r_i)+\tau+\kappa} \\ &+ (1-\lambda^{\frac{1}{2}(s-r_3)})(1-\lambda^{\frac{1}{2}(s-r_1)+\tau} \\ &- \lambda^{\frac{1}{2}(s-r_2)+\kappa}) - \lambda^{\frac{1}{2}(s-r_3)+\tau+\kappa}(\lambda^{\frac{1}{2}(s-r_1)} + \lambda^{\frac{1}{2}(s-r_2)}) \\ = &- (1-\lambda^{\tau+\kappa})\tilde{g}(\lambda) + (1-\lambda^{\frac{1}{2}(s-r_3)})(1-\lambda^{\frac{1}{2}(s-r_1)+\tau} - \lambda^{\frac{1}{2}(s-r_2)+\kappa} \\ &- \lambda^{\tau+\kappa} + \lambda^{\frac{1}{2}(s-r_1)+\tau+\kappa} + \lambda^{\frac{1}{2}(s-r_2)+\tau+\kappa}) \\ = &- (1-\lambda^{\tau+\kappa})\tilde{g}(\lambda) \\ &+ (1-\lambda^{\frac{1}{2}(s-r_3)})\{(1-\lambda^{\tau})(1-\lambda^{\frac{1}{2}(s-r_2)+\kappa}) + \lambda^{\tau}(1-\lambda^{\kappa})(1-\lambda^{\frac{1}{2}(s-r_1)})\}. \end{aligned}$$

Let $d_1 = \tau$, $d_2 = \frac{1}{2}(s-r_2)+\kappa$, $d_3 = \frac{1}{2}(s-r_3)$ and $\hat{d}_1 = \kappa$, $\hat{d}_2 = \frac{1}{2}(s-r_1)$, $\hat{d}_3 = \frac{1}{2}(s-r_3)$. Let

$$\rho_1 = \sum_{i=1}^{3}(d_i - 1) = s - \frac{1}{2}(r_2+r_3) + \tau + \kappa - 3 \quad \text{and}$$

$$\hat{\rho}_1 = \sum_{i=1}^{3} (\hat{d}_i - 1) = s - \frac{1}{2}(r_1 + r_3) + \kappa - 3.$$

Then

$$c(\lambda) = \sum_{t=0}^{\rho_1} c_t \lambda^t = \prod_{i=1}^{3} (1 + \lambda + \lambda^2 + \cdots + \lambda^{d_i - 1})$$

and

$$\hat{c}(\lambda) = \sum_{t=0}^{\hat{\rho}_1} \hat{c}_t \lambda^t = \prod_{i=1}^{3} (1 + \lambda + \lambda^2 + \cdots + \lambda^{\hat{d}_i - 1})$$

are the Hilbert series of the standard complete intersection k-algebras $R/(w_0^{d_1}, w_1^{d_2}, w_2^{d_3})$ and $R/(w_0^{\hat{d}_1}, w_1^{\hat{d}_2}, w_2^{\hat{d}_3})$, respectively. This proves (1). In the similar way of the proof of Theorem 4.3, $\sigma + 3 = s + \tau + \kappa$ and $\eta + 3 = s$. This implies that $\sigma = \eta + \tau + \kappa$. This proves (2). We note that

$$\sum_{i=0}^{\sigma} h_i \lambda^i = \lambda^{\tau + \kappa} \sum_{i=0}^{\eta} g_i \lambda^i + c(\lambda) + \lambda^{\tau} \hat{c}(\lambda). \tag{4.10}$$

(3) follows from (4.10). □

The following example illustrates Theorem 4.6.

Example 4.7. Let $m = 2$. $\mathbf{h} = (1, 3, 6, 9, 10, 5, 1)$ is a PI-sequence of type 3 defined by a skew-symmetrizable matrix G_2 in (3.3) given as follows

$$G_2 = \begin{bmatrix} 0 & 0 & 0 & w_1^2 & w_0 w_1 & 0 & w_1 w_2 \\ 0 & 0 & 0 & -w_0^2 & 0 & -w_0 w_2 & 0 \\ 0 & 0 & 0 & 0 & w_0^2 w_1 & 0 & w_0 w_1^2 \\ -w_1 & w_0 & 0 & 0 & w_2 & 0 & 0 \\ -w_0 & 0 & -w_0 & -w_2 & 0 & w_1 & 0 \\ 0 & w_2 & 0 & 0 & -w_1 & 0 & w_0 \\ -w_2 & 0 & -w_1 & 0 & 0 & -w_0 & 0 \end{bmatrix},$$

where $v_2 = w_0 w_1$. $I_2 = (x_1, x_2, \ldots, x_7)$ is a perfect ideal of grade 3 with type 3, where $x_i = \mathcal{A}(G_2)_i / v_2$ for $i = 1, 2, 3, \ldots, 7$ (see (3.4)). So $\tau = \deg u_1 = \deg w_0 = 1$ and $\kappa = \deg u_2 = \deg w_1 = 1$. The Hilbert function of R/I_2 is $\mathbf{h} = (h_0, h_1, h_2, h_3, h_4, h_5, h_6) = (1, 3, 6, 9, 10, 5, 1)$. Hence $\sigma = 6$. We can get an 7×7 alternating matrix T in (3.7) from G_2. Let $\tilde{I}_2$ be the ideal generated by the maximal order pfaffians of T. Since I_2 has grade 3 and $I_2 \subset \tilde{I}_2$, $\tilde{I}_2$ is a Gorenstein ideal of grade 3. The Hilbert function of $R/\tilde{I}_2$ is $\mathbf{g} = (g_0, g_1, g_2, g_3, g_4) = (1, 3, 6, 3, 1)$. Hence $\eta = 4$. Thus $\sigma = 6 = 4 + 1 + 1 = \eta + \tau + \kappa$. We know that $d_1 = 1$, $d_2 = 4$, $d_3 = 3$ and $\hat{d}_1 = 1$, $\hat{d}_2 = 3$, $\hat{d}_3 = 3$. Since $\mathbf{c} = (c_0, c_1, c_2, c_3, c_4, c_5) = (1, 2, 3, 3, 2, 1)$ and $\hat{\mathbf{c}} = (\hat{c}_0, \hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{c}_4) = (1, 2, 3, 2, 1)$ are CI-sequences having the type $(1, 4, 3)$

and $(1,3,3)$, respectively, $\rho_1 = 3+2 = 5$ and $\hat{\rho}_1 = 2+2 = 4$. So it follows that

$$\sum_{i=0}^{6} h_i\lambda^i = \lambda^{\tau+\kappa}\sum_{i=0}^{4} g_i\lambda^i + \sum_{i=0}^{\rho_1} c_i\lambda^i + \lambda^{\tau}\sum_{i=0}^{\hat{\rho}_1}\hat{c}_i\lambda^i.$$

Finally we characterize the Hilbert function of a zero dimensional standard k-algebra R/I_3, where I_3 is a homogeneous perfect ideal of grade 3 with type 4 defined by a skew-symmetrizable matrix G_3 in (3.5). (2) of Theorem 3.5 says that I_3 is a perfect ideal of grade 3 with type 4 linked to a homogeneous almost complete intersection of grade 3 with even type by a regular sequence.

Theorem 4.8. *Let $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_\sigma)$ be a sequence of nonnegative integers with $h_1 = 3$ and $h_\sigma \neq 0$. If $\mathbf{h}$ is the Hilbert function of the zero dimensional standard k-algebra R/I_3, where I_3 is a homogeneous perfect ideal of grade 3 with type 4 defined by a skew-symmetizable matrix G_3 in (3.5), then there exist a Gorenstein sequence $\mathbf{g} = (g_0, g_1, g_2, \ldots, g_\eta)$ with $g_1 = 3$ and two CI-sequences $\mathbf{c} = (c_0, c_1, c_2, \ldots, c_{\rho_1})$, and $\hat{\mathbf{c}} = (\hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{\hat{\rho}_1})$ with $2 \leq c_1, \hat{c}_1 \leq 3$ satisfying the following three properties*:

1. *$\mathbf{c}$ and $\hat{\mathbf{c}}$ have the types $(\tau+q_1, \kappa+q_2, \nu+q_3)$ and (q_1, q_2, q_3), respectively, where q_i is the degree of the i-th generator for the Gorenstein ideal $\tilde{I}_3$ of grade 3 corresponding to $\mathbf{g}$ for $i = 1, 2, 3, \ldots, n+3$,*
2. *$\sigma = \eta + \tau + \kappa + \nu$,*
3. *$h_\sigma = 1$ and*

$$h_i = \begin{cases} c_i & \text{if } 0 \leq i \leq \tau+\kappa+\nu-1 \\ g_{i-\tau-\kappa-\nu} + c_i - \hat{c}_{i-\tau-\kappa-\nu} & \text{if } \tau+\kappa+\nu \leq i < \sigma, \end{cases}$$

where we set $g_i = 0$ if $\eta < i \leq \sigma$, $c_i = 0$ if $\rho_1 < i \leq \sigma$ and $\hat{c}_i = 0$ if $\hat{\rho}_1 < i \leq \sigma$.

