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On Minimal Unknotting Crossing Data for Closed Toric Braids

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Abstract. Unknotting numbers for torus knots and links are well known. In this paper, we present a new approach to determine the position of unknotting number crossing changes in a toric braid such that the closure of the resultant braid is equivalent to the trivial knot or link. Further we give unknotting numbers of more than 600 knots.

1. Introduction

The *unknotting number* $u(K)$ of a knot K is the minimum number of crossing changes required to convert K into the trivial knot taken over all knot diagrams representing K. In [5, 6], Kronheimer and Mrowka used gauge theory to prove that the unknotting number of an algebraic knot is equal to the genus of the Milnor fiber. A consequence of this result provides the unknotting number of a torus knot of type (p, q) to be $(p-1)(q-1)/2$. In general, it is not true that some minimum crossing diagram of a knot K contains $u(K)$ crossings whose under/over crossing change makes it unknot [1, 7]. But it is interesting to observe that for $p < q$ a minimal diagram, the closure of a toric braid of type $B(p, q) = (\sigma_1\sigma_2 \cdots \sigma_{p-1})^q$, of a torus knot of type (p, q) contains $(p-1)(q-1)/2$ crossings whose change from over to under gives a trivial knot diagram. A well-established technique for unknotting a knot diagram is that of converting it to an ascending or descending diagram. If a knot K with c crossings has a bridge with b crossings, then we can convert K into an ascending or descending diagram with less than or equal to $(c-b)/2$ crossing changes. In the case of torus knots $K(p, q)$ given by the closure of $(\sigma_{p-1}\sigma_{p-2} \cdots \sigma_1)^q$, we have $c = q(p-1)$ and $b = p-1$. Thus we can convert $K(p, q)$ into an ascending or

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descending diagram with $(p-1)(q-1)/2$ crossing changes. This method is very easy to understand but very time consuming in deciding the exact position of these $u(K)$ crossings as there is no particular pattern in the selected crossings. In this paper, we present a new approach to unknot torus knots and extend the same approach to torus links. This new approach is not only symmetric in nature but also has many applications. In particular, this approach is helpful in finding unknotting numbers of many knots discussed in Section 5. As an another application, we provided a sharp upper bound for region unknotting number of torus knots [8].

In Section 2, we introduce a few new terminologies like crossing data, unknotting crossing data, minimal unknotting crossing data and some basic results. Here we assign natural numbers to all crossings in a braid and denote this finite sequence of numbers as *Crossing Data*. Essentially, finite subsequence of crossing data having $(p-1)(q-1)/2$ numbers corresponding to the crossing changes to make $K(p,q)$ an ascending or descending diagram is a minimal unknotting crossing data for $K(p,q)$. In this paper, by symmetry/symmetric we mean the positions of crossings corresponding to minimal unknotting crossing data in $B(p,q)$.

In Section 3, we provide a new approach to find minimal unknotting crossing data for all torus knots. This minimal unknotting crossing data is essentially the symmetric choice of $(p-1)(q-1)/2$ crossings in $K(p,q)$ whose change from over to under makes it unknot.

In Section 4, we extend this new approach to find minimal unknotting crossing data for torus links. Further, using minimal unknotting crossing data and unknotting sequences for torus knots that we provided in [9], we find unknotting numbers of more than 600 knots in Section 5.

2. Preliminaries

Throughout this paper we represent braids as the product of elementary braids $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ such that the height functional value is different at every crossing in the braid diagram. Thus we can assign natural numbers to the crossings starting from the first crossing from the top to the last crossing at the bottom. Also, we consider torus knots of type $K(p,q)$ as the closure of toric braids $B(p,q)$.

Definition 2.1. *Crossing data* for any n-braid β with k crossings is a finite sequence of natural numbers $\langle 1, 2, \ldots, k\rangle$. For example, as shown in Figure 1, the crossing data for the braid $\beta = \sigma_1^2\sigma_2\sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_2\sigma_3\sigma_1^{-1}$ is $\langle 1, 2, 3, \ldots, 10\rangle$.

Definition 2.2. An *unknotting crossing data* for an n-braid, β, is a subsequence to the crossing data of β such that if we make crossing change at these crossing positions then the closure of this braid is equivalent to unknot or unlink.

Definition 2.3. If the number of elements in an unknotting crossing data of a braid β is equal to the unknotting number of K, where K is the closure of β, then we say such an unknotting crossing data as *minimal unknotting crossing data* for both the braid β and the knot K.

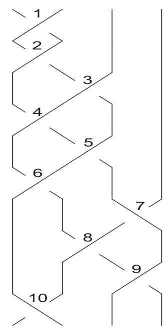

Figure 1: Braid β with different height functional value at each crossing

Throughout the paper we use " $\sim$ " for braid equivalence (related with braid relations), " $\sim_M$ " for Markov equivalence and " $=$ " when either diagram remains same but used different notations for the expression or diagrams are related with Reidemeister moves. For any braid $\beta = \sigma_1^{e_1} \cdots \sigma_n^{e_n}$, where $e_i = \pm 1$, the closure of β is denoted by $cl(\sigma_1^{e_1} \cdots \sigma_n^{e_n})$. The following lemmas are useful in proving the main results of Section 3 and 4.

Lemma 2.4. *For every $n \geq 1$, let $\eta_1 = \sigma_1\sigma_2 \cdots \sigma_n$, $\eta_2 = \sigma_1\sigma_2 \cdots \sigma_{n-1}\sigma_n^{-1}$, $\ldots$, $\eta_n = \sigma_1\sigma_2^{-1} \cdots \sigma_{n-1}^{-1}\sigma_n^{-1}$. Then the $(n+1)$-braid $\eta_1\eta_2 \cdots \eta_n$ is equivalent to $\sigma_n\sigma_{n-1} \cdots \sigma_2\sigma_1$.*

Proof. We prove this lemma by using mathematical induction. For $n = 1$, it is obvious to observe that the 2-braid $\eta_1 = \sigma_1$.
For $n = 2$, the 3-braid $\eta_1\eta_2 = \sigma_1\sigma_2\sigma_1\sigma_2^{-1} \sim \sigma_2\sigma_1\sigma_2\sigma_2^{-1} = \sigma_2\sigma_1$.
Using induction, assume that the result holds for $n = k$. Now by considering the case for $n = k+1$, the $(k+2)$-braid

$$\eta_1\eta_2 \cdots \eta_{k+1} = \sigma_1\sigma_2 \cdots \sigma_{k+1}\sigma_1\sigma_2 \cdots \sigma_{k+1}^{-1} \cdots \sigma_1\sigma_2^{-1} \cdots \sigma_{k+1}^{-1}.$$

Using simple fundamental relations such as $\sigma_j\sigma_i = \sigma_i\sigma_j$ for $|j-i| \geq 2$,
$\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1} = \sigma_i^{-1}\sigma_{i+1}\sigma_i$, we observe that

$$\begin{aligned}
\eta_1\eta_2 \cdots \eta_{k+1} &\sim \sigma_1\sigma_2 \cdots \sigma_k\sigma_1\sigma_2 \cdots \sigma_{k-1}\underbrace{\sigma_{k+1}\sigma_k\sigma_{k+1}^{-1}} \cdots \sigma_1\sigma_2^{-1} \cdots \sigma_{k+1}^{-1} \\
&\sim \sigma_1\sigma_2 \cdots \sigma_k\sigma_1\sigma_2 \cdots \sigma_{k-1}\sigma_k^{-1}\sigma_{k+1}\sigma_k \cdots \sigma_1\sigma_2^{-1} \cdots \sigma_{k+1}^{-1} \\
&\sim \sigma_1\sigma_2 \cdots \sigma_k\sigma_1\sigma_2 \cdots \sigma_{k-1}\sigma_k^{-1}\sigma_1 \cdots \sigma_{k-2}\sigma_{k+1}\underbrace{\sigma_k\sigma_{k-1}\sigma_k^{-1}}\sigma_{k+1}^{-1} \cdots \sigma_1\sigma_2^{-1} \cdots \sigma_{k+1}^{-1} \\
&\sim \sigma_1\sigma_2 \cdots \sigma_k\sigma_1\sigma_2 \cdots \sigma_{k-1}\sigma_k^{-1}\sigma_1 \cdots \sigma_{k-2}\sigma_{k+1}\sigma_{k-1}^{-1}\sigma_k\sigma_{k-1}\sigma_{k+1}^{-1} \cdots \sigma_1\sigma_2^{-1} \cdots \sigma_{k+1}^{-1} \\
&\sim \sigma_1\sigma_2 \cdots \sigma_k\sigma_1\sigma_2 \cdots \sigma_{k-1}\sigma_k^{-1}\sigma_1 \cdots \sigma_{k-2}\sigma_{k-1}^{-1}\underbrace{\sigma_{k+1}\sigma_k\sigma_{k+1}^{-1}}\sigma_{k-1} \cdots \sigma_1\sigma_2^{-1} \cdots \sigma_{k+1}^{-1} \\
&\sim \sigma_1\sigma_2 \cdots \sigma_k\sigma_1\sigma_2 \cdots \sigma_{k-1}\sigma_k^{-1}\sigma_1 \cdots \sigma_{k-2}\sigma_{k-1}^{-1}\sigma_k^{-1}\sigma_{k+1}\sigma_k\sigma_{k-1} \cdots \sigma_1\sigma_2^{-1} \cdots \sigma_{k+1}^{-1}.
\end{aligned}$$

Finally, the above braid is equivalent to

$$\sigma_1\sigma_2\cdots\sigma_k\sigma_1\sigma_2\cdots\sigma_{k-1}\sigma_k^{-1}\cdots\sigma_1\sigma_2^{-1}\cdots\sigma_k^{-1}\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_k^{-1}\sigma_{k+1}\sigma_k\cdots\sigma_1. \tag{2.1}$$

Now, by induction hypothesis

$$\sigma_1\sigma_2\cdots\sigma_k\sigma_1\sigma_2\cdots\sigma_{k-1}\sigma_k^{-1}\cdots\sigma_1\sigma_2^{-1}\cdots\sigma_k^{-1}\sim\sigma_k\sigma_{k-1}\cdots\sigma_2\sigma_1.$$

Hence the braid in (2.1) is

$$\sim\sigma_k\sigma_{k-1}\cdots\sigma_2\sigma_1\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_k^{-1}\sigma_{k+1}\sigma_k\cdots\sigma_1\sim\sigma_{k+1}\sigma_k\cdots\sigma_1.$$

Therefore $\eta_1\eta_2\cdots\eta_n\sim\sigma_n\sigma_{n-1}\cdots\sigma_2\sigma_1$ is true for all positive integers n. □

Lemma 2.5. *For every n, the $(n+1)$-braid*

$$\sigma_1\sigma_2\cdots\sigma_n\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_n^{-1}\sigma_1\sigma_2\cdots\sigma_{n-1}^{-1}\sigma_n^{-1}\cdots\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_n^{-1}$$

is a trivial $(n+1)$-braid.

Proof. Directly follows from Lemma 2.4. □

Lemma 2.6. *If $1\le j<i\le n$, then*

$$\sigma_i^{e}(\rho_j\rho_{j+1}\cdots\rho_{i-2})\sigma_{i-1}^{e_{i-1}}\sigma_i^{e_i}(\rho_{i+1}\cdots\rho_n)\sim(\rho_j\rho_{j+1}\cdots\rho_{i-2})\sigma_{i-1}^{e_i}\sigma_i^{e_{i-1}}(\rho_{i+1}\cdots\rho_n)\sigma_{i-1}^{e}$$

if either $e=e_{i-1}$ or $e_{i-1}=e_i$, where e,e_{i-1},e_i have values ±1 and ρ_k is either σ_k or σ_k^{-1}.