Proof. The proof is similar to that of Theorem 4.6. □

The following example illustrates Theorem 4.8.

Example 4.9. Let $m = 2$. $\mathbf{h} = (1, 3, 6, 10, 12, 12, 6, 1)$ be a PI sequence of type 4 defined by a skew-symmetrizable matrix G_3 in (3.5) given by

$$G_3 = \begin{bmatrix} 0 & 0 & 0 & w_0w_1w_2 & 0 & w_1w_2^2 & w_1^2w_2 \\ 0 & 0 & 0 & -w_0w_2^2 & -w_0^2w_2 & -w_0w_2^2 & 0 \\ 0 & 0 & 0 & 0 & w_0w_1^2 & w_0^2w_1 & w_0w_1^2 \\ -w_0 & w_2 & 0 & 0 & 0 & w_1 & w_0 \\ 0 & w_0 & -w_1 & 0 & 0 & 0 & w_2 \\ -w_2 & w_2 & -w_0 & -w_1 & 0 & 0 & 0 \\ -w_1 & 0 & -w_1 & -w_0 & -w_2 & 0 & 0 \end{bmatrix},$$

where $v_3 = w_0w_1w_2$. $I_3 = (x_1, x_2, \ldots, x_7)$ is a perfect ideal of grade 3 with type 4, where $x_i = \mathcal{A}(G_3)_i/v_3$ for $i = 1, 2, 3, \ldots, 7$ (see (3.6)). So $\tau = \deg w_0 = 1, \kappa = \deg w_1 = 1$ and $\nu = \deg w_2 = 1$. The minimal free resolution of R/I_3

is described in [5]. The Hilbert function of R/I_3 is $\mathbf{h} = (h_0, h_1, h_2, \ldots, h_7) = (1, 3, 6, 10, 12, 12, 6, 1)$. Hence $\sigma = 7$. The same argument mentioned in Example 4.7 gives us a Gorenstein ideal $\tilde{I}_3$ of grade 3 and the Hilbert function of $R/\tilde{I}_3$ is $\mathbf{g} = (g_0, g_1, g_2, g_3, g_4) = (1, 3, 6, 3, 1)$. Hence $\eta = 4$ and $\sigma = 7 = 4 + 1 + 1 + 1 = \eta + \tau + \kappa + \nu$. We know that $\hat{d}_1 = 3, \hat{d}_2 = 3$ and $\hat{d}_3 = 3$. Since $\mathbf{c} = (1, 3, 6, 10, 12, 12, 10, 6, 3, 1)$ and $\hat{\mathbf{c}} = (1, 3, 6, 7, 6, 3, 1)$ are CI-sequences having types $(4, 4, 4)$ and $(3, 3, 3)$, respectively, $\rho_1 = 9$ and $\hat{\rho}_1 = 6$. So it follows that

$$\sum_{i=0}^{7} h_t \lambda^i = \lambda^{\tau+\kappa+\nu} \sum_{i=0}^{4} g_i \lambda^i + \sum_{i=0}^{\rho_1} c_i \lambda^i - \lambda^{\tau+\kappa+\nu} \sum_{i=0}^{\hat{\rho}_1} \hat{c}_i \lambda^i.$$

5. Unimodality of Gorenstein sequence defined by skew-symmetrizable matrix G_i

In this section we let $R = k[w_0, w_1, w_2, w_3]$ be the polynomial ring mentioned in the abstract ($m = 3$) and we assume that every perfect ideal P in R and every regular sequence in P are homogeneous. Let I_i be a perfect ideal of grade 3 defined by G_i in Section 3 for $i = 0, 1, 2, 3$ and let $a = a_1, a_2, a_3$ be a regular sequence in I_i. Let H_i be the sum of two perfect ideals I_i and J_i which are geometrically linked by a. We define a sequence $\mathbf{h} = (1, 4, h_2, \ldots, h_s)$ of nonnegative integers with $h_s \neq 0$ to be a *Gorenstein sequence defined by a skew-symmetrizable matrix* G_i if $\mathbf{h}$ is the Hilbert function of R/H_i for an integer $i (0 \leq i \leq 3)$. In this section we use the results described in Section 3, Proposition 5.1 and Lemmas 5.2, 5.3, 5.4 below to prove that for $i = 1, 2, 3$ Gorenstein sequence defined by a skew-symmetrizable matrix G_i is unimodal. For $i = 0$ it follows from Theorem 4.1 that Gorenstein sequence $\mathbf{h}$ defined by a skew-symmetrizable matrix G_0 is unimodal.

Proposition 5.1. *Let I_0 and J_0 be a Gorenstein ideal of grade* 3 *and an almost complete intersection of grade* 3 *with type t in R, respectively, which are geometrically linked by a regular sequence $a = a_1, a_2, a_3$ in $I_0 \cap J_0$. Then if $H_0 = I_0 + J_0$, then $h(R/H_0) = (1, 4, h_2, \ldots, h_{\sigma_0})$ is unimodal.*

Proof. Since I_0 is a Gorenstein ideal of grade 3, by Theorem 2.1 [2] $I_0 = (Y_1, Y_2, \ldots, Y_n)$ for some $n \times n$ alternating matrix Y, where Y_i is the maximal order pfaffians of Y for every i. Since I_0 and J_0 are geometrically linked, $H_0 = (Y_1, Y_2, \ldots, Y_n, w)$ for some homogeneous element w in R. Hence we have $H_0 = (I_0, w)$. Let $\deg w = e$ and let $\mathbf{g} = (g_0, g_1, g_2, \ldots, g_\eta, g_{\eta+1}, \ldots)$ be the Hilbert function of R/I_0 with $\sigma(R/I_0) = \eta$. Since w is regular on R/I_0, it follows from Theorems 2.1 [7] and 3.1 [17] that $\sigma(R/(a)) = \eta + e$ and $\sigma(R/H_0) = \sigma_0 = \eta + e - 1$. Since $H_0 = (I_0, w)$ and $\deg w = e$, we have

$$h_i = \begin{cases} g_i & \text{if } 0 \leq i \leq e - 1 \\ g_i - g_{i-e} & \text{if } e \leq i \leq \sigma_0. \end{cases} \tag{5.1}$$

We want to show that $h_i - h_{i-1} \geq 0$ for an integer i with $1 \leq i \leq [\sigma_0/2]$. We have two cases: either $e \leq [\sigma_0/2]$ or $e > [\sigma_0/2]$.

Case (a) $e > [\sigma_0/2]$.
In this case if $1 \leq i \leq [\sigma_0/2]$, then $i < e$. Hence $h_i - h_{i-1} = g_i - g_{i-1} \geq 0$. The inequality follows from Proposition 2.2 since R/I_0 is a one dimensional standard Gorenstein k-algebra. So we get the desired result.

Case (b) $e \leq [\sigma_0/2]$.
If i is an integer with $1 \leq i \leq e-1$, then $h_i - h_{i-1} = g_i - g_{i-1} \geq 0$. The same argument of case (a) gives us the desired result. If $i = e$, then $h_i - h_{i-1} = g_i - g_{i-1} - 1 \geq 0$. The inequality also follows from the same argument. Now we assume that $e < i \leq [\sigma_0/2]$. Since $H_{R/I_0}(\lambda) = \sum_{i=0}^{\infty} g_i \lambda^i$ is the Hilbert series of a one dimensional standard Gorenstein k-algebra, $(1-\lambda)H_{R/I_0}(\lambda)$ is the Hilbert series of a zero dimensional standard Gorenstein k-algebra. Hence $(g_0, g_1 - g_0, g_2 - g_1, \ldots, g_\eta - g_{\eta-1})$ is a Gorenstein sequence with $g_1 - g_0 \leq 3$. It follows from Theorem 4.1 that for each k with $1 \leq k \leq [\eta/2]$

$$g_k - g_{k-1} - (g_{k-1} - g_{k-2}) \geq 0, \text{ where } g_l = 0 \text{ for } l < 0.$$