Proof. Proof follows by using the braid relations

$$\sigma_j^{e_j}\rho_i=\rho_i\sigma_j^{e_j}\ \textit{if}\ |i-j|>1\ \textit{and}\ \sigma_{i+1}^{\alpha_1}\sigma_i^{\alpha_2}\sigma_{i+1}^{\alpha_3}=\sigma_i^{\alpha_3}\sigma_{i+1}^{\alpha_2}\sigma_i^{\alpha_1}$$

if either $\alpha_1=\alpha_2$ or $\alpha_2=\alpha_3$, where $\alpha_1,\alpha_2,\alpha_3$ *and* e_j *are* ±1. □

Lemma 2.7. *For any two integers p,a with $p>a$ and $(p,a)=1$, let $\eta_i=\rho_1\rho_2\cdots\rho_{p-a-1}\sigma_{p-a}\cdots\sigma_{p-i}\sigma_{p-i+1}^{-1}\cdots\sigma_{p-1}^{-1}$. Then the knot $K'=cl(\eta_1\eta_2\cdots\eta_a)$ is equivalent to $cl(\rho_1\rho_2\cdots\rho_{p-a-1})^a$.*

Proof. Let

$$\eta_i^{(j)}=\rho_1\rho_2\cdots\rho_{p-a-1}\sigma_{p-a}\cdots\sigma_{p-j-i+1}\sigma_{p-i-j+2}^{-1}\cdots\sigma_{p-j-1}^{-1}\sigma_{p-j}^{-1},$$

for $1\le j\le a-1$ and $1\le i\le a-j+1$.
With this, K' can be represented as closure of

$$(\rho_1\rho_2\cdots\rho_{p-a-1}\sigma_{p-a}\cdots\sigma_{p-2}\sigma_{p-1})(\rho_1\rho_2\cdots\rho_{p-a-1}\sigma_{p-a}\cdots\sigma_{p-2}\sigma_{p-1}^{-1})\eta_3\cdots\eta_a.$$

Hence

$$K'\sim cl((\rho_1\rho_2\cdots\rho_{p-a-1}\sigma_{p-a}\cdots\sigma_{p-2})(\rho_1\rho_2\cdots\rho_{p-a-1}\sigma_{p-a}\cdots\sigma_{p-2}^{-1})\sigma_{p-1}\sigma_{p-2}\eta_3\cdots\eta_a)$$

$$\begin{aligned}
&\sim cl(\eta_1^{(2)}\eta_2^{(2)}\sigma_{p-1}\sigma_{p-2}\eta_3\cdots\eta_a)\\
&\sim cl(\eta_1^{(2)}\eta_2^{(2)}\eta_3^{(2)}\sigma_{p-1}\sigma_{p-2}\sigma_{p-3}\eta_4\cdots\eta_a)\\
&\sim cl(\eta_1^{(2)}\eta_2^{(2)}\cdots\eta_{a-1}^{(2)}\eta_a^{(2)}\sigma_{p-1}\sigma_{p-2}\cdots\sigma_{p-a}).
\end{aligned}$$

Since $\eta_a^{(2)}=\rho_1\rho_2\cdots\rho_{p-a-1}\sigma_{p-a}^{-1}\sigma_{p-a+1}^{-1}\cdots\sigma_{p-2}^{-1}$, we have

$$K'\sim_M cl(\eta_1^{(2)}\eta_2^{(2)}\cdots\eta_{a-1}^{(2)}\rho_1\rho_2\cdots\rho_{p-a-1}).$$

Similarly, we obtain

$$K'\sim_M cl(\eta_1^{(3)}\eta_2^{(3)}\cdots\eta_{a-2}^{(3)}(\rho_1\rho_2\cdots\rho_{p-a-1})^2);$$

$$K'\sim_M cl(\eta_1^{(4)}\eta_2^{(4)}\cdots\eta_{a-3}^{(4)}(\rho_1\rho_2\cdots\rho_{p-a-1})^3);$$

and so on.
Finally, we have $K'\sim_M cl(\rho_1\rho_2\cdots\rho_{p-a-1})^a$. □

Remark 2.8. In Lemma 2.7, if $\rho_i=\sigma_i\ \forall i$, then

$$\begin{aligned}
\eta_1\eta_2\cdots\eta_a &= (\sigma_1\sigma_2\cdots\sigma_{p-2}\sigma_{p-1})(\sigma_1\sigma_2\cdots\sigma_{p-2}\sigma_{p-1}^{-1})\cdots(\sigma_1\sigma_2\cdots\sigma_{p-a}\sigma_{p-a+1}^{-1}\cdots\sigma_{p-2}^{-1}\sigma_{p-1}^{-1})\\
&\sim(\sigma_1\sigma_2\cdots\sigma_{p-a-1})(\sigma_1\sigma_2\cdots\sigma_{p-a-1})\cdots(\sigma_1\sigma_2\cdots\sigma_{p-a-1}) \qquad (2.2)\\
&= B(p-a,a).
\end{aligned}$$

Observe that, if we change crossings in $B(p,a)$, as in (2.2), we obtain a braid which is Markov equivalent to $B(p-a,a)$.

Lemma 2.9. *For every a, the braid*

$$(\sigma_1\sigma_2\cdots\sigma_{a-1})^a\sigma_a\sigma_{a-1}\sigma_{a-2}\cdots\sigma_1\sim(\sigma_1\sigma_2\cdots\sigma_a)^a.$$

Proof. Observe that the given braid

$$\begin{aligned}
&(\sigma_1\sigma_2\cdots\sigma_{a-1})^a\sigma_a\sigma_{a-1}\sigma_{a-2}\cdots\sigma_1\\
&=(\sigma_1\sigma_2\cdots\sigma_{a-1})^{a-1}(\sigma_1\sigma_2\cdots\sigma_{a-1})\sigma_a\sigma_{a-1}\sigma_{a-2}\cdots\sigma_1\\
&=(\sigma_1\sigma_2\cdots\sigma_{a-1})^{a-1}(\sigma_1\sigma_2\cdots\sigma_{a-1}\sigma_a)\sigma_{a-1}\sigma_{a-2}\cdots\sigma_1.
\end{aligned}$$

Now using Lemma 2.6, we have

$$\begin{aligned}
(\sigma_1\sigma_2\cdots\sigma_{a-1})^a\sigma_a\sigma_{a-1}\cdots\sigma_1 &\sim(\sigma_1\sigma_2\cdots\sigma_{a-1})^{a-1}\sigma_a\sigma_{a-1}\cdots\sigma_2(\sigma_1\sigma_2\cdots\sigma_{a-1}\sigma_a)\\
&\sim(\sigma_1\sigma_2\cdots\sigma_{a-1})^{a-2}\sigma_a\sigma_{a-1}\cdots\sigma_3(\sigma_1\sigma_2\cdots\sigma_{a-1}\sigma_a)^2.
\end{aligned}$$

Continuing this way by using Lemma 2.6, we get

$$(\sigma_1\sigma_2\cdots\sigma_{a-1})^a\sigma_a\sigma_{a-1}\sigma_{a-2}\cdots\sigma_1\sim(\sigma_1\sigma_2\cdots\sigma_a)^a. \qquad \square$$

Remark 2.10. We can generalize Lemma 2.9 as

$$(\sigma_1\sigma_2\cdots\sigma_{a-1})^b\sigma_a\sigma_{a-1}\sigma_{a-2}\cdots\sigma_{a-b+1}\sim(\sigma_1\sigma_2\cdots\sigma_a)^b.$$

3. A Method for Determining Minimal Unknotting Crossing Data

In this section, we provide a new approach to find the position of the $(p-1)(q-1)/2$ crossings in $K(p,q)$ such that crossing change at these positions provides an unknot. In particular, we provide minimal unknotting crossing data for the following two classes of torus knots $K(p,q)$:

1. When $q \equiv 1$ or $(p-1)$ $(mod\ p)$,
2. otherwise.

Theorem 3.1. *Let $K(p,q)$ be a torus knot, where $q \equiv 1$ or $p-1$ $(mod\ p)$ and for each i, $1 \leq i \leq q$, let m_i be a non negative integer such that $i-1 \equiv m_i$ $(mod\ p)$. If K' is obtained from $K(p,q)$ by changing last m_i crossings in each i^{th} factor $\sigma_1\sigma_2\cdots\sigma_{p-1}$ of $K(p,q)$, then K' is equivalent to trivial knot. Moreover, the corresponding crossings give minimal unknotting crossing data for $K(p,q)$.*

Proof. Let K' be obtained from $K(p,q)$ by making crossing changes as mentioned in the hypothesis. Then consider the following cases:

Case 1. When $q \equiv 1$ $(mod\ p)$.
In this case, we have $q = mp+1$ for some m and

$$\begin{aligned}
K' &= cl(\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}}_{1}\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}^{-1}}_{2}\cdots\underbrace{\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{p-1}^{-1}}_{mp}\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}}_{mp+1})\\
&= cl((\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}}_{1}\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}^{-1}}_{2}\cdots\underbrace{\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{p-1}^{-1}}_{p})^m\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}})\\
&= cl(\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}})\quad\text{by Lemma 2.5}
\end{aligned}$$

Now, we show that the number of crossing changes is equal to the unknotting number of $K(p,q)$. From [5, 6], we know that the unknotting number for these torus knots is equal to

$$\frac{(p-1)(q-1)}{2}=\frac{(p-1)mp}{2}.$$

Observe that the number of crossing changes made is equal to

$$m\cdot\sum_{i=0}^{p-1} i=\frac{m\cdot(p-1)p}{2}=\frac{(p-1)mp}{2},$$

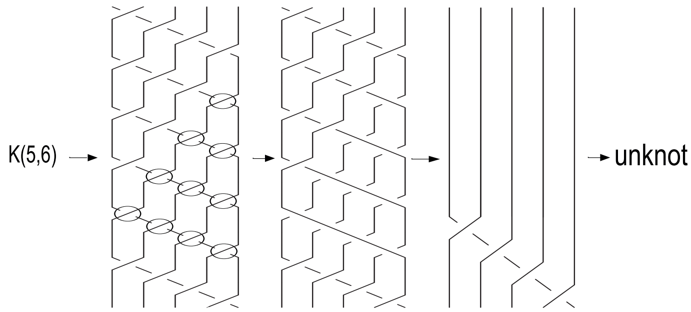

Figure 2: Unknotting procedure for $K(5,6)$

which is same as the unknotting number of $K(p,q)$.

In Figure 2, we showed this procedure pictorially for $K(5,6)$. It is easy to observe that the crossing data $\langle 8,11,12,14,15,16,17,18,19,20\rangle$ is a minimal unknotting crossing data for $K(5,6)$.

Case 2. When $q \equiv p-1$ $(mod\ p)$.
In this case, we have $q = mp-1$ for some m. Also,
$K' = cl(\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}}_{1}\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}^{-1}}_{2}\cdots\underbrace{\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{p-1}^{-1}}_{(m-1)p}\underbrace{\sigma_1\sigma_2\cdots\sigma_{p-1}}_{(m-1)p+1}\cdots\underbrace{\sigma_1\sigma_2^{-1}\cdots\sigma_{p-1}^{-1}}_{mp-1})$
$= cl(\alpha^{m-1}\beta)$, where

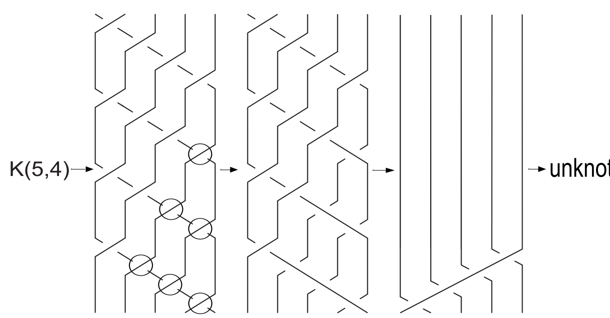

Figure 3: Unknotting procedure for $K(5,4)$

$$\alpha = \sigma_1\sigma_2\cdots\sigma_{p-1}\sigma_1\sigma_2\cdots\sigma_{p-1}^{-1}\sigma_1\sigma_2\cdots\sigma_{p-2}^{-1}\sigma_{p-1}^{-1}\cdots\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{p-1}^{-1},$$

$$and\ \beta = \sigma_1\sigma_2\cdots\sigma_{p-1}\sigma_1\sigma_2\cdots\sigma_{p-1}^{-1}\sigma_1\sigma_2\cdots\sigma_{p-2}^{-1}\sigma_{p-1}^{-1}\cdots\sigma_1\sigma_2^{-1}\cdots\sigma_{p-1}^{-1}.$$

By Lemma 2.5, we get

$$K' = cl(\alpha^{m-1}\beta) \sim cl(\beta).$$

By Lemma 2.4, we have $\beta \sim \sigma_{p-1}\sigma_{p-2}\cdots\sigma_2\sigma_1$, whose closure is the trivial knot.
Since $q = mp-1$, the unknotting number of $K(p,q)$ is equal to $\dfrac{(p-1)(mp-2)}{2}$.

Observe that K' is obtained from $K(p,q)$ by making $(m-1)\cdot\sum_{i=0}^{p-1} i + \sum_{i=0}^{p-2} i = \frac{(p-1)(mp-2)}{2}$ number of crossing changes. Hence the finite sequence of numbers corresponding to these crossing changes is a minimal unknotting crossing data for $K(p,q)$.

In Figure 3, we showed this procedure pictorially for $K(5,4)$. It is easy to observe that the crossing data $\langle 8, 11, 12, 14, 15, 16\rangle$ is a minimal unknotting crossing data for $K(5,6)$. □

It is interesting to observe that the procedure specified in Theorem 3.1 is applicable only for the case when $q \equiv 1$ or $p-1$ $(mod\ p)$. More generally, we have the following:

Theorem 3.2. *Let $K(p,q)$ be a torus knot and for each i, $1 \leq i \leq q$, m_i be a non negative integer such that $i-1 \equiv m_i$ $(mod\ p)$. Then the following statements are equivalent:*

(1) *The unknotting number of $K(p,q)$ is equal to $\sum_{i=1}^{q} m_i$.*

(2) *$q \equiv 1$ or $p-1$ $(mod\ p)$.*

(3) *Subsequence of the crossing data obtained by considering last m_i crossings in each i^{th} factor $\sigma_1\sigma_2\cdots\sigma_{p-1}$ of $K(p,q)$ is a minimal unknotting crossing data for $K(p,q)$.*

Proof. (1) $\Rightarrow$ (2). For a given torus knot $K(p,q)$, we can write $q = mp + a$ for some integers m and a with $0 < a < p$. Observe that $\sum_{i=1}^{q} m_i = m \cdot \sum_{i=0}^{p-1} i + \sum_{i=0}^{a-1} i = \frac{mp(p-1)}{2} + \frac{a(a-1)}{2}$, and the unknotting number of $K(p,q)$ is equal to $\frac{mp(p-1)}{2} + \frac{(p-1)(a-1)}{2}$. Therefore, we have only two solutions for a, i.e., either $a = 1$ or $a = p-1$. Hence $q \equiv 1$ or $p-1$ $(mod\ p)$. (2) $\Rightarrow$ (3) is nothing but Theorem 3.1 and (3) $\Rightarrow$ (1) directly follows from the definition of minimal unknotting crossing data. □

Now consider the case when $q \equiv a$ $(mod\ p)$, where $a \neq 1$ and $a \neq p-1$. To find minimal unknotting crossing data, we first observe the following:

Remark 3.3. For any pair p, q with $q = mp + a$ for some $m \geq 0$, $a < p$ and $(p,q) = 1$, let m_i $(1 \leq i \leq q)$ be a non negative integer such that $i-1 \equiv m_i$ $(mod\ p)$. Then, by changing last m_i crossings in each i^{th} factor $\sigma_1\sigma_2\cdots\sigma_{p-1}$ of $B(p,q)$, we get a new braid $B' = \alpha^m\beta$, where
$\alpha = (\sigma_1\sigma_2\cdots\sigma_{p-2}\sigma_{p-1})(\sigma_1\sigma_2\cdots\sigma_{p-2}\sigma_{p-1}^{-1})\cdots(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{p-2}^{-1}\sigma_{p-1}^{-1})$ and

$\beta = (\sigma_1\sigma_2\cdots\sigma_{p-2}\sigma_{p-1})(\sigma_1\sigma_2\cdots\sigma_{p-2}\sigma_{p-1}^{-1})\cdots(\sigma_1\sigma_2\cdots\sigma_{p-(a-1)}^{-1}\sigma_{p-(a-2)}^{-1}\cdots\sigma_{p-2}^{-1}\sigma_{p-1}^{-1})$.
By Lemma 2.5, α is a trivial $p-$braid. Hence it is easy to observe that the closure of $\alpha^m\beta$ is equivalent to closure of β. Also by Remark 2.8, β is Markov equivalent to $B(p-a,a)$. Hence B' is Markov equivalent to $B(p-a,a)$. Thus we can find a minimal unknotting crossing data for $K(p,q)$ if we can find an unknotting crossing data for $B(p-a,a)$ such that the number of elements in this unknotting crossing data is equal to $u(K(p,q))\setminus\#(\text{crossings changed in } B(p,q))$, which is equal to

$$\frac{(p-a-1)(a-1)}{2}.$$

Observe that this is the unknotting number of $K(p-a,a)$.

The above observation will be utilized to find minimal unknotting crossing data for any torus knot $K(p,q)$, where $q \not\equiv 1$ or $p-1$ $(mod\ p)$. Since the proof of this procedure contains complicated braid representations and lengthy computations, we provide the procedure for finding minimal unknotting crossing data by using examples.

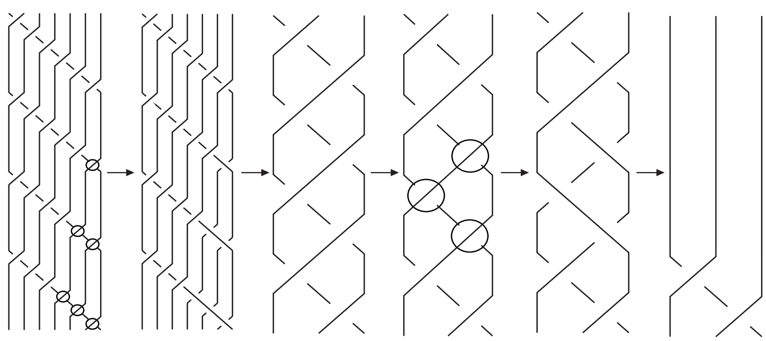

Figure 4: Unknotting procedure for $K(7,4)$

Example 3.4. Procedure to find minimal unknotting crossing data for the torus knot $K(7,4)$ is as follows:
By Lemma 2.7, it is easy to observe that after changing last $m_i (1 \leq i \leq 4)$ crossings in each i^{th} factor, i.e., $\langle 12,17,18,22,23,24\rangle$, in $B(7,4)$ we get $B' = (\sigma_1\sigma_2)^4 = B(3,4)$. Now in $B(3,4)$, $q \equiv 1$ $(mod\ p)$ so by Theorem 3.1, after changing last $m_i(1 \leq i \leq 4)$ crossings in each i^{th} factor, i.e., $\langle 4,5,6\rangle$, in $B(3,4)$ we get a braid whose closure is the trivial knot. This whole procedure is shown pictorially in Figure 4. By Lemma 2.7, the corresponding crossings in $B(7,4)$, $\langle 8,12,13,14,17,18,22,23,24\rangle$, is a minimal unknotting crossing data for $K(7,4)$. Figure 5 shows a minimal unknotting crossing data for $K(7,4)$.

In general, any torus knot $K(p,q)$ can be transformed into the trivial knot by following the same procedure, i.e., for each new pair p,q, change last m_i crossings in i^{th} factor $\sigma_1\sigma_2\cdots\sigma_{p-1}$ in $B(p,q)$. By Lemma 2.7, the corresponding crossings

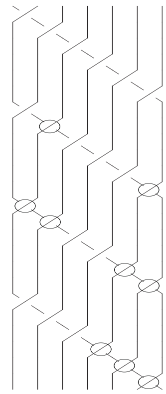

Figure 5: A minimal unknotting crossing data for the torus knot $K(7,4)$

in $K(p,q)$ provide a minimal unknotting crossing data for $K(p,q)$. Observe that if $p<q$, the new pair p',q' obtained from p,q will be $p'=p$ and $q'=q-mp$ such that $q'<p$ and if $q<p$, then p',q' will be $q'=q$ and $p'=p-mq$ such that $p'<q$. Hence, Euclid's algorithm ensures that the number of steps in the above unknotting procedure will be finite and will end when we get a pair p',q' such that $q'\equiv 1$ or $p'-1$ $(mod\ p')$.

Example 3.5. A minimal unknotting crossing data for $K(13,3)$ is $\langle 15,18,21,24,26,27,29,30,32,33,35,36\rangle$.

4. Minimal Unknotting Crossing Data for Torus Links

In this section, we extend the method of finding minimal unknotting crossing data for torus knots to torus links.

Theorem 4.1. *Let $K(p,p)$ be a torus link, then the finite subsequence of crossing data corresponding to last $i-1$ crossings in each i^{th} factor $\sigma_1\sigma_2\cdots\sigma_{p-1}$ of $B(p,p)$ is a minimal unknotting crossing data for $K(p,p)$.*

Proof. It follows from Lemma 2.5 that the finite subsequence of the crossing data corresponding to the last $i-1$ crossings in each i^{th} factor $\sigma_1\sigma_2\cdots\sigma_{p-1}$ of $B(p,p)$ is an unknotting crossing data for $K(p,p)$. Note that the total number of crossings changed (or number of elements of the subsequence) is equal to $\displaystyle\sum_{i=0}^{p-1} i=\frac{(p-1)p}{2}$, and the unknotting number of $K(p,p)$ is $\dfrac{(p-1)(p-1)+(p,p)-1}{2}=\dfrac{(p-1)p}{2}$.