Thus for each i with $e < i \leq [\sigma_0/2]$ it follows from (5.1) that

$$\begin{aligned} h_i - h_{i-1} &= g_i - g_{i-e} - (g_{i-1} - g_{i-1-e}) = g_i - g_{i-1} - (g_{i-e} - g_{i-1-e}) \\ &= g_i - g_{i-1} - (g_{i-1} - g_{i-2}) + (g_{i-1} - g_{i-2}) - (g_{i-2} - g_{i-3}) \\ &\quad + (g_{i-2} - g_{i-3}) - (g_{i-3} - g_{i-4}) + \cdots + (g_{i-e+1} - g_{i-e}) \\ &\quad - (g_{i-e} - g_{i-1-e}) \geq 0. \end{aligned}$$

This completes the proof. □

Now we prove that a Gorenstein sequence defined by a skew-symmetrizable matrix G_i in (3.1) or (3.3) or (3.5) is unimodal for $i = 1, 2, 3$. For this purpose we need some lemmas. We can get a Gorenstein ideal of grade 3 from G_i. Let $\tilde{G}_i$ be an alternating matrix obtained from G_i for $i = 1, 2, 3$ (see Examples 3.2 and 3.4) and $\tilde{I}_i = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m)$ a Gorenstein ideal of grade 3 generated by the maximal order pfaffians of $\tilde{G}_i$. Let $\tilde{a} = \tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ be a regular sequence in $\tilde{I}_i$ defined as follows for $i = 1, 2, 3$: Let k be an integer with $3 \leq k \leq m$, where $m = n$ or $m = n + 3$.

$$\tilde{a} = \begin{cases} \tilde{x}_1, \tilde{x}_2, \tilde{x}_k & \text{if } i = 1 \\ \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 & \text{if } i = 2 \\ \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 & \text{if } i = 3. \end{cases}$$

Let $\tilde{J}_i = (\tilde{a}) : \tilde{I}_i$. Then $\tilde{J}_i$ is an almost complete intersection of grade 3 for $i = 1, 2, 3$. We assume that entries of $\tilde{G}_i$ and all u_j are homogeneous in the ideal $\mathfrak{m} = (w_0, w_1, w_2, w_3)$ of R.

Lemma 5.2. *With the notation above, we assume that*

(1) *I_i and J_i are linked by a, and*
(2) *I_i has grade* 3.

If $\tilde{I}_i$ and $\tilde{J}_i$ are geometrically linked by $\tilde{a}$, then I_i and J_i are geometrically linked by a.

Proof. Since $\tilde{I}_i$ is an ideal generated by the maximal order pfaffians of $\tilde{G}_i$, the grade of $\tilde{I}_i$ is less than or equal to 3. Since I_i has grade 3 and I_i is properly contained in $\tilde{I}_i$, $\tilde{I}_i$ has grade 3. Theorem 2.1 [2] implies that $\tilde{I}_i$ is Gorenstein. Hence $\tilde{J}_i = (\tilde{a}) : \tilde{I}_i = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, w)$, where w is an element defined in (3.5) [9]. It follows from the definitions of G_i and $\tilde{G}_i$ that if $i = 1$, then $a_1 = \tilde{x}_1, a_2 = \tilde{x}_2$ and $a_3 = u_1\tilde{x}_k$ (see Example 3.2), if $i = 2$, then $a_1 = u_1\tilde{x}_1, a_2 = u_2\tilde{x}_2$ and $a_3 = \tilde{x}_3$, and if $i = 3$, then $a_1 = u_1\tilde{x}_1, a_2 = u_2\tilde{x}_2$ and $a_3 = u_3\tilde{x}_3$ (see Example 3.4). Let $J_i = (a) : I_i$. It is easy to show that $\tilde{J}_i = J_i$ for $i = 1, 2, 3$. Since $\tilde{I}_i$ and $\tilde{J}_i$ are geometrically linked, $\tilde{I}_i \cap \tilde{J}_i = (\tilde{a})$. We want to show that $I_i \cap J_i = (a)$. Since $\tilde{I}_i \cap \tilde{J}_i = (\tilde{a})$, w is not contained in $\tilde{I}_i$. Since $I_i \subseteq \tilde{I}_i$, w is not contained in I_i. Since $\tilde{J}_i = J_i$, $I_i \cap J_i = (a)$. Thus I_i and J_i are geometrically linked by a. □

Let H be a Gorenstein ideal of grade 4 expressed as the sum of two perfect ideals of grade 3 geometrically linked by a regular sequence z. Then the Hilbert function of R/H is characterized as follows.

Lemma 5.3. *Let R be the polynomial ring mentioned in this section. Let I and J be perfect ideals of grade 3 algebraically linked by a regular sequence z and $H = I + J$. Then*

$$H_{R/H}(\lambda) = H_{R/I}(\lambda) + H_{R/J}(\lambda) - H_{R/(z)}(\lambda) \tag{5.2}$$

if and only if I and J are geometrically linked.

Proof. Let us consider a short exact sequence

$$0 \longrightarrow R/I \cap J \longrightarrow R/I \oplus R/J \longrightarrow R/(I + J) \longrightarrow 0. \tag{5.3}$$

Suppose that I and J are not geometrically linked. Then $I \cap J \neq (z)$. Since $(z) \subseteq I$ and $(z) \subseteq J$, $(z) \subsetneqq I \cap J$. Let $H_{(z)}(\lambda)$ and $H_{I\cap J}(\lambda)$ be the Hilbert series of (z) and $I \cap J$, respectively. Then

$$H_{(z)}(\lambda) \neq H_{I\cap J}(\lambda).$$

On the other hand, we get the following from (5.2) and (5.3)

$$H_{R/I\cap J}(\lambda) = H_{R/(z)}(\lambda). \tag{5.4}$$

We also obtain the following from (2.1) and (5.4)

$$H_{(z)}(\lambda) = H_{I\cap J}(\lambda).$$

This is a contradiction. Conversely, we assume that I and J are geometrically linked by the regular sequence z. Then $I \cap J = (z)$ and

$$H_{R/I}(\lambda) + H_{R/J}(\lambda) = H_{R/I\oplus R/J}(\lambda) = H_{R/(z)}(\lambda) + H_{R/H}(\lambda).$$

Hence we have the desired result. □

We remark that Lemma 5.3 is true for a Gorenstein ideal H of grade $m+1$ in the polynomial ring R mentioned in the abstract such that H is the sum of perfect ideals I and J of grade m geometrically linked by a regular sequence z.

Let $\mathbf{c} = (c_0, c_1, c_2, \ldots)$ and $\hat{\mathbf{c}} = (\hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots)$ be the Hilbert functions of one dimensional standard complete intersection k-algebras $R/(w_1^{d_1}, w_2^{d_2}, w_3^{d_3})$ and $R/(w_1^{\hat{d}_1}, w_2^{\hat{d}_2}, w_3^{\hat{d}_3})$, where $d = (d_1, d_2, d_3)$ and $\hat{d} = (\hat{d}_1, \hat{d}_2, \hat{d}_3)$ are types of CI-sequences mentioned in Theorems 4.3 or 4.6 or 4.8. Let $\mathbf{a} = (\check{x}_0, \check{x}_1, \check{x}_2, \ldots)$ and $\tilde{\mathbf{a}} = (\hat{x}_0, \hat{x}_1, \hat{x}_2, \ldots)$ be the Hilbert functions of one dimensional standard complete intersection k-algebras $R/(a)$ and $R/(\tilde{a})$, where a and $\tilde{a}$ are the regular sequences mentioned above. Let $\check{d}_i = \deg x_i$ and $\tilde{d}_i = \deg \tilde{x}_i$ for $i = 1, 2, 3$. Let ρ_1 and $\hat{\rho}_1$ be the integers mentioned in Theorems 4.3 or 4.6 or 4.8.

Lemma 5.4. *With the notation above we let $\check{\rho}_1$ and $\tilde{\rho}_1$ be positive integers defined as follows*

$$\check{\rho}_1 = \sum_{i=1}^{3} (\check{d}_i - 1) \ \ \textit{and} \ \ \tilde{\rho}_1 = \sum_{i=1}^{3} (\tilde{d}_i - 1).$$

Then

(1) $k_i = c_i - (\check{x}_i - \check{x}_{i-1}) \geq 0$ *for* $0 \leq i \leq [(\check{\rho}_1 - 1)/2]$.

(2) $k_i = (c_i - c_{i-1}) - (\check{x}_i - \check{x}_{i-1}) + (\hat{x}_i - \hat{x}_{i-1}) \geq 0$ *for* $1 \leq i \leq [(\check{\rho}_1 - 1)/2]$.

Proof. (1) Let $d = (d_1, d_2, d_3)$ be the type of a CI-sequence mentioned in Theorem 4.3. The proof for the case that $d = (d_1, d_2, d_3)$ is the type of a CI-sequence mentioned in Theorems 4.6 or 4.8 is similar to that of this case. We prove only this case. So a and $\tilde{a}$ are regular sequences in I_1 and $\tilde{I}_1$ in Example 3.2, respectively. Hence $d_i = q_i$ for $i = 1, 2$ and $d_3 = \tau$, $\check{d}_i = q_i$ for $i = 1, 2$ and $\check{d}_3 = q_k + \tau$, and $\tilde{d}_i = q_i$ for $i = 1, 2$ and $\tilde{d}_3 = q_k$ for some integer k $(3 \leq k \leq n)$. It follows from Corollary 3.3 [17] that the Hilbert series of $R/(c), R/(a)$ and $R/(\tilde{a})$ are

(5.5)

$$H_{R/(c)}(\lambda) = \sum_{i=0}^{\infty} c_i \lambda^i = \frac{\prod_{i=1}^{3}(1 - \lambda^{d_i})}{(1-\lambda)^4}, \ H_{R/(a)}(\lambda) = \sum_{i=0}^{\infty} \check{x}_i \lambda^i = \frac{\prod_{i=1}^{3}(1 - \lambda^{\check{d}_i})}{(1-\lambda)^4},$$

$$H_{R/(\tilde{a})}(\lambda) = \sum_{i=0}^{\infty} \hat{x}_i \lambda^i = \frac{\prod_{i=1}^{3}(1 - \lambda^{\tilde{d}_i})}{(1-\lambda)^4}.$$

Let

$$c(\lambda) = \prod_{i=1}^{3}(1 + \lambda + \cdots + \lambda^{d_i - 1}), \quad \tilde{x}(\lambda) = \prod_{i=1}^{3}(1 + \lambda + \cdots + \lambda^{\tilde{d}_i - 1}) \quad \text{and}$$

$$x(\lambda) = \prod_{i=1}^{3}(1 + \lambda + \cdots + \lambda^{\check{d}_i - 1})$$

be the polynomials in λ. Then $c(\lambda) = \sum_{l=0}^{\rho_1}(c_l - c_{l-1})\lambda^l$ and $x(\lambda) = \sum_{l=0}^{\check{\rho}_1}(\check{x}_l - \check{x}_{l-1})\lambda^l$. Since $d_i = \check{d}_i = q_i$ for $i = 1, 2$, $c(\lambda)$ and $x(\lambda)$ have a common factor

$f(\lambda) = \prod_{i=1}^{2}(1+\lambda+\lambda^2+\cdots+\lambda^{q_i-1})$. Let $f(\lambda) = f_0 + f_1\lambda + f_2\lambda^2 + \cdots + f_\gamma\lambda^\gamma$, where $\gamma = q_1 + q_2 - 2$. Then $f(\lambda)$ is symmetric, that is, $f_i = f_{\gamma-i}$. Let $c(\lambda) = m_0 + m_1\lambda + m_2\lambda^2 + \cdots + m_{\rho_1}\lambda^{\rho_1}$ and $x(\lambda) = n_0 + n_1\lambda + n_2\lambda^2 + \cdots + n_{\check{\rho}_1}\lambda^{\check{\rho}_1}$. Since $R/(c)$ is one dimensional and $\sigma(R/(c)) = \rho_1$,

$$c_i = c_{i-1} + m_i = c_{i-2} + m_{i-1} + m_i = \cdots = \sum_{l=0}^{i} m_l \quad \text{for } i = 0,1,2,\ldots,\rho_1,$$

and $c_i = c_{\rho_1}$ for $i = \rho_1+1, \rho_1+2, \ldots,$ where $c_l = 0$ and $m_l = 0$ if $l < 0$. It is sufficient to show that $k_i = c_i - (\check{x}_i - \check{x}_{i-1}) = \sum_{l=0}^{i} m_l - n_i \geq 0$ for $i = 0,1,2,\ldots,[(\check{\rho}_1-1)/2]$. Let i be an integer with $0 \leq i \leq [(\check{\rho}_1-1)/2]$ and let $p = \min\{\gamma, i\}$ be an integer. If $i \leq \tau - 1$, then $m_i = n_i = \sum_{l=0}^{p} f_l$. Hence $k_i = \sum_{l=0}^{i} m_l - n_i \geq 0$ for $i = 0,1,2,\ldots,\tau-1$. For $i = \tau, \tau+1, \ldots, \rho_1$ we have

$$m_i = \sum_{l=i-\tau+1}^{p} f_l.$$

If $i \leq q_k + \tau - 1$, then $n_i = \sum_{l=0}^{p} f_l$. For $i = q_k+\tau, q_k+\tau+1, \ldots, [\check{\rho}_1/2]$ we have

$$n_i = \sum_{l=i-q_k-\tau+1}^{p} f_l.$$

Hence it is easy to show that $k_i = \sum_{l=0}^{i} m_l - n_i \geq 0$ for $i = \tau, \tau+1, \ldots, \rho_1 - 1$ and $k_i = \sum_{l=0}^{\rho_1} m_l - n_i \geq 0$ for $i = \rho_1, \rho_1+1, \ldots, [\check{\rho}_1/2]$.

(2) We notice that $c(\lambda), \tilde{x}(\lambda)$ and $x(\lambda)$ have a common factor $f(\lambda)$ given in the proof of (1). So we have

$$\begin{aligned} &c(\lambda) - x(\lambda) + \tilde{x}(\lambda) \\ &= \sum_{l=0}^{\rho_1}(c_l - c_{l-1})\lambda^l - \sum_{l=0}^{\check{\rho}_1}(\check{x}_l - \check{x}_{l-1})\lambda^l + \sum_{l=0}^{\tilde{\rho}_1}(\hat{x}_l - \hat{x}_{l-1})\lambda^l \\ &= f(\lambda)\{(1+\lambda+\lambda^2+\cdots+\lambda^{\tau-1}) - (1+\lambda+\lambda^2+\cdots+\lambda^{q_k+\tau-1}) \\ &\quad + (1+\lambda+\lambda^2+\cdots+\lambda^{q_k-1})\}. \end{aligned}$$

If $\tau \leq q_k$, then $[(\check{\rho}_1-1)/2] \leq \tilde{\rho}_1$ and if $\tau > q_k$, then $[(\check{\rho}_1-1)/2] \leq \rho_1$. This implies that

$$(c_l - c_{l-1}) - (\check{x}_l - \check{x}_{l-1}) + (\hat{x}_l - \hat{x}_{l-1}) \geq 0 \quad \text{for } l = 0,1,2,\ldots,[(\check{\rho}_1-1)/2]. \ \square$$

Now we use the results mentioned in Section 3, Proposition 5.1 and Lemma 5.4 to prove that if I_i and J_i are geometrically linked by a regular sequence $a = a_1, a_2, a_3$, then a Gorenstein sequence $h(R/H_i)$ is unimodal for $i = 1,2,3$, where $H_i = I_i + J_i$.

Theorem 5.5. *With the notation above, we let I_i be a perfect ideal of grade 3 for $i = 1,2,3$. If $\tilde{I}_i$ and J_i are geometrically linked by a regular sequence $\tilde{a}$, then a Gorenstein sequence $h(R/H_i) = (1,4,h_2,\ldots,h_s)$ is unimodal.*

Proof. Let $H_0 = \tilde{I}_i + \tilde{J}_i$. Since $\tilde{I}_i$ and $\tilde{J}_i$ are geometrically linked by $\tilde{a}$ for $i = 1, 2, 3$, by Proposition 5.1 $h(R/H_0)$ is unimodal. The assumption and Lemma 5.2 implies that I_i and J_i are geometrically linked by a regular sequence a. We note that $\tilde{J}_i = J_i$ for $i = 1, 2, 3$. We prove that a Gorenstein sequence $h(R/H_i)$ is unimodal for only $i = 1$. For the case of $i = 2, 3$ the proof is similar to that of the case of $i = 1$. As shown in the proof of Theorem 4.3, $\sigma(R/I_1) = \eta + \tau$, where $\tau = \deg u_1$. Let e be the integer mentioned in the proof of Proposition 5.1. Since $\sigma(R/(\tilde{a})) = \eta + e$, we have $\sigma(R/(a)) = \eta + e + \tau$. It follows from Theorem 2.1 [7] that $\sigma(R/H_1) = \eta + e + \tau - 1 = \check{\rho}_1 - 1$, where $\check{\rho}_1$ is an integer in Lemma 5.4. Now we set

$$H_{R/\tilde{I}_1}(\lambda) = \sum_{i=0}^{\infty} g_i\lambda^i,\ \ H_{R/(\tilde{a})}(\lambda) = \sum_{i=0}^{\infty} \hat{x}_i\lambda^i,\ \ H_{R/H_0}(\lambda) = \sum_{i=0}^{\eta+e-1} h_i\lambda^i,$$
$$H_{R/I_1}(\lambda) = \sum_{i=0}^{\infty} p_i\lambda^i,\ \ H_{R/(a)}(\lambda) = \sum_{i=0}^{\infty} \check{x}_i\lambda^i,\ \ H_{R/H_1}(\lambda) = \sum_{i=0}^{\eta+e+\tau-1} l_i\lambda^i.$$