Thus the chosen finite subsequence of crossing data is a minimal unknotting crossing data for $K(p,p)$. □

Remark 4.2. For every p,q with $(p,q)=d$, if $q\equiv a(mod\ p)$, then $(p,a)=$

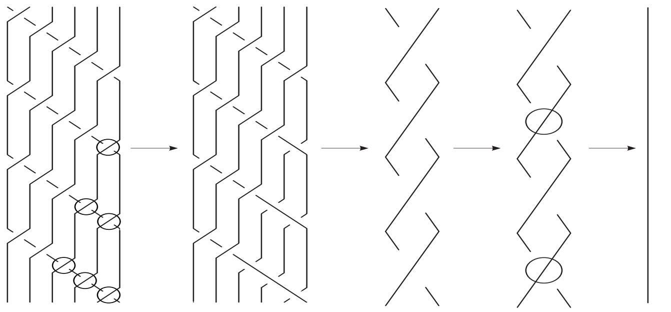

Figure 6: Unknotting procedure for $K(6,4)$

$(p-a,a)=d$ and

$$B(p,q)=(\sigma_1\sigma_2\cdots\sigma_{p-1})^q=(\sigma_1\sigma_2\cdots\sigma_{p-1})^{mp+a} \quad for\ some\ m\geq 0.$$

Let m_i $(1\leq i\leq q)$ be a non negative integer such that $i-1\equiv m_i$ $(mod\ p)$. Then, by changing last m_i crossings in each i^{th} factor $\sigma_1\sigma_2\cdots\sigma_{p-1}$ in $B(p,q)$, we obtain a braid β' equivalent to $\alpha^m\beta$, where
$\alpha=(\sigma_1\sigma_2\cdots\sigma_{p-2}\sigma_{p-1})(\sigma_1\sigma_2\cdots\sigma_{p-2}\sigma_{p-1}^{-1})\cdots(\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{p-2}^{-1}\sigma_{p-1}^{-1})$ and
$\beta=(\sigma_1\sigma_2\cdots\sigma_{p-2}\sigma_{p-1})(\sigma_1\sigma_2\cdots\sigma_{p-2}\sigma_{p-1}^{-1})\cdots(\sigma_1\sigma_2\cdots\sigma_{p-(a-1)}^{-1}\sigma_{p-(a-2)}^{-1}\cdots\sigma_{p-2}^{-1}\sigma_{p-1}^{-1})$.
By Lemma 2.7 and Theorem 4.1, α is a trivial p-braid and β is Markov equivalent to $B(p-a,a)$. Thus we can find a minimal unknotting crossing data for $K(p,q)$ if we can find an unknotting crossing data for $B(p-a,a)$ whose cardinality is equal to $u(K(p,q))\setminus\#(\text{crossings changed in } B(p,q))=\frac{(p-1)(q-1)+d-1}{2}-\frac{mp(p-1)}{2}-\frac{a(a-1)}{2}=\frac{(p-a-1)(a-1)+d-1}{2}$. Observe that this is the unknotting number of $K(p-a,a)$.

Example 4.3. A minimal unknotting crossing data for $K(6,4)$, as shown in Figure 7, is $\langle 6,10,14,15,16,18,19,20\rangle$. The complete procedure to find minimal unknotting crossing data for $K(6,4)$ is shown pictorially in Figure 6.

Now we will provide an unknotting procedure for any torus link of type $K(p,q)$, where $(p,q)=d$ and $q=mp+a$. Unknotting procedure for torus links remains same as for torus knots. Specifically, let $p_1=p$ and $q_1=q$, then define q_{i+1} as the remainder when q_i is divided by p_i and let s_i be the quotient, then

$$q_{i+1}=q_i-s_ip_i.$$

Similarly, define p_{i+1} as the remainder when p_i is divided by q_{i+1} and let t_i be the quotient, then

$$p_{i+1}=p_i-t_iq_{i+1}.$$

According to Euclid's algorithm there exists $n\in\mathbb{N}$ such that q_{n+1} or $p_{n+1}=0$ and the number of steps in Euclid's algorithm is an upper bound for n.

For $i=1,2,\ldots,n$, consider $K(p_i,q_i),K(p_i-q_{i+1},q_{i+1}),K(p_i-2q_{i+1},q_{i+1}),\cdots,K(p_i-(t_i-1)q_{i+1},q_{i+1})$(if $t_i\geq 1$) . Note that, for each i, if we change crossings,

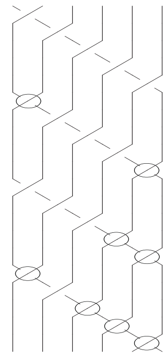

Figure 7: A minimal unknotting crossing data for the torus link $K(6,4)$

corresponding to the last $m_{i_j}(1 \leq i_j \leq q_i$ and $i_j - 1 \equiv m_{i_j}$ $(mod\ p_i))$ crossings of each factor $\sigma_1\sigma_2\cdots\sigma_{p_i-1}$ of $K(p_i,q_i)$, the last $m_{i_j}(1 \leq i_j \leq q_{i+1}$ and $i_j - 1 \equiv m_{i_j}$ $(mod\ p_i - q_{i+1}))$ crossings of each factor $\sigma_1\sigma_2\cdots\sigma_{p_i-q_{i+1}-1}$ of $K(p_i-q_{i+1},q_{i+1})$ and so on upto the last $m_{i_j}(1 \leq i_j \leq q_{i+1}$ and $i_j - 1 \equiv m_{i_j}$ $(mod\ p_i - (t_i-1)q_{i+1}))$ crossings of each factor $\sigma_1\sigma_2\cdots\sigma_{p_i-(t_i-1)q_{i+1}-1}$ of $K(p_i-(t_i-1)q_{i+1},q_{i+1})$, in $K(p_i,q_i)$, then the resultant knot is equivalent to $K(p_{i+1},q_{i+1})$ by Lemma 2.7 and Theorem 4.1. After changing the crossings in the same manner for each i in $K(p_i,q_i)$, by Lemma 2.7, Theorem 4.1 and Remark 4.2, the corresponding crossings in $K(p,q)$ provide a minimal unknotting crossing data for $K(p,q)$.

Minimal unknotting crossing data for some torus knots obtained by the unknotting method discussed in Section 3 are given in Table 1.

5. Unknotting Numbers of Knots

In this section we provide unknotting numbers of some knots by showing them in an unknotting sequence of some torus knots. Also, we discuss how to find knot types from the diagrams obtained by some random crossing changes in minimal unknotting crossing data of torus knots.

Definition 5.1. An *unknotting sequence* for a knot or a link K is a finite sequence of knots or links,

$$K = K_n, K_{n-1}, K_{n-2}, \cdots, K_1, K_0$$

such that

(1) the unknotting number of K_i is i, i.e., $u(K_i) = i$ for $0 \leq i \leq n$ and

(2) two consecutive knots or links of the sequence are related by one crossing change.

Definition 5.2. A knot K is said to be *unknotted* via K' if K' lies in some unknotting sequence of K.

Torus Knot	Minimal Unknotting Crossing Data
$K(4,7)$	$6, 8, 9, 10, 11, 12, 18, 20, 21$
$K(5,6)$	$8, 11, 12, 14, 15, 16, 17, 18, 19, 20$
$K(5,7)$	$8, 11, 12, 14, 15, 16, 17, 18, 19, 20, 26, 28$
$K(5,8)$	$8, 11, 12, 14, 15, 16, 17, 18, 19, 20, 25, 28, 31, 32$
$K(7,4)$	$8, 12, 13, 14, 17, 18, 22, 23, 24$
$K(7,5)$	$7, 12, 17, 18, 19, 22, 23, 24, 27, 28, 29, 30$
$K(8,5)$	$9, 14, 15, 16, 20, 21, 26, 27, 28, 30, 32, 33, 34, 35$

Table 1: Minimal unknotting crossing data

Theorem 5.3. *Let X be a minimal unknotting crossing data in a diagram D of a knot K, and Y be any subsequence of X. If K' is a knot with a diagram D' obtained from D by making crossing changes corresponding to the crossing data of Y, then $u(K') = u(K) \setminus \#(Y)$.*

Proof. Since X is a minimal unknotting crossing data of K, $\#(X) = u(K) = n$ (say). Let Y be any subset of X such that $\#(Y) = m$ ($1 \leq m \leq n$). Observe that every permutation of X gives an unknotting sequence of K. Now consider an unknotting sequence corresponding to the permutation in which the first m elements are chosen from the crossing data of Y. Then K' will be the knot at the $(n-m)^{th}$ position of this unknotting sequence of K.

Hence $u(K') = n - m = u(K) - \#(Y)$. □

Thus, by Theorem 5.3, after changing m crossings from *minimal unknotting crossing data* in the torus knot $K(p,q)$, the unknotting number of the knot-type of the resultant diagram is equal to $u(K(p,q)) - m$. The knot-type for this resultant diagram can be found using knot programs *LinKnot* [4] and *Knotscape* [3]. One can use a braid representation of the resultant diagram as the input for *LinKnot* and can find its minimal Dowker-Thistlethwaite notation (DT code) as an output. Then the knot program *Knotscape* locates the knot-type in H-T knot table [2] using DT code as an input. For example, a minimal unknotting crossing data for $K(5,7)$ is $\langle 8, 11, 12, 14, 15, 16, 17, 18, 19, 20, 26, 28\rangle$. From this minimal unknotting crossing data of $K(5,7)$, if we make crossing changes at $\langle 11, 12, 14, 15, 17, 19, 20, 26, 28\rangle$ crossings, the resultant diagram is the closure of quasitoric braid diagram $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1}\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_4^{-1}\sigma_1\sigma_2$ $\sigma_3\sigma_4\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}$. Using this braid representation as an input, the knot program *LinKnot* provides its minimal Dowkar notation as $\{\{10\}, \{4, 8, 12, 2, 16, 6, 18, 20, 10, 14\}\}$. By using this Dowkar notaion in *Knotscape*, we can identify this knot as ${}_a10_8$.

Since the unknotting number of $K(5,7)$ is 12 and ${}_a10_8$ is obtained by making 9 crossing changes from the above minimal unknotting crossing data of $K(5,7)$, $u({}_a10_8) = u(K(5,7)) - 9 = 3$.

Consider the torus knot $K(5,6)$ and its minimal unknotting crossing data $\langle 8, 11, 12, 14, 15, 16, 17, 18, 19, 20\rangle$. The crossings corresponding to this minimal unknotting crossing data are shown in Figure 8. If we make crossing changes at $\langle 8, 11, 12, 16, 18, 20\rangle$ in $K(5, 6)$, shown in Figure 9, the resultant diagram will be a diagram of ${}_n14_{14274}$. So $u({}_n14_{14274}) = u(K(5, 6)) - 6 = 4$. Similarly, if we make $\langle 8, 11, 12, 16, 20\rangle$ crossing changes in $K(5, 6)$, shown in Figure 10, the resultant diagram will be a diagram of ${}_n14_{24498}$. So $u({}_n14_{24498}) = u(K(5, 6)) - 5 = 5$.

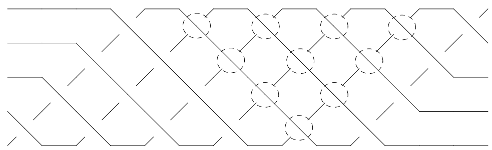

Figure 8: A minimal unknotting crossing data for $K(5, 6)$

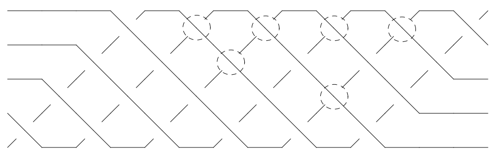

Figure 9: ${}_n14_{14274}$

Some observations for unknotting numbers of knots are given in Tables 2 - 16. Description of each column of these tables 2 - 16 are as follows: First column represents the H-T notaion of the knot K. Unknotting number of K is shown in the second column. Third column shows the torus knot, whose unknotting sequence contains K. Fourth column provide the subsequence of minimal unknotting crossing data of torus knot, whose crossing changes results in K. Each row gives information about the unknotting number of a particular knot and the way to obtain it. For example, unknotting number of ${}_n14_{11242}$ is 5, since this knot can be obtained in an unknotting sequence of $K(5, 7)$ by making crossing changes at $\langle 11, 12, 14, 16, 17, 18, 19\rangle$ from the minimal unknotting crossing data of $K(5, 7)$.