We note that
(5.6)
$$H_{R/\tilde{I}_1}(\lambda) = \sum_{i=0}^{\infty} g_i\lambda^i = \frac{\tilde{g}(\lambda)}{(1-\lambda)^4} \quad \text{and} \quad H_{R/I_1}(\lambda) = \sum_{i=0}^{\infty} p_i\lambda^i = \frac{\tilde{h}(\lambda)}{(1-\lambda)^4},$$

where $\tilde{g}(\lambda)$ and $\tilde{h}(\lambda)$ are polynomials in (4.3) and (4.4), respectively. $(p_0, p_1 - p_0, p_2 - p_1, \ldots, p_{\eta+\tau} - p_{\eta+\tau-1})$ is a Brown sequence defined by a skew-symmetrizable matrix G_1 in (3.1) with $p_1 - p_0 = 3$ and $(g_0, g_1 - g_0, g_2 - g_1, \ldots, g_\eta - g_{\eta-1})$ is a Gorenstein sequence with $g_1 - g_0 = 3$. As shown in the proof of Theorem 4.3, there exists a CI-sequence $\mathbf{c} = (c_0, c_1, \ldots, c_{\rho_1})$ such that

$$p_i - p_{i-1} = \begin{cases} c_i & \text{if } 0 \le i \le \tau - 1 \\ g_{i-\tau} - g_{i-\tau-1} + c_i & \text{if } \tau \le i \le \eta + \tau. \end{cases} \tag{5.7}$$

Then it follows from Lemma 5.3 that

$$H_{R/H_0}(\lambda) = H_{R/\tilde{I}_1}(\lambda) + H_{R/\tilde{J}_1}(\lambda) - H_{R/(\tilde{a})}(\lambda),$$
$$H_{R/H_1}(\lambda) = H_{R/I_1}(\lambda) + H_{R/J_1}(\lambda) - F_{R/(a)}(\lambda).$$

This implies that

$$H_{R/H_1}(\lambda) - H_{R/H_0}(\lambda) = H_{R/I_1}(\lambda) - H_{R/\tilde{I}_1}(\lambda) - H_{R/(a)}(\lambda) + H_{R/(\tilde{a})}(\lambda).$$

Hence we want to show that for an integer i with $1 \le i \le [(\check{\rho}_1 - 1)/2]$ we have

$$l_i - l_{i-1} = (h_i - h_{i-1}) + (p_i - p_{i-1}) - (g_i - g_{i-1}) - (\check{x}_i - \check{x}_{i-1}) + (\hat{x}_i - \hat{x}_{i-1}) \ge 0.$$

We have two cases: either $\tau \le e - 1$ or $\tau > e - 1$.

Case (a) $\tau \le e - 1$. We have three parts: (i) $1 \le i \le \tau - 1$ or (ii) $\tau \le i \le e - 1$ or (iii) $e \le i \le [(\check{\rho}_1 - 1)/2]$.

(i) $0 \le i \le \tau - 1$. Then it follows from (5.1), (5.7), Proposition 5.1, and (1) of Lemma 5.4 that

$$l_i - l_{i-1} = c_i - (\check{x}_i - \check{x}_{i-1}) + (\hat{x}_i - \hat{x}_{i-1}) \ge 0,$$

where we set $l_k = 0, \hat{x}_k = 0$ and $\check{x}_k = 0$ if $k < 0$.

(ii) $\tau \le i \le e - 1$. Then in the similar way of (i) we have

$$l_i - l_{i-1} = (g_{i-\tau} - g_{i-\tau-1}) + c_i - (\check{x}_i - \check{x}_{i-1}) + (\hat{x}_i - \hat{x}_{i-1}) \ge 0.$$

The inequality follows from (2) of Proposition 2.2 and (1) of Lemma 5.4.

(iii) $e < i \le [(\check{\rho}_1 - 1)/2]$. In this case we have

$$l_i - l_{i-1} = (g_{i-\tau} - g_{i-\tau-1}) - (g_{i-e} - g_{i-e-1}) + c_i - (\check{x}_i - \check{x}_{i-1}) + (\hat{x}_i - \hat{x}_{i-1}) \ge 0.$$

Since $\tau < e$ and $(g_0, g_1 - g_0, g_2 - g_1, \dots, g_\eta - g_{\eta-1})$ is a Gorenstein sequence with $g_1 - g_0 = 3$, it follows from Theorem 4.1 that $(g_{i-\tau} - g_{i-\tau-1}) - (g_{i-e} - g_{i-e-1}) \ge 0$. Hence (1) of Lemma 5.4 gives us that $l_i - l_{i-1} \ge 0$.

Case (b) $\tau > e - 1$. We have three parts: (i) $1 \le i \le e - 1$ or (ii) $e \le i \le \tau - 1$ or (iii) $\tau \le i \le [(\check{\rho}_1 - 1)/2]$. The proof of (i) is similar to that of (i) of case (a). To prove (ii) and (iii) we need to show the following statement:

(5.8) $\hat{x}_{i-e} - \hat{x}_{i-e-1} - (g_{i-e} - g_{i-e-1}) \ge 0$ for $i = e, e+1, \dots, [(\check{\rho}_1 - 1)/2]$.

Since $R/\tilde{I}_1$ is a one dimensional standard Gorenstein k-algebra with the Hilbert function $\mathbf{g}$, it follows from (5.6) that $\bar{\mathbf{g}} = (g_0, g_1 - g_0, g_2 - g_1, \dots, g_\eta - g_{\eta-1})$ is a Gorenstein sequence with $g_1 - g_0 = 3$. Hence there is a Gorenstein ideal K of grade 3 in the polynomial ring $\bar{R} = k[p, q, r]$ with indeterminates p, q, r and $\deg p = \deg q = \deg r = 1$ such that

$$H_{\bar{R}/K}(\lambda) = \frac{\tilde{g}(\lambda)}{(1-\lambda)^3} = \sum_{i=0}^{\eta} (g_i - g_{i-1})\lambda^i, \text{ where } g_k = 0 \text{ if } k < 0.$$

Similarly, $\bar{\tilde{a}} = (\hat{x}_0, \hat{x}_1 - \hat{x}_0, \hat{x}_2 - \hat{x}_1, \dots, \hat{x}_{\tilde{\rho}_1} - \hat{x}_{\tilde{\rho}_1 - 1})$ is a CI-sequence with $\hat{x}_1 - \hat{x}_0 = 3$ which has type (q_1, q_2, q_k). Since $(\tilde{a}) \subseteq \tilde{I}_1$, without loss of generality, by (4.3), (5.5) and Proposition 2.1 we may assume that there is a complete intersection L of grade 3 in K such that

$$H_{\bar{R}/L}(\lambda) = \frac{\prod_{i=1}^{3}(1 - \lambda^{\tilde{d}_i})}{(1-\lambda)^3} = \sum_{i=0}^{\tilde{\rho}_1} (\hat{x}_i - \hat{x}_{i-1})\lambda^i, \text{ where } \hat{x}_k = 0 \text{ if } k < 0.$$

Since L is a complete intersection which is properly contained in K, it follows from (2.1) that $\hat{x}_{i-e} - \hat{x}_{i-e-1} - (g_{i-e} - g_{i-e-1}) \ge 0$ for $i = e, e+1, \dots, [(\check{\rho}_1 - 1)/2]$.

(ii) $e \le i \le \tau - 1$. In this case we have

$$\begin{aligned} l_i - l_{i-1} = & -(g_{i-e} - g_{i-e-1}) + c_i - (\check{x}_i - \check{x}_{i-1}) + (\hat{x}_i - \hat{x}_{i-1}) \\ = & (\hat{x}_{i-e} - \hat{x}_{i-e-1}) - (g_{i-e} - g_{i-e-1}) + c_i - (\check{x}_i - \check{x}_{i-1}) \\ & + (\hat{x}_i - \hat{x}_{i-1}) - (\hat{x}_{i-e} - \hat{x}_{i-e-1}) \ge 0. \end{aligned}$$

It follows from Theorem 4.1 that $(\hat{x}_i - \hat{x}_{i-1}) - (\hat{x}_{i-e} - \hat{x}_{i-e-1}) \geq 0$ for $i = e, e+1, \ldots, [(\check{\rho}_1 - 1)/2]$. Hence the inequality follows from (5.8) and (1) of Lemma 5.4.