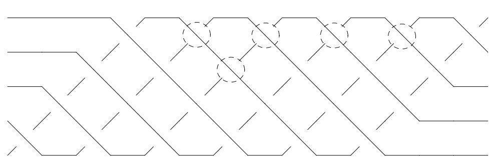

Figure 10: ${}_n14_{24498}$

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
${}_a10_8$	3	$K(5,7)$	$11, 12, 14, 15, 17, 19, 20, 26, 28$
${}_a10_{13}$	3	$K(5,7)$	$8, 11, 12, 15, 17, 18, 19, 26, 28$
${}_a10_{40}$	3	$K(5,7)$	$11, 12, 14, 15, 17, 18, 19, 26, 28$
${}_n10_6$	3	$K(5,8)$	$12, 14, 15, 17, 18, 19, 20, 25, 28, 31, 32$
${}_n10_7$	3	$K(5,8)$	$11, 12, 15, 16, 17, 19, 20, 25, 28, 31, 32$
${}_n10_{11}$	2	$K(5,7)$	$11, 12, 14, 16, 17, 18, 19, 20, 26, 28$
${}_n10_{16}$	2	$K(5,7)$	$8, 12, 14, 16, 17, 18, 19, 20, 26, 28$
${}_n10_{21}$	4	$K(5,7)$	$15, 16, 17, 18, 19, 20, 26, 28$
${}_n10_{22}$	3	$K(5,8)$	$8, 12, 15, 17, 18, 19, 20, 25, 28, 31, 32$
${}_n10_{27}$	4	$K(5,7)$	$14, 15, 17, 18, 19, 20, 26, 28$
${}_n10_{30}$	3	$K(5,7)$	$8, 11, 12, 14, 15, 17, 18, 19, 26$
${}_n10_{31}$	3	$K(5,8)$	$12, 14, 15, 16, 17, 18, 19, 20, 25, 31, 32$
${}_n10_{36}$	4	$K(5,7)$	$14, 16, 17, 18, 19, 20, 26, 28$
${}_n10_{42}$	2	$K(5,7)$	$8, 11, 12, 14, 15, 16, 17, 19, 26, 28$
${}_a11_{94}$	3	$K(5,8)$	$8, 11, 12, 14, 16, 18, 19, 20, 25, 31, 32$
${}_a11_{186}$	3	$K(5,8)$	$8, 11, 12, 14, 15, 16, 18, 19, 20, 25, 31$
${}_a11_{191}$	3	$K(5,8)$	$8, 11, 12, 14, 15, 18, 19, 20, 25, 31, 32$
${}_a11_{227}$	3	$K(5,7)$	$8, 11, 12, 14, 15, 18, 19, 26, 28$
${}_a11_{234}$	4	$K(5,8)$	$8, 11, 14, 15, 18, 19, 25, 28, 31, 32$
${}_a11_{240}$	4	$K(5,7)$	$11, 14, 16, 17, 18, 19, 26, 28$
${}_a11_{263}$	4	$K(7,4)$	$8, 12, 14, 17, 18$
${}_a11_{334}$	4	$K(5,8)$	$8, 11, 12, 14, 15, 18, 19, 25, 31, 32$
${}_a11_{338}$	4	$K(5,7)$	$11, 14, 15, 16, 17, 18, 19, 26$
${}_a11_{367}$	5	$K(5,8)$	$8, 11, 12, 14, 15, 18, 19, 31, 32$
${}_n11_{77}$	4	$K(5,8)$	$8, 12, 14, 15, 16, 17, 19, 28, 31, 32$
${}_n11_{93}$	3	$K(5,8)$	$8, 12, 14, 15, 18, 19, 20, 25, 28, 31, 32$
${}_n11_{95}$	2	$K(5,7)$	$11, 12, 15, 16, 17, 18, 19, 20, 26, 28$
${}_n11_{105}$	2	$K(5,7)$	$8, 11, 12, 14, 16, 17, 18, 19, 26, 28$
${}_n11_{118}$	2	$K(5,7)$	$8, 11, 12, 16, 17, 18, 19, 20, 26, 28$
${}_n11_{136}$	3	$K(5,7)$	$11, 12, 14, 15, 18, 19, 20, 26, 28$
${}_a12_{276}$	4	$K(7,4)$	$8, 14, 17, 18, 22$
${}_n12_{74}$	4	$K(5,7)$	$11, 14, 16, 18, 19, 20, 26, 28$
${}_n12_{91}$	4	$K(5,7)$	$8, 11, 16, 17, 19, 20, 26, 28$
${}_n12_{96}$	3	$K(5,7)$	$8, 12, 15, 16, 17, 19, 20, 26, 28$
${}_n12_{105}$	4	$K(5,7)$	$12, 15, 16, 17, 19, 20, 26, 28$
${}_n12_{110}$	3	$K(5,7)$	$8, 11, 12, 15, 17, 19, 20, 26, 28$
${}_n12_{136}$	4	$K(5,8)$	$8, 11, 12, 14, 16, 17, 18, 20, 25, 28$
${}_n12_{153}$	4	$K(5,7)$	$14, 15, 16, 17, 18, 19, 26, 28$
${}_n12_{166}$	4	$K(5,7)$	$12, 14, 15, 16, 17, 18, 19, 26$
${}_n12_{187}$	4	$K(5,8)$	$8, 12, 14, 15, 16, 17, 18, 20, 31, 32$
${}_n12_{234}$	3	$K(5,7)$	$8, 12, 14, 16, 18, 19, 20, 26, 28$
${}_n12_{242}$	5	$K(5,7)$	$11, 12, 14, 15, 17, 18, 28$
${}_n12_{243}$	4	$K(5,7)$	$8, 12, 15, 17, 18, 19, 20, 26$

Table 2: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
$_n12_{244}$	4	$K(5,7)$	$11, 12, 14, 15, 17, 18, 26, 28$
$_n12_{289}$	3	$K(5,7)$	$11, 12, 14, 15, 16, 18, 19, 20, 26$
$_n12_{292}$	4	$K(5,7)$	$12, 14, 15, 16, 17, 18, 20, 26$
$_n12_{305}$	4	$K(5,7)$	$12, 15, 16, 17, 18, 19, 26, 28$
$_n12_{308}$	3	$K(5,7)$	$8, 11, 12, 14, 15, 18, 19, 20, 26$
$_n12_{328}$	4	$K(5,7)$	$12, 14, 15, 16, 19, 20, 26, 28$
$_n12_{338}$	4	$K(5,7)$	$11, 12, 14, 18, 19, 20, 26, 28$
$_n12_{341}$	3	$K(5,7)$	$8, 11, 12, 15, 16, 17, 18, 19, 26$
$_n12_{374}$	4	$K(5,7)$	$8, 12, 15, 18, 19, 20, 26, 28$
$_n12_{386}$	4	$K(5,7)$	$11, 14, 15, 18, 19, 20, 26, 28$
$_n12_{417}$	4	$K(5,7)$	$11, 12, 14, 16, 17, 18, 19, 28$
$_n12_{426}$	4	$K(5,7)$	$12, 14, 15, 16, 17, 19, 20, 28$
$_n12_{472}$	5	$K(5,7)$	$11, 12, 14, 15, 16, 26, 28$
$_n12_{474}$	4	$K(5,7)$	$8, 11, 12, 15, 17, 18, 19, 26$
$_n12_{502}$	4	$K(5,7)$	$11, 12, 14, 15, 16, 19, 20, 28$
$_n12_{503}$	3	$K(5,7)$	$8, 11, 12, 15, 16, 18, 19, 26, 28$
$_n12_{518}$	4	$K(5,8)$	$8, 11, 14, 15, 16, 17, 18, 20, 25, 28$
$_n12_{528}$	3	$K(5,8)$	$8, 11, 12, 15, 18, 19, 20, 25, 28, 31, 32$
$_n12_{574}$	5	$K(5,7)$	$11, 12, 14, 15, 17, 18, 19$
$_n12_{576}$	4	$K(5,7)$	$11, 12, 14, 15, 17, 18, 19, 26$
$_n12_{591}$	4	$K(5,7)$	$12, 15, 17, 18, 19, 20, 26, 28$
$_n12_{594}$	3	$K(5,7)$	$8, 11, 12, 15, 18, 19, 20, 26, 28$
$_n12_{640}$	4	$K(5,7)$	$12, 14, 15, 18, 19, 20, 26, 28$
$_n12_{647}$	4	$K(5,7)$	$12, 14, 15, 17, 18, 20, 26, 28$
$_n12_{660}$	3	$K(5,8)$	$8, 12, 14, 15, 16, 18, 19, 20, 25, 31, 32$
$_n12_{674}$	3	$K(5,7)$	$11, 12, 14, 16, 17, 18, 20, 26, 28$
$_n12_{679}$	5	$K(5,7)$	$8, 14, 16, 18, 19, 20, 28$
$_n12_{680}$	4	$K(5,7)$	$8, 12, 15, 17, 19, 20, 26, 28$
$_n12_{682}$	3	$K(5,7)$	$11, 12, 14, 16, 17, 19, 20, 26, 28$
$_n12_{688}$	5	$K(5,7)$	$11, 12, 14, 15, 16, 17, 20$
$_n12_{689}$	4	$K(5,7)$	$8, 12, 15, 16, 17, 18, 19, 26$
$_n12_{694}$	4	$K(5,7)$	$11, 14, 15, 16, 19, 20, 26, 28$
$_n12_{725}$	5	$K(5,7)$	$11, 12, 14, 15, 16, 20, 28$
$_n12_{750}$	3	$K(5,8)$	$11, 12, 14, 16, 17, 18, 20, 25, 28, 31, 32$
$_n12_{830}$	3	$K(5,8)$	$8, 11, 12, 16, 17, 18, 20, 25, 28, 31, 32$
$_n12_{850}$	4	$K(5,7)$	$11, 12, 15, 17, 18, 19, 20, 26$
$_n12_{882}$	3	$K(5,7)$	$11, 12, 14, 16, 17, 18, 19, 20, 26$
$_n12_{888}$	5	$K(5,8)$	$8, 11, 14, 15, 16, 17, 18, 20, 28$
$_a13_{1073}$	4	$K(7,4)$	$8, 14, 17, 22, 23$
$_a13_{2705}$	4	$K(7,5)$	$7, 17, 19, 22, 23, 27, 28, 29$
$_a13_{3092}$	5	$K(7,5)$	$17, 19, 22, 23, 27, 28, 29$
$_a13_{4878}$	6	$K(7,5)$	$17, 22, 23, 27, 28, 29$
$_n13_{241}$	5	$K(5,7)$	$8, 16, 17, 19, 20, 26, 28$

Table 3: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
${}_n13_{300}$	5	$K(5,8)$	$8,11,12,16,17,18,20,25,28$
${}_n13_{588}$	3	$K(5,7)$	$8,12,14,16,17,19,20,26,28$
${}_n13_{596}$	4	$K(5,7)$	$8,12,15,17,18,19,26,28$
${}_n13_{604}$	5	$K(5,7)$	$8,14,17,19,20,26,28$
${}_n13_{631}$	3	$K(5,8)$	$11,12,15,16,17,18,20,25,28,31,32$
${}_n13_{730}$	4	$K(5,7)$	$12,14,15,17,19,20,26,28$
${}_n13_{770}$	4	$K(5,7)$	$8,11,12,14,19,20,26,28$
${}_n13_{792}$	4	$K(5,8)$	$8,12,14,15,16,17,19,25,31,32$
${}_n13_{833}$	4	$K(5,8)$	$8,11,12,16,17,19,20,25,31,32$
${}_n13_{883}$	4	$K(5,7)$	$8,11,16,18,19,20,26,28$
${}_n13_{915}$	3	$K(5,7)$	$11,12,14,15,16,17,19,26,28$
${}_n13_{930}$	4	$K(5,8)$	$8,11,12,14,18,19,20,25,28,32$
${}_n13_{981}$	5	$K(7,5)$	$7,12,19,22,23,24,28$
${}_n13_{986}$	3	$K(5,8)$	$8,11,12,15,17,18,20,25,28,31,32$
${}_n13_{1104}$	5	$K(5,8)$	$8,11,12,15,16,18,25,28,32$
${}_n13_{1176}$	5	$K(5,7)$	$8,14,15,16,18,20,28$
${}_n13_{1291}$	5	$K(5,8)$	$8,11,12,15,17,18,20,31,32$
${}_n13_{1320}$	5	$K(5,8)$	$8,11,12,15,17,18,20,25,28$
${}_n13_{1643}$	4	$K(5,7)$	$11,12,14,15,17,19,20,26$
${}_n13_{1696}$	3	$K(5,7)$	$11,15,16,17,18,19,20,26,28$
${}_n13_{1779}$	3	$K(5,7)$	$8,12,14,16,17,18,19,20,26$
${}_n13_{1905}$	4	$K(5,7)$	$8,12,14,15,18,19,26,28$
${}_n13_{1907}$	3	$K(5,7)$	$8,12,14,15,16,17,19,26,28$
${}_n13_{1928}$	4	$K(5,7)$	$12,14,15,17,18,19,26,28$
${}_n13_{2019}$	4	$K(5,7)$	$8,11,12,15,17,18,26,28$
${}_n13_{2102}$	3	$K(5,7)$	$8,12,16,17,18,19,20,26,28$
${}_n13_{2113}$	3	$K(5,7)$	$8,11,12,15,16,17,18,26,28$
${}_n13_{2192}$	4	$K(5,7)$	$8,15,16,17,18,19,26,28$
${}_n13_{2405}$	5	$K(5,8)$	$8,11,14,15,16,17,18,20,31$
${}_n13_{2710}$	4	$K(7,4)$	$8,13,14,22,23$
${}_n13_{2941}$	4	$K(5,7)$	$8,11,14,16,18,19,26,28$
${}_n13_{2987}$	3	$K(5,7)$	$11,12,15,16,18,19,20,26,28$
${}_n13_{3034}$	3	$K(5,7)$	$8,11,12,16,17,18,19,20,26$
${}_n13_{3054}$	4	$K(5,7)$	$12,14,15,17,18,19,20,26$
${}_n13_{3111}$	4	$K(5,7)$	$11,12,14,15,19,20,26,28$
${}_n13_{3137}$	4	$K(5,7)$	$8,11,14,15,18,19,26,28$
${}_n13_{3206}$	3	$K(5,7)$	$11,12,14,16,17,18,19,26,28$
${}_n13_{3354}$	4	$K(5,7)$	$12,14,15,16,17,18,26,28$
${}_n13_{3527}$	3	$K(5,7)$	$8,11,12,14,15,16,19,26,28$
${}_n13_{4116}$	3	$K(5,8)$	$8,12,15,16,17,18,20,25,28,31,32$
${}_n13_{4143}$	3	$K(5,7)$	$11,12,14,15,16,18,19,26,28$
${}_n13_{4178}$	4	$K(7,5)$	$7,17,18,19,23,24,27,30$
${}_n13_{4242}$	4	$K(5,8)$	$8,11,14,15,16,19,20,25,31,32$

Table 4: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
$_n13_{4569}$	4	$K(5,7)$	$8,12,15,16,17,18,26,28$
$_n13_{4587}$	5	$K(5,7)$	$8,12,15,16,17,18,28$
$_n13_{4635}$	4	$K(5,8)$	$8,11,12,14,19,20,25,28,31,32$
$_n13_{4639}$	4	$K(5,8)$	$8,11,12,16,19,20,25,28,31,32$
$_n13_{5016}$	5	$K(5,8)$	$8,11,12,15,18,25,28,31,32$
$_n14_{2125}$	4	$K(5,7)$	$11,12,15,17,19,20,26,28$
$_n14_{2559}$	5	$K(5,8)$	$8,11,12,14,15,19,25,28,32$
$_n14_{3405}$	3	$K(5,7)$	$11,12,15,16,17,19,20,26,28$
$_n14_{3478}$	3	$K(5,7)$	$8,11,12,16,17,19,20,26,28$
$_n14_{4701}$	5	$K(5,7)$	$11,12,14,15,20,26,28$
$_n14_{4738}$	5	$K(5,8)$	$8,11,12,14,15,19,25,31,32$
$_n14_{5339}$	5	$K(7,5)$	$7,17,19,23,24,28,30$
$_n14_{6022}$	6	$K(5,8)$	$8,11,14,15,16,18,28,32$
$_n14_{6023}$	5	$K(5,6)$	$8,12,16,17,19$
$_n14_{6024}$	5	$K(5,7)$	$11,12,14,15,17,18,26$
$_n14_{6690}$	3	$K(5,8)$	$8,11,15,16,17,19,20,25,28,31,32$
$_n14_{7513}$	4	$K(5,7)$	$8,12,15,16,19,20,26,28$
$_n14_{7543}$	5	$K(5,7)$	$8,15,17,18,20,26,28$
$_n14_{7549}$	4	$K(5,7)$	$8,11,12,15,17,19,20,26$
$_n14_{7554}$	3	$K(5,7)$	$11,12,15,16,17,18,19,26,28$
$_n14_{7560}$	5	$K(5,7)$	$8,15,17,19,20,26,28$
$_n14_{7725}$	5	$K(5,7)$	$8,14,16,18,19,26,28$
$_n14_{7838}$	5	$K(5,7)$	$11,12,14,15,17,26,28$
$_n14_{7841}$	3	$K(5,8)$	$8,12,15,16,17,19,20,25,28,31,32$
$_n14_{7846}$	3	$K(5,8)$	$8,12,14,15,16,17,19,20,25,31,32$
$_n14_{8330}$	3	$K(5,7)$	$8,12,14,16,17,18,19,26,28$
$_n14_{8335}$	4	$K(5,6)$	$8,11,12,15,18,19$
$_n14_{8360}$	3	$K(5,8)$	$8,11,12,15,17,19,20,25,28,31,32$
$_n14_{8403}$	4	$K(5,8)$	$8,11,14,15,18,19,20,25,28,32$
$_n14_{8510}$	5	$K(5,8)$	$8,11,12,14,16,17,18,20,32$
$_n14_{8994}$	4	$K(7,5)$	$7,17,19,23,24,27,29,30$
$_n14_{9211}$	4	$K(5,8)$	$8,12,14,15,16,17,18,20,25,28$
$_n14_{9225}$	4	$K(7,5)$	$7,17,18,19,22,23,28,29$
$_n14_{9274}$	5	$K(5,8)$	$8,11,12,14,15,18,25,28,32$
$_n14_{9515}$	4	$K(5,7)$	$12,14,15,16,17,19,20,26$
$_n14_{9525}$	5	$K(5,7)$	$8,15,16,17,18,20,28$
$_n14_{9537}$	4	$K(5,7)$	$8,12,15,16,17,19,20,26$
$_n14_{9578}$	5	$K(5,7)$	$8,11,16,17,18,19,28$
$_n14_{9729}$	3	$K(5,7)$	$11,12,14,16,18,19,20,26,28$
$_n14_{9742}$	5	$K(5,7)$	$8,12,15,17,19,20,28$
$_n14_{10290}$	3	$K(5,7)$	$8,11,12,16,17,18,19,26,28$
$_n14_{10320}$	5	$K(5,8)$	$8,11,14,15,16,17,19,25,31$
$_n14_{10352}$	4	$K(5,6)$	$8,11,12,16,18,19$

Table 5: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
$_n14_{10526}$	5	$K(5,8)$	$8, 11, 14, 15, 16, 18, 19, 20, 28$
$_n14_{10586}$	6	$K(7,4)$	$8, 12, 14, 23$
$_n14_{10943}$	5	$K(5,8)$	$8, 11, 12, 14, 16, 18, 19, 25, 31$
$_n14_{11242}$	5	$K(5,7)$	$11, 12, 14, 16, 17, 18, 19$
$_n14_{11559}$	4	$K(5,7)$	$8, 11, 12, 14, 15, 18, 19, 26$
$_n14_{11655}$	4	$K(5,7)$	$11, 12, 14, 15, 16, 19, 20, 26$
$_n14_{11799}$	5	$K(5,8)$	$8, 11, 12, 14, 17, 19, 25, 28, 32$
$_n14_{11804}$	4	$K(5,7)$	$8, 11, 16, 17, 18, 19, 26, 28$
$_n14_{12201}$	6	$K(5,8)$	$8, 11, 14, 15, 16, 18, 19, 28$
$_n14_{12461}$	5	$K(5,7)$	$8, 12, 15, 18, 19, 20, 26$
$_n14_{12557}$	5	$K(5,7)$	$11, 12, 14, 15, 16, 19, 20$
$_n14_{12558}$	4	$K(5,7)$	$8, 11, 12, 15, 16, 18, 19, 26$
$_n14_{12695}$	5	$K(5,8)$	$8, 11, 12, 16, 18, 19, 25, 31, 32$
$_n14_{12701}$	5	$K(5,7)$	$8, 12, 14, 15, 18, 19, 28$
$_n14_{13103}$	4	$K(5,8)$	$8, 11, 15, 18, 19, 20, 25, 28, 31, 32$
$_n14_{13108}$	4	$K(5,8)$	$8, 11, 12, 15, 18, 19, 20, 25, 28, 32$
$_n14_{13125}$	4	$K(5,8)$	$8, 11, 12, 14, 15, 18, 20, 25, 31, 32$
$_n14_{13641}$	4	$K(5,7)$	$8, 12, 15, 16, 17, 19, 20, 28$
$_n14_{13659}$	4	$K(5,7)$	$11, 12, 15, 16, 17, 19, 20, 28$
$_n14_{13683}$	5	$K(5,7)$	$8, 14, 17, 18, 19, 26, 28$
$_n14_{13754}$	3	$K(5,7)$	$8, 11, 12, 14, 16, 18, 19, 20, 28$
$_n14_{13912}$	4	$K(5,7)$	$8, 11, 15, 16, 19, 20, 26, 28$
$_n14_{13917}$	5	$K(5,7)$	$8, 14, 15, 16, 19, 20, 26$
$_n14_{13918}$	4	$K(5,8)$	$8, 11, 12, 14, 18, 19, 20, 25, 31, 32$
$_n14_{13982}$	4	$K(5,8)$	$8, 11, 14, 15, 19, 20, 25, 28, 31, 32$
$_n14_{14025}$	5	$K(5,7)$	$8, 14, 15, 18, 19, 26, 28$
$_n14_{14269}$	5	$K(5,7)$	$8, 15, 18, 19, 20, 26, 28$
$_n14_{14356}$	5	$K(5,7)$	$8, 15, 16, 17, 18, 19, 20$
$_n14_{14431}$	5	$K(5,8)$	$8, 11, 12, 14, 16, 18, 19, 20, 31$
$_n14_{14560}$	5	$K(5,8)$	$8, 11, 14, 15, 16, 18, 19, 20, 31$
$_n14_{14620}$	5	$K(5,8)$	$8, 11, 12, 14, 15, 18, 25, 31, 32$
$_n14_{14776}$	4	$K(5,7)$	$11, 12, 14, 15, 18, 19, 20, 26$
$_n14_{15241}$	4	$K(5,7)$	$8, 11, 12, 15, 16, 17, 18, 26$
$_n14_{15552}$	4	$K(5,7)$	$8, 11, 12, 14, 15, 19, 20, 26$
$_n14_{15798}$	4	$K(5,7)$	$11, 14, 15, 16, 18, 19, 20, 26$
$_n14_{15856}$	6	$K(5,8)$	$8, 11, 14, 15, 16, 18, 28, 31$
$_n14_{16016}$	4	$K(5,7)$	$8, 11, 15, 16, 17, 19, 20, 26$
$_n14_{16147}$	5	$K(5,7)$	$11, 12, 14, 15, 17, 20, 26$
$_n14_{16235}$	4	$K(5,8)$	$8, 11, 12, 14, 15, 18, 20, 25, 28, 32$
$_n14_{16364}$	5	$K(5,7)$	$8, 14, 15, 18, 19, 20, 26$
$_n14_{16367}$	4	$K(5,7)$	$8, 11, 12, 15, 18, 19, 20, 26$
$_n14_{16415}$	4	$K(5,7)$	$8, 11, 15, 16, 18, 19, 26, 28$
$_n14_{16755}$	4	$K(5,7)$	$8, 11, 12, 14, 18, 19, 26, 28$