(iii) $\tau \leq i \leq [(\check{\rho}_1 - 1)/2]$. Since $(\check{x}_0, \check{x}_1 - \check{x}_0, \ldots, \check{x}_{\check{\rho}_1} - \check{x}_{\check{\rho}_1 - 1})$ and $(\hat{x}_0, \hat{x}_1 - \hat{x}_0, \ldots, \hat{x}_{\hat{\rho}_1} - \hat{x}_{\hat{\rho}_1 - 1})$ are CI-sequences with $\check{x}_1 - \check{x}_0 = 3$ and $\hat{x}_1 - \hat{x}_0 = 3$ which have type $(q_1, q_2, q_k + \tau)$ and (q_1, q_2, q_k), respectively, we have

$$\check{x}_{i-e} - \check{x}_{i-e-1} - (\hat{x}_{i-e} - \hat{x}_{i-e-1}) \geq 0 \ \text{ for } \ i = \tau, \tau+1, \ldots, [(\check{\rho}_1 - 1)/2].$$

Hence it follows from (5.8) that

$$\check{x}_{i-e} - \check{x}_{i-e-1} - (g_{i-e} - g_{i-e-1}) \geq 0 \ \text{ for } \ i = \tau, \tau+1, \ldots, [(\check{\rho}_1 - 1)/2].$$

Since $i - 1 \geq i - e$, it follows from Theorem 4.1 and (1) of Lemma 5.4 that $c_{i-1} - (\check{x}_{i-e} - \check{x}_{i-e-1}) \geq 0$ for $i = \tau, \tau+1, \ldots, [(\check{\rho}_1 - 1)/2]$. Hence we have

$$\begin{aligned} l_i - l_{i-1} = & -(g_{i-e} - g_{i-e-1}) + (g_{i-\tau} - g_{i-\tau-1}) + c_i - (\check{x}_i - \check{x}_{i-1}) + (\hat{x}_i - \hat{x}_{i-1}) \\ = & \ (\check{x}_{i-e} - \check{x}_{i-e-1}) - (g_{i-e} - g_{i-e-1}) + c_{i-1} - (\check{x}_{i-e} - \check{x}_{i-e-1}) \\ & + (g_{i-\tau} - g_{i-\tau-1}) + (c_i - c_{i-1}) - (\check{x}_i - \check{x}_{i-1}) + (\hat{x}_i - \hat{x}_{i-1}) \geq 0. \end{aligned}$$

The last inequality follows from (2) of Lemma 5.4. This completes our proof. □

The following three examples demonstrate Theorem 5.5. In the first example we construct a unimodal Gorenstein sequence defined by skew-symmetrizable matrix G_1 in (3.1).

Example 5.6. $\mathbf{h} = (1, 4, 10, 20, 34, 52, 71, 84, 84, 71, 52, 34, 20, 10, 4, 1)$ is a unimodal Gorenstein sequence defined by a 7×7 skew-symmetrizable matrix G_1 in (3.1) given as follows

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & w_2^3 & w_2 w_3^2 & w_1^2 w_2 & 0 \\ 0 & 0 & w_2 w_3^2 & 0 & w_1^2 w_2 & w_2^3 & w_0^2 w_2 \\ 0 & -w_3^2 & 0 & w_3^2 & w_1^2 & w_0^2 & 0 \\ -w_2^2 & 0 & -w_3^2 & 0 & w_0^2 & 0 & w_1^2 \\ -w_3^2 & -w_1^2 & -w_1^2 & -w_0^2 & 0 & 0 & w_2^2 \\ -w_1^2 & -w_2^2 & -w_0^2 & 0 & 0 & 0 & w_3^2 \\ 0 & -w_0^2 & 0 & -w_1^2 & -w_2^2 & -w_3^2 & 0 \end{bmatrix},$$

where $v_1 = w_2$. Then $I_1 = (x_1, x_2, \ldots, x_7)$ is a perfect ideal of grade 3 with type 2, where $x_i = \mathcal{A}(G_1)_i / v_1$ for $i = 1, 2, \ldots, 7$. Let $\tilde{G}_1 = Y$ be a 7×7 alternating matrix obtained from G_1. Then $\tilde{I}_1 = (Y_1, Y_2, \ldots, Y_7)$ is a Gorenstein ideal of grade 3. An easy computation by CoCoA 4.7.5 shows that $a = x_1, x_2, x_3$ is a regular sequence. $J_1 = (a) : I_1$ is an almost complete intersection of grade 3 with type 4. Since $I_1 \cap J_1 = (a)$, I_1 and J_1 are geometrically linked by a. Then $H_1 = I_1 + J_1$ is a Gorenstein ideal of grade 4 and the Hilbert function of R/H_1 is $\mathbf{h}$. It is easy to show that $\mathbf{h}$ unimodal.

In the second example we construct a unimodal Gorenstein sequence defined by a skew-symmetrizable matrix G_2 in (3.3).

Example 5.7. $\mathbf{h} = (1, 4, 10, 20, 34, 52, 71, 88, 100, 100, 88, 71, 52, 34, 20, 10, 4, 1)$ is a unimodal Gorenstein sequence defined by a 7×7 skew-symmetrizable matrix G_2 in (3.3) given as follows

$$G_2 = \begin{bmatrix} 0 & 0 & 0 & w_0^2w_2^2 & w_2^4 & w_2^2w_3^2 & 0 \\ 0 & 0 & 0 & -w_1w_2^2 & -w_0^2w_1 & -w_1^3 & -w_1w_3^2 \\ 0 & 0 & 0 & 0 & w_1w_2^2w_3^2 & w_1w_2^4 & w_1^3w_2^2 \\ -w_0^2 & w_2^2 & 0 & 0 & w_1^2 & w_3^2 & w_2^2 \\ -w_2^2 & w_0^2 & w_3^2 & -w_1^2 & 0 & w_2^2 & 0 \\ -w_3^2 & w_1^2 & w_2^2 & -w_3^2 & -w_2^2 & 0 & w_0^2 \\ 0 & w_3^2 & w_1^2 & -w_2^2 & 0 & -w_0^2 & 0 \end{bmatrix},$$

where $v_2 = w_1w_2^2$. Then $I_2 = (x_1, x_2, \ldots, x_7)$ is a perfect ideal of grade 3 with type 3, where $x_i = \mathcal{A}(G_2)_i/v_2$ for $i = 1, 2, \ldots, 7$. We can get a 7×7 alternating matrix T in (3.7) from G_2. $\tilde{I}_2 = (T_1, T_2, \ldots, T_7)$ is a Gorenstein ideal of grade 3. An easy computation by CoCoA 4.7.5 shows that $a = x_1, x_2, x_3$ is a regular sequence. $J_2 = (a) : I_2$ is an almost complete intersection of grade 3 with type 4. Since $I_2 \cap J_2 = (a)$, I_2 and J_2 are geometrically linked by a. Then $H_2 = I_2 + J_2$ is a Gorenstein ideal of grade 4 and the Hilbert function of R/H_2 is $\mathbf{h}$. We can easily check that $\mathbf{h}$ is unimodal.

In final example we construct a unimodal Gorenstein sequence defined by a skew-symmetrizable matrix G_3 in (3.5).

Example 5.8. $\mathbf{h} = (1, 4, 10, 20, 34, 52, 71, 88, 100, 104, 104, 100, 88, 71, 52, 34, 20, 10, 4, 1)$ is a unimodal Gorenstein sequence defined by a 7×7 skew-symmetrizable matrix G_3 in (3.5) defined by

$$G_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & w_1w_2w_3^2 & w_1^3w_2 & w_1w_2^3 \\ 0 & 0 & 0 & -w_0^5w_2 & -w_0^3w_2^3 & -w_0^3w_2w_3^2 & 0 \\ 0 & 0 & 0 & w_0^3w_1w_3^2 & w_0^5w_1 & 0 & w_0^3w_1^3 \\ 0 & w_0^2 & -w_3^2 & 0 & w_1^2 & w_2^2 & w_3^2 \\ -w_3^2 & w_2^2 & -w_0^2 & -w_1^2 & 0 & w_3^2 & 0 \\ -w_1^2 & w_3^2 & 0 & -w_2^2 & -w_3^2 & 0 & w_0^2 \\ -w_2^2 & 0 & -w_1^2 & -w_3^2 & 0 & -w_0^2 & 0 \end{bmatrix},$$

where $v_3 = w_0^2w_1w_2$. Then $I_3 = (x_1, x_2, \ldots, x_7)$ is a perfect ideal of grade 3 with type 4, where $x_i = \mathcal{A}(G_3)_i/v_3$ for $i = 1, 2, \ldots, 7$. The same argument mentioned above gives us that $\tilde{I}_3 = (T_1, T_2, \ldots, T_7)$ is a Gorenstein ideal of grade 3. An easy computation by CoCoA 4.7.5 shows that $a = x_1, x_2, x_3$ is a regular sequence. $J_3 = (a) : I_3$ is an almost complete intersection of grade 3 with type 4. Since $I_3 \cap J_3 = (a)$, I_3 and J_3 are geometrically linked by a. Then $H_3 = I_3 + J_3$ is a Gorenstein ideal of grade 4 and the Hilbert function of R/H_3 is $\mathbf{h}$. We can easily check that $\mathbf{h}$ is unimodal.