Table 6: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
${}_n14_{16872}$	4	$K(5,6)$	$11, 12, 14, 16, 18, 19$
${}_n14_{17160}$	5	$K(5,7)$	$8, 16, 17, 18, 19, 26, 28$
${}_n14_{17191}$	5	$K(5,7)$	$8, 14, 15, 16, 18, 19, 28$
${}_n14_{17435}$	5	$K(5,8)$	$8, 11, 14, 15, 16, 19, 20, 25, 28$
${}_n14_{17880}$	4	$K(5,8)$	$8, 11, 14, 15, 18, 19, 20, 25, 31, 32$
${}_n14_{17916}$	5	$K(7,5)$	$7, 12, 19, 22, 23, 28, 29$
${}_n14_{18079}$	6	$K(5,8)$	$8, 11, 14, 15, 16, 18, 19, 31$
${}_n14_{18194}$	4	$K(5,7)$	$8, 11, 15, 18, 19, 20, 26, 28$
${}_n14_{18217}$	5	$K(5,8)$	$8, 11, 14, 15, 16, 19, 20, 28, 31$
${}_n14_{18339}$	5	$K(5,7)$	$8, 11, 14, 15, 18, 26, 28$
${}_n14_{18363}$	5	$K(5,7)$	$8, 14, 15, 16, 18, 26, 28$
${}_n14_{18372}$	5	$K(5,7)$	$8, 14, 15, 17, 19, 20, 26$
${}_n14_{18417}$	4	$K(5,6)$	$8, 11, 12, 15, 16, 19$
${}_n14_{18550}$	4	$K(5,6)$	$8, 11, 12, 15, 16, 18$
${}_n14_{18605}$	4	$K(5,7)$	$11, 12, 15, 18, 19, 20, 26, 28$
${}_n14_{19738}$	5	$K(5,8)$	$8, 11, 12, 14, 15, 18, 19, 25, 31$
${}_n14_{19763}$	5	$K(5,8)$	$8, 11, 12, 14, 18, 19, 25, 31, 32$
${}_n14_{19768}$	5	$K(5,7)$	$8, 11, 14, 15, 19, 26, 28$
${}_n14_{19778}$	5	$K(5,8)$	$8, 11, 14, 15, 16, 18, 20, 28, 31$
${}_n14_{19894}$	5	$K(5,8)$	$8, 11, 14, 15, 16, 18, 20, 31, 32$
${}_n14_{19905}$	4	$K(7,5)$	$7, 17, 18, 19, 23, 27, 28, 29$
${}_n14_{19947}$	5	$K(5,8)$	$8, 11, 12, 14, 15, 18, 20, 28, 31$
${}_n14_{19973}$	4	$K(5,6)$	$8, 11, 12, 15, 16, 17$
${}_n14_{20441}$	6	$K(7,5)$	$17, 19, 22, 23, 24, 28$
${}_n14_{20613}$	6	$K(7,5)$	$17, 19, 24, 27, 28, 29$
${}_n14_{20639}$	5	$K(5,7)$	$8, 14, 16, 17, 18, 26, 28$
${}_n14_{20750}$	5	$K(5,7)$	$8, 11, 14, 16, 18, 26, 28$
${}_n14_{20757}$	5	$K(5,8)$	$8, 11, 12, 15, 18, 19, 25, 28, 32$
${}_n14_{20761}$	5	$K(7,5)$	$7, 17, 18, 23, 27, 29, 30$
${}_n14_{21024}$	5	$K(5,7)$	$8, 12, 15, 18, 20, 26, 28$
${}_n14_{21324}$	6	$K(5,8)$	$8, 11, 12, 17, 19, 28, 31, 32$
${}_n14_{21325}$	5	$K(5,7)$	$8, 15, 16, 17, 18, 19, 26$
${}_n14_{21785}$	5	$K(5,7)$	$8, 14, 15, 16, 18, 19, 26$
${}_n14_{21881}$	6	$K(5,8)$	$8, 11, 14, 15, 16, 17, 28, 32$
${}_n14_{21883}$	4	$K(5,7)$	$8, 12, 14, 16, 18, 19, 20, 28$
${}_n14_{22068}$	5	$K(5,7)$	$8, 11, 12, 15, 17, 26, 28$
${}_n14_{22343}$	4	$K(5,7)$	$11, 12, 14, 16, 17, 18, 20, 28$
${}_n14_{22361}$	5	$K(5,6)$	$11, 12, 16, 17, 19$
${}_n14_{22448}$	5	$K(5,6)$	$8, 11, 12, 15, 19$
${}_n14_{23835}$	5	$K(5,7)$	$8, 11, 12, 19, 20, 26, 28$
${}_n14_{23842}$	4	$K(5,7)$	$8, 11, 12, 15, 19, 20, 26, 28$
${}_n14_{23958}$	5	$K(5,7)$	$8, 12, 15, 16, 18, 26, 28$
${}_n14_{24000}$	4	$K(5,7)$	$12, 14, 16, 18, 19, 20, 26, 28$

Table 7: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
$_n14_{24006}$	5	$K(5,7)$	$8,11,12,15,17,18,26$
$_n14_{24016}$	5	$K(5,6)$	$11,12,16,18,19$
$_n14_{24120}$	5	$K(5,7)$	$8,12,15,16,17,20,28$
$_n14_{24170}$	4	$K(5,7)$	$12,14,16,17,18,20,26,28$
$_n14_{24177}$	5	$K(5,6)$	$8,11,15,18,19$
$_n14_{24498}$	5	$K(5,6)$	$8,12,16,19,20$
$_n14_{24551}$	6	$K(5,7)$	$12,14,18,19,20,26$
$_n14_{24763}$	6	$K(5,8)$	$8,11,12,14,17,18,19,31$
$_n14_{24778}$	5	$K(5,6)$	$8,12,15,16,17$
$_n14_{25987}$	5	$K(5,7)$	$8,15,16,17,18,20,26$
$_n14_{26185}$	5	$K(5,7)$	$8,11,14,15,17,18,26$
$_n14_{26215}$	4	$K(5,8)$	$8,11,12,15,18,19,20,25,31,32$
$_n14_{27153}$	4	$K(5,8)$	$8,11,12,14,17,18,20,25,31,32$
$_n14_{27156}$	4	$K(5,8)$	$8,12,14,15,16,17,19,20,25,32$
$_n14_{27179}$	6	$K(5,7)$	$12,14,17,18,20,28$
$_n14_{27214}$	6	$K(5,7)$	$15,16,17,19,20,28$
$_n14_{27233}$	6	$K(5,8)$	$8,11,12,14,16,18,31,32$
$_n14_{27326}$	5	$K(5,6)$	$8,16,17,18,20$
$_n14_{27342}$	4	$K(5,8)$	$8,11,14,15,17,18,19,20,25,31$
$_n15_{5190}$	5	$K(7,5)$	$7,12,18,19,22,24,30$
$_n15_{5951}$	5	$K(7,5)$	$7,12,18,19,22,23,29$
$_n15_{6144}$	5	$K(7,5)$	$7,12,18,19,22,29,30$
$_n15_{7047}$	5	$K(5,8)$	$8,11,12,17,19,20,25,28,32$
$_n15_{9534}$	5	$K(7,5)$	$7,12,18,19,23,27,30$
$_n15_{11622}$	2	$K(5,7)$	$8,11,12,15,16,17,19,20,26,28$
$_n15_{11690}$	6	$K(5,8)$	$8,11,12,17,19,20,28,31$
$_n15_{11691}$	4	$K(5,7)$	$12,14,16,17,19,20,26,28$
$_n15_{11725}$	5	$K(5,7)$	$8,12,15,17,20,26,28$
$_n15_{11754}$	5	$K(5,7)$	$8,12,17,18,19,26,28$
$_n15_{11787}$	6	$K(5,8)$	$8,11,12,17,19,20,25,28$
$_n15_{11813}$	4	$K(5,8)$	$8,11,12,14,16,17,18,25,28,32$
$_n15_{11941}$	3	$K(5,8)$	$8,12,14,15,17,19,20,25,28,31,32$
$_n15_{11995}$	6	$K(5,8)$	$8,11,12,16,17,18,25,28$
$_n15_{11996}$	4	$K(5,7)$	$11,12,14,16,17,20,26,28$
$_n15_{12113}$	4	$K(5,8)$	$8,11,12,16,17,18,20,25,28,32$
$_n15_{12266}$	5	$K(5,6)$	$11,15,16,18,19$
$_n15_{12301}$	5	$K(5,7)$	$8,12,14,17,18,19,26$
$_n15_{12324}$	5	$K(5,8)$	$8,11,12,15,19,25,28,31,32$
$_n15_{12325}$	6	$K(5,8)$	$8,11,12,17,19,20,31,32$
$_n15_{12332}$	5	$K(5,8)$	$8,11,12,14,17,19,20,25,32$
$_n15_{12642}$	5	$K(5,8)$	$8,11,12,16,17,19,20,25,31$
$_n15_{12951}$	5	$K(5,7)$	$8,14,16,17,18,19,26$
$_n15_{12957}$	3	$K(5,7)$	$8,11,12,14,16,19,20,26,28$

Table 8: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
$_n15_{13123}$	5	$K(7,5)$	$7,12,18,24,27,28,30$
$_n15_{13342}$	4	$K(7,4)$	$8,12,14,22,24$
$_n15_{13465}$	6	$K(5,8)$	$8,11,12,14,16,18,25,28$
$_n15_{13551}$	5	$K(7,5)$	$7,12,18,19,22,23,30$
$_n15_{13583}$	4	$K(5,8)$	$8,11,12,15,16,17,18,25,28,32$
$_n15_{14445}$	5	$K(7,5)$	$7,17,19,22,23,24,28$
$_n15_{14867}$	5	$K(5,8)$	$8,11,12,16,17,19,20,28,32$
$_n15_{15531}$	5	$K(7,5)$	$7,17,18,19,22,24,30$
$_n15_{15791}$	5	$K(5,8)$	$8,11,12,17,19,20,25,31,32$
$_n15_{18290}$	4	$K(7,4)$	$8,14,18,22,23$
$_n15_{18579}$	5	$K(5,8)$	$8,11,12,16,18,19,20,28,32$
$_n15_{22010}$	4	$K(5,7)$	$8,12,14,16,17,20,26,28$
$_n15_{22078}$	6	$K(5,8)$	$8,11,14,15,16,17,20,32$
$_n15_{22172}$	5	$K(5,7)$	$8,12,15,17,18,19,26$
$_n15_{22279}$	4	$K(5,7)$	$8,12,14,16,19,20,26,28$
$_n15_{22340}$	6	$K(5,8)$	$8,11,14,15,16,17,20,31$
$_n15_{22479}$	5	$K(5,7)$	$8,14,15,16,17,19,26$
$_n15_{22503}$	5	$K(5,8)$	$8,11,12,14,18,19,20,28,32$
$_n15_{22790}$	5	$K(5,8)$	$8,11,12,14,15,19,20,25,31$
$_n15_{22968}$	5	$K(5,6)$	$8,11,15,19,20$
$_n15_{23154}$	4	$K(5,8)$	$8,11,12,15,16,17,18,25,31,32$
$_n15_{23225}$	6	$K(5,8)$	$8,11,12,15,17,19,25,28$
$_n15_{23298}$	4	$K(5,8)$	$8,12,14,15,16,17,19,20,28,31$
$_n15_{23603}$	6	$K(5,8)$	$8,11,12,15,16,18,25,28$
$_n15_{24293}$	5	$K(7,5)$	$7,17,23,24,27,28,30$
$_n15_{24391}$	4	$K(7,5)$	$7,17,18,19,23,27,29,30$
$_n15_{24804}$	5	$K(7,5)$	$7,17,18,22,24,28,30$
$_n15_{25153}$	5	$K(5,8)$	$8,11,12,14,16,18,20,28,32$
$_n15_{25605}$	5	$K(7,5)$	$7,17,19,24,27,28,29$
$_n15_{26021}$	5	$K(5,8)$	$8,11,12,18,19,20,25,28,32$
$_n15_{26519}$	4	$K(5,8)$	$8,11,12,15,17,18,20,25,31,32$
$_n15_{26952}$	4	$K(5,8)$	$8,11,15,17,18,20,25,28,31,32$
$_n15_{27229}$	6	$K(5,8)$	$8,11,12,16,18,19,28,32$
$_n15_{27461}$	4	$K(5,8)$	$8,11,12,15,17,18,20,25,28,32$
$_n15_{30327}$	6	$K(5,8)$	$8,11,14,15,16,17,20,28$
$_n15_{30328}$	5	$K(5,6)$	$12,14,16,17,18$
$_n15_{30333}$	4	$K(5,7)$	$8,12,14,16,18,20,26,28$
$_n15_{30419}$	6	$K(5,8)$	$8,11,12,15,18,19,25,28$
$_n15_{32158}$	6	$K(5,8)$	$8,11,12,18,19,25,28,32$
$_n15_{33401}$	5	$K(5,8)$	$8,11,12,15,17,19,25,28,32$
$_n15_{33411}$	5	$K(5,8)$	$8,11,12,15,16,17,19,28,32$
$_n15_{38208}$	5	$K(5,7)$	$8,14,15,16,18,20,26$
$_n15_{40125}$	5	$K(5,7)$	$8,11,15,17,18,26,28$