The following lemma gives us the relation between the Hilbert functions of perfect ideals of grade 3 linked by a regular sequence.

Lemma 5.9 ([6]). *Let R be the polynomial ring mentioned above. Let I and J be perfect ideals of grade 3 linked by a regular sequence z. Let $\sigma = \sigma(R/(z))$. Then we have*

$$h(R/(z), i) = h(R/J, i) + h(R/I, \sigma - i) \quad \textit{for} \quad i = 0, 1, 2, \ldots, \sigma.$$

We remark that Lemma 5.9 is true for perfect ideals I and J of grade m in the polynomial ring R mentioned in the abstract.

For a perfect ideal K in R we define $H(R/K, i) = 0$ for $i < 0$ and $\Delta H(R/K, i) = H(R/K, i) - H(R/K, i-1)$ for $i = 0, 1, 2, \ldots$. By combining Theorem 5.5 with Lemmas 5.3 and 5.9 we get following proposition.

Proposition 5.10. *Let I and J be perfect ideals of grade 3 geometrically linked by a regular sequence $z = z_1, z_2, z_3$. Let I_i and J_i be the perfect ideals of grade 3 geometrically linked by a regular sequence $a = a_1, a_2, a_3$ mentioned above for $i = 0, 1, 2, 3$. Let $H = I + J$ and $\sigma = h(R/(z))$. We assume that $\deg a_k = \deg z_k$ for $k = 1, 2, 3$. If $\Delta H(R/I_i, j) - \Delta H(R/I_i, \sigma - j) \leq \Delta H(R/I, j) - \Delta H(R/I, \sigma - j)$ for $j = 0, 1, 2, \ldots, [(\sigma - 1)/2]$, then $h(R/H) = (1, 4, h_2, \ldots, h_s)$ is unimodal.*

Proof. The proof follows from Theorem 5.5 and Lemmas 5.3 and 5.9. □

Let $\mathcal{G}_p(4)$ be the set of Gorenstein sequences $h(R/H) = (1, 4, h_2, \ldots, h_s)$, where H is the sum of perfect ideals of grade 3 geometrically linked by a regular sequence z. First we give an example which shows that $\mathbf{h}_1 = (1, 4, 1)$ and $\mathbf{h}_2 = (1, 4, 4, 1)$ belong to $\mathcal{G}_p(4)$.

Example 5.11. (1) Let $X = (x_{ij})$ be an 2×4 matrix defined as follows

$$X = \begin{bmatrix} w_0^p & w_1^p & w_2^p & w_3^p \\ w_1^q & w_2^q & w_3^q & w_0^q \end{bmatrix}.$$

Let $p = q = 1$. Then $I = I_2(X)$ is a perfect ideal of grade 3 with type 3 and the minimal free resolution of R/I described in [12]. Let X_{ij} be the determinant of a 2×2 submatrix of X formed by columns i and j. Then an easy computation by CoCoA 4.5.7 shows that $z = X_{12}, X_{13}, X_{34}$ is a regular sequence and that $J = (z) : I$ is a perfect ideal of grade 3. Furthermore, I and J are geometrically linked by z. Thus $H = I + J$ is a Gorenstein ideal of grade 4 such that $h(R/H) = (1, 4, 1)$.

(2) Let $T = (t_{ij})$ be an 5×5 alternating matrix defined as follows

$$T = \begin{bmatrix} 0 & 0 & -w_1 & -w_0 & w_2 \\ 0 & 0 & -w_3 & -w_1 & w_0 \\ w_1 & w_3 & 0 & w_2 & 0 \\ w_0 & w_1 & -w_2 & 0 & w_3 \\ -w_2 & -w_0 & 0 & -w_3 & 0 \end{bmatrix}.$$

Then $I_0 = \mathrm{Pf}_4(T)$ is a Gorenstein ideal of grade 3. Let T_i be the pfaffian of 4×4 alternating submatrix of T obtained by deleting the i-th column and row from T. It is easy to shows that $a = (w_0 + w_3)T_1, T_2, T_5$ is a regular sequence.

Since I_0 is Gorenstein, $J_0 = (a) : I_0 = (a, w_2(w_0 + w_3))$ is an almost complete intersection of grade 3. Since $w_2(w_0+w_3)$ is not contained in I_0, $I_0 \cap J_0 = (a)$. So I_0 and J_0 are geometrically linked by (a) and $H_0 = I_0 + J_0 = (I_0, w_2(w_0 + w_3))$ is a Gorenstein ideal of grade 4 such that $h(R/H_0) = (1, 4, 4, 1)$.

As a result of Proposition 5.10, we show that a Gorenstein sequence $h(R/H)$ in $\mathcal{G}_p(4)$ which falls into one of the following three cases is unmodal.

Corollary 5.12. *With notation in Proposition* 5.10 *we let* $\sigma^* = \sigma(R/I)$ *and* $\sigma - \sigma^* = \alpha^*$. *We assume that* (p) $\sigma^* \leq [(\sigma - 1)/2]$ *or* (q) $[(\sigma - 1)/2] < \sigma^*$ *and* $[(\sigma - 1)/2] < \alpha^*$ *or* (r) $\alpha^* \leq [(\sigma - 1)/2] < \sigma^*$ *and* $\Delta H(R/I, i) - \Delta H(R/I, \sigma - i) \geq 0$ *for* $i = \alpha^*, \alpha^* + 1, \ldots, [(\sigma - 1)/2]$. *Then* $h(R/H) = (1, 4, h_2, \ldots, h_s)$ *is unimodal.*

Proof. Let $z = z_1, z_2, z_3$ be a regular sequence in $I \cap J$ and e_i the degree of z_i for $i = 1, 2, 3$. Then $\sigma = \sum_{i=1}^{3} e_i - 3$. It follows from Theorem 2.1 [7] that $\sigma(R/H) = \sigma - 1$. So $s = \sigma - 1$. Since $h_1 = 4$, we may assume that $e_i \geq 2$ for $i = 1, 2, 3$. If $\sigma = 3$, then $h(R/H) = (1, 4, 1)$ and if $\sigma = 4$, then $h(R/H) = (1, 4, 4, 1)$. If $\sigma = 5$ and if $h(R/H) = (1, 4, n, 4, 1)$ belongs to $\mathcal{G}_p(4)$, then $4 \leq n \leq 10$ (see [7]). We have nothing to prove. Hence we assume that $\sigma \geq 6$. First we show that $h(R/H) = (1, 4, h_2, \ldots, h_s)$ is unimodal for case (p). Since $\sigma^* = \sigma(R/I)$, we have $\Delta H(R/I, k) = 0$ for $k > \sigma^*$. Let i be an integer with $0 \leq i \leq [(\sigma - 1)/2]$. Since $\sigma^* \leq [(\sigma - 1)/2]$, we have

$$\sigma^* \leq [(\sigma - 1)/2] < \sigma - [(\sigma - 1)/2] \leq \sigma - i.$$

This implies that $\Delta H(R/I, \sigma - i) = 0$ for $i = 0, 1, 2, \ldots, [(\sigma - 1)/2]$. Hence

$$\begin{aligned} &\Delta H(R/I, i) - \Delta H(R/I, \sigma - i) \geq 1 \quad \text{if } i = 0, 1, 2, \ldots, \sigma^*, \\ &\Delta H(R/I, i) - \Delta H(R/I, \sigma - i) = 0 \quad \text{if } i = \sigma^* + 1, \ldots, [(\sigma - 1)/2]. \end{aligned}$$

The inequality follows from the fact that R/I is a one dimensional standard k-algebra. Let $T = (t_{ij})$ be an 5×5 alternating matrix in (2) of Example 5.11. Then $I_0 = \mathrm{Pf}_4(T)$ is a Gorenstein ideal of grade 3 and the Hilbert series $H_{R/I_0}(\lambda)$ of R/I_0 is

$$H_{R/I_0}(\lambda) = 1 + 4\lambda + \sum_{k=2}^{\infty} 5\lambda^k.$$

Hence $\sigma_0 = \sigma(R/I_0) = 2$. We note that $u_i = e_i - 2$ is an nonnegative integer for $i = 1, 2, 3$. Let T_i be the pfaffian of 4×4 alternating submatrix of T obtained by deleting the i-th row and column from T. A simple computation by CoCoA 4.7.5 shows that $a = (w_0 + w_3)^{u_1} T_1, (w_1 + w_2)^{u_2} T_2, (w_0 + w_1)^{u_3} T_5$ is a regular sequence. Then $J_0 = (a) : I_0 = (a, w)$ is an almost complete intersection for some element $w \in R$. A direct computation from (3.5) [9] says that w is not contained in I_0. Hence $I_0 \cap J_0 = (a)$. So I_0 and J_0 are geometrically linked by (a) and $H_0 = I_0 + J_0 = (I_0, w)$ is a Gorenstein ideal of grade 4. Hence if

$\Delta(R/I_0, i) = \Delta H(R/I_0, i) - \Delta H(R/I_0, \sigma - i)$, then we have

$$\Delta(R/I_0, i) = \begin{cases} 1 & \text{if } i = 0 \\ 3 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \\ 0 & \text{if } i = 3, 4, \ldots, [(\sigma - 1)/2]. \end{cases}$$

Since $\sigma^* > \sigma_0 = 2$, Proposition 5.10 implies that $h(R/H)$ is unimodal. This completes the proof for case (p). Now we show the two cases (q) and (r). For case (q) we have $\Delta H(R/I, \sigma - i) = 0$ for $i = 0, 1, 2, \ldots, \alpha^* - 1$. Since $[(\sigma - 1)/2] < \alpha^*$, the same argument mentioned in the case (p) completes the proof. For case (r) the proof is similar to that of case (q). □

A Gorenstein sequence $h(R/H) = (1, 4, 10, 20, 35, 56, 56, 56, 56, 56, 35, 20, 10, 4, 1)$ in the following example belongs to case (p) of Corollary 5.12.

Example 5.13. $\mathbf{h} = (1, 4, 10, 20, 35, 56, 56, 56, 56, 56, 35, 20, 10, 4, 1)$ is a unimodal Gorenstein sequence in (p) of Corollary 5.12. Let $X = (x_{ij})$ be the 6×8 matrix defined as follows

$$X = \begin{bmatrix} w_0 & w_1 & w_2 & w_3 & 0 & w_0 & w_1 & w_2 \\ 0 & w_0 & w_1 & w_2 & w_3 & 0 & w_0 & w_1 \\ w_2 & 0 & w_0 & w_1 & w_2 & w_3 & 0 & w_0 \\ w_1 & w_2 & 0 & w_0 & w_1 & w_2 & w_3 & 0 \\ w_0 & w_1 & w_2 & 0 & w_0 & w_1 & w_2 & w_3 \\ w_3 & w_0 & w_1 & w_2 & 0 & w_0 & w_1 & w_2 \end{bmatrix}.$$

A direct computation by CoCoA 4.7.5, Algebra system shows that $I = I_6(X)$ is a perfect ideal of grade 3. The Hilbert series $H_{R/I}(\lambda)$ of R/I is

$$H_{R/I}(\lambda) = 1 + 4\lambda + 10\lambda^2 + 20\lambda^3 + 35\lambda^4 + \sum_{k=5}^{\infty} 56\lambda^k.$$

This shows that $\sigma^* = \sigma(R/I) = 5$. Let X_{ij} be the determinant of the 6×6 submatrix of X obtained by deleting two columns i, j from X. Then $z = X_{12}, X_{68}, X_{78}$ is a regular sequence. It is well-known that $J = (z) : I$ is a perfect ideal of grade 3. A simple computation by CoCoA 4.7.5 shows that I and J are geometrically linked by z. Hence $H = I + J$ is a Gorenstein ideal of grade 4. Since the degrees of X_{ij} are all 6, it follows from Proposition 2.1 that $\sigma = \sigma(R/(z)) = 15$. So we have $[(\sigma - 1)/2] = 7$ and $\sigma^* \leq [(\sigma - 1)/2]$. Thus it follows from case (p) of Corollary 5.12 that $h(R/H) = (1, 4, 10, 20, 35, 56, 56, 56, 56, 56, 35, 20, 10, 4, 1)$ is a unimodal Gorenstein sequence.

A Gorenstein sequence $h(R/H) = (1, 4, 10, 20, 35, 46, 46, 35, 20, 10, 4, 1)$ in the following example belongs to case (q) of Corollary 5.12.

Example 5.14. $\mathbf{h} = (1,4,10,20,35,46,46,35,20,10,4,1)$ is a unimodal Gorenstein sequence in (q) of Corollary 5.12. Let $X = (x_{ij})$ be the 3×5 matrix defined as follows

$$X = \begin{bmatrix} w_0^2 & w_1^2 & w_2^2 & w_3^2 & 0 \\ 0 & w_0^2 & w_1^2 & w_2^2 & w_3^2 \\ w_3 & 0 & w_0 & w_1 & w_2 \end{bmatrix}.$$

A direct computation by CoCoA 4.7.5 shows that $I = I_3(X)$ is a perfect ideal of grade 3. The Hilbert series $H_{R/I}(\lambda)$ of R/I is

$$H_{R/I}(\lambda) = 1 + 4\lambda + 10\lambda^2 + 20\lambda^3 + 35\lambda^4 + 46\lambda^4 + \sum_{k=6}^{\infty} 49\lambda^k.$$

This shows that $\sigma^* = \sigma(R/I) = 6$. Let X_{ij} be the determinant of the 3×3 submatrix of X obtained by deleting two columns i, j from X. Then $z = X_{12}, X_{13}, X_{34}$ is a regular sequence. It is well-known that $J = (z) : I$ is a perfect ideal of grade 3. A simple computation by CoCoA 4.7.5 shows that I and J are geometrically linked by z. Hence $H = I + J$ is a Gorenstein ideal of grade 4. Since the degrees of X_{ij} are all 5, it follows from Proposition 2.1 that $\sigma = \sigma(R/(z)) = 12$. So we have $[(\sigma-1)/2] = 5$ and $[(\sigma-1)/2] < \sigma^*$ and $[(\sigma-1)/2] < \alpha^* = 6$. Thus it follows from case (q) of Corollary 5.12 that $h(R/H) = (1,4,10,20,35,46,46,35,20,10,4,1)$ is a unimodal Gorenstein sequence.

A Gorenstein sequence $\mathbf{h} = (1,4,10,20,35,56,75,84,75,56,35,20,10,4,1)$ in the following example belongs to case (r) of Corollary 5.12.

Example 5.15. $\mathbf{h} = (1,4,10,20,35,56,75,84,75,56,35,20,10,4,1)$ is a unimodal Gorenstein sequence in (r) of Corollary 5.12. Let $X = (x_{ij})$ be an 2×4 matrix in (1) of Example 5.11 and let $p = q = 3$. The similar argument mentioned in Example 5.11 gives us that $I = I_2(X)$ is a perfect ideal of grade 3 with type 3 and that $z = X_{12}, X_{13}, X_{34}$ is a regular sequence. Then $J = (z) : I$ is a perfect ideal of grade 3. Furthermore, I and J are geometrically linked by z. Thus $H = I + J$ is a Gorenstein ideal of grade 4. We show that $h(R/H) = (1,4,\ldots,h_s)$ is unimodal. Since the degrees of X_{ij} are all 6, we have $\sigma = \sigma(R/(z)) = 15$ and $[(\sigma-1)/2] = 7$. Simple computation by CoCoA 4.5.7 says that

$$H_{R/I}(\lambda) = 1+4\lambda+10\lambda^2+20\lambda^3+35\lambda^4+56\lambda^5+78\lambda^6+96\lambda^7+105\lambda^8+\sum_{k=9}^{\infty} 108\lambda^k.$$

Then $\sigma^* = \sigma(R/I) = 9$ and $\alpha^* = 6$. This implies that $\alpha^* \le [(\sigma-1)/2] \le \sigma^*$ and $\Delta H(R/I,i) - \Delta H(R/I,\sigma-i) \ge 0$ for $i = \alpha^*, \alpha^*+1, \ldots, 7$. Hence it follows from (r) of Corollary 5.11 that $h(R/H) = (1,4,10,20,35,56,75,84,75,56,35,20,10,4,1)$ is a unimodal Gorenstein sequence.

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