Table 9: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
$_n15_{40183}$	5	$K(5,7)$	$8, 15, 17, 18, 19, 26, 28$
$_n15_{40211}$	6	$K(5,8)$	$8, 11, 14, 15, 16, 17, 25, 32$
$_n15_{40214}$	4	$K(5,7)$	$12, 16, 17, 18, 19, 20, 26, 28$
$_n15_{40293}$	5	$K(5,7)$	$8, 11, 16, 18, 19, 20, 26$
$_n15_{40693}$	4	$K(5,7)$	$8, 11, 15, 16, 17, 18, 26, 28$
$_n15_{41009}$	5	$K(5,6)$	$11, 15, 16, 17, 19$
$_n15_{41024}$	4	$K(5,7)$	$11, 12, 14, 16, 17, 18, 20, 26$
$_n15_{41183}$	4	$K(5,7)$	$11, 15, 17, 18, 19, 20, 26, 28$
$_n15_{41185}$	6	$K(5,8)$	$8, 11, 14, 15, 16, 18, 25, 32$
$_n15_{41228}$	5	$K(5,7)$	$8, 12, 15, 18, 19, 26, 28$
$_n15_{41316}$	6	$K(5,7)$	$11, 17, 18, 20, 26, 28$
$_n15_{41337}$	5	$K(5,7)$	$8, 12, 14, 15, 18, 20, 26$
$_n15_{41357}$	4	$K(5,8)$	$8, 12, 14, 15, 16, 17, 19, 20, 25, 28$
$_n15_{43553}$	4	$K(5,7)$	$11, 16, 17, 18, 19, 20, 26, 28$
$_n15_{43686}$	6	$K(5,8)$	$8, 11, 12, 17, 18, 20, 28, 32$
$_n15_{43833}$	6	$K(5,7)$	$11, 17, 19, 20, 26, 28$
$_n15_{45028}$	4	$K(5,7)$	$11, 12, 14, 16, 17, 19, 26, 28$
$_n15_{47690}$	4	$K(5,7)$	$8, 12, 16, 17, 18, 20, 26, 28$
$_n15_{49408}$	3	$K(5,7)$	$8, 11, 12, 14, 16, 18, 19, 26, 28$
$_n15_{52191}$	5	$K(5,8)$	$8, 11, 12, 14, 16, 18, 20, 28, 31$
$_n15_{52210}$	3	$K(5,8)$	$8, 11, 12, 15, 16, 17, 19, 20, 25, 31, 32$
$_n15_{52885}$	4	$K(5,8)$	$8, 11, 14, 15, 16, 17, 18, 20, 25, 31$
$_n15_{52931}$	4	$K(5,7)$	$8, 12, 14, 16, 17, 18, 19, 20$
$_n15_{52940}$	4	$K(5,7)$	$8, 12, 16, 17, 18, 19, 20, 28$
$_n15_{53158}$	5	$K(5,8)$	$8, 11, 12, 14, 16, 18, 20, 31, 32$
$_n15_{53614}$	4	$K(7,4)$	$8, 12, 14, 22, 23$
$_n15_{53717}$	6	$K(5,8)$	$8, 11, 14, 15, 16, 17, 19, 28$
$_n15_{54809}$	5	$K(5,7)$	$11, 12, 14, 15, 18, 26, 28$
$_n15_{57679}$	5	$K(5,7)$	$8, 12, 15, 17, 18, 26, 28$
$_n15_{57753}$	4	$K(5,7)$	$11, 12, 16, 17, 18, 20, 26, 28$
$_n15_{57962}$	5	$K(5,8)$	$8, 11, 12, 14, 17, 19, 25, 31, 32$
$_n15_{58010}$	5	$K(5,8)$	$8, 11, 12, 16, 17, 19, 25, 31, 32$
$_n15_{58121}$	4	$K(5,7)$	$11, 12, 14, 16, 18, 20, 26, 28$
$_n15_{62359}$	6	$K(5,8)$	$8, 11, 14, 15, 16, 18, 31, 32$
$_n15_{64341}$	4	$K(5,7)$	$8, 12, 16, 17, 18, 19, 20, 26$
$_n15_{64541}$	4	$K(5,7)$	$8, 11, 12, 15, 18, 19, 26, 28$
$_n15_{68293}$	5	$K(5,6)$	$12, 14, 15, 17, 19$
$_n15_{68932}$	4	$K(7,4)$	$8, 14, 18, 22, 24$
$_n15_{69285}$	4	$K(7,4)$	$8, 12, 14, 18, 22$
$_n15_{70447}$	6	$K(7,4)$	$8, 14, 17, 23$
$_n15_{73400}$	3	$K(5,7)$	$8, 11, 12, 14, 16, 18, 19, 20, 26$
$_n15_{76133}$	6	$K(7,4)$	$8, 14, 17, 22$
$_n15_{76150}$	4	$K(5,7)$	$11, 12, 15, 16, 18, 20, 26, 28$

Table 10: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
$_n15_{76348}$	4	$K(5,8)$	$8, 11, 12, 14, 16, 17, 18, 25, 31, 32$
$_n15_{77252}$	4	$K(5,7)$	$11, 15, 16, 18, 19, 20, 26, 28$
$_n15_{78503}$	5	$K(5,7)$	$8, 14, 15, 17, 18, 19, 26$
$_n15_{78826}$	4	$K(5,7)$	$8, 12, 14, 15, 16, 19, 26, 28$
$_n15_{80794}$	6	$K(5,8)$	$8, 11, 12, 14, 15, 20, 28, 32$
$_n15_{85988}$	5	$K(5,7)$	$8, 11, 15, 17, 18, 19, 26$
$_n15_{85999}$	5	$K(5,6)$	$8, 12, 16, 17, 18$
$_n15_{86672}$	5	$K(5,7)$	$11, 12, 14, 15, 16, 18, 26$
$_n15_{86757}$	4	$K(5,7)$	$11, 12, 16, 17, 18, 19, 20, 26$
$_n15_{87652}$	5	$K(5,7)$	$8, 11, 16, 18, 19, 20, 28$
$_n15_{87876}$	4	$K(5,7)$	$8, 12, 16, 18, 19, 20, 26, 28$
$_n15_{90644}$	4	$K(5,7)$	$12, 14, 16, 17, 18, 19, 26, 28$
$_n15_{94073}$	5	$K(5,7)$	$8, 12, 14, 17, 18, 26, 28$
$_n15_{96262}$	4	$K(5,7)$	$8, 12, 14, 16, 18, 19, 20, 26$
$_n15_{99775}$	4	$K(5,7)$	$11, 12, 14, 15, 16, 19, 26, 28$
$_n15_{100681}$	5	$K(5,8)$	$8, 11, 12, 15, 18, 19, 25, 31, 32$
$_n15_{101247}$	5	$K(5,7)$	$8, 11, 14, 18, 19, 26, 28$
$_n15_{106007}$	5	$K(5,7)$	$8, 11, 12, 14, 18, 26, 28$
$_n15_{106491}$	5	$K(5,8)$	$8, 11, 14, 15, 16, 18, 20, 25, 32$
$_n15_{110621}$	4	$K(5,7)$	$11, 12, 14, 16, 17, 18, 19, 26$
$_n15_{110804}$	4	$K(5,7)$	$12, 14, 16, 17, 18, 19, 20, 26$
$_n15_{115629}$	5	$K(5,8)$	$8, 11, 12, 14, 18, 19, 20, 25, 32$
$_n15_{117854}$	6	$K(7,5)$	$7, 12, 18, 19, 22, 24$
$_n15_{118394}$	4	$K(5,8)$	$8, 11, 12, 15, 16, 18, 20, 25, 28, 32$
$_n15_{119483}$	6	$K(5,8)$	$8, 11, 12, 16, 17, 19, 20, 32$
$_n15_{122907}$	6	$K(5,8)$	$8, 11, 12, 15, 16, 19, 28, 32$
$_n15_{125141}$	5	$K(5,8)$	$8, 11, 12, 15, 17, 18, 19, 25, 32$
$_n15_{125225}$	4	$K(7,4)$	$8, 14, 22, 23, 24$
$_n15_{125261}$	4	$K(5,7)$	$12, 14, 15, 16, 17, 19, 26, 28$
$_n15_{125424}$	4	$K(5,7)$	$11, 12, 14, 15, 17, 19, 26, 28$
$_n15_{125444}$	4	$K(5,8)$	$8, 11, 12, 15, 18, 20, 25, 28, 31, 32$
$_n15_{126163}$	4	$K(5,7)$	$8, 12, 14, 16, 17, 19, 26, 28$
$_n15_{126168}$	4	$K(5,7)$	$8, 11, 12, 14, 16, 17, 18, 26$
$_n15_{128700}$	4	$K(5,7)$	$8, 11, 12, 14, 16, 18, 26, 28$
$_n15_{128785}$	5	$K(5,8)$	$8, 11, 12, 16, 18, 20, 28, 31, 32$
$_n15_{131593}$	4	$K(7,5)$	$7, 17, 19, 23, 27, 28, 29, 30$
$_n15_{131659}$	4	$K(5,8)$	$8, 11, 12, 14, 15, 16, 18, 20, 25, 31$
$_n15_{133868}$	5	$K(5,8)$	$8, 11, 12, 16, 17, 18, 20, 28, 32$
$_n15_{133872}$	5	$K(7,5)$	$7, 17, 18, 19, 23, 28, 30$
$_n15_{134029}$	5	$K(5,8)$	$8, 11, 14, 15, 16, 19, 20, 25, 31$
$_n15_{134089}$	4	$K(5,8)$	$8, 11, 12, 15, 16, 17, 18, 20, 25, 31$
$_n15_{134791}$	4	$K(5,7)$	$11, 12, 14, 15, 18, 19, 26, 28$
$_n15_{136727}$	4	$K(5,7)$	$11, 12, 14, 16, 17, 18, 26, 28$

Table 11: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
$_n15_{136931}$	4	$K(5,8)$	$8,11,12,15,16,18,20,25,31,32$
$_n15_{137251}$	4	$K(5,7)$	$11,14,15,16,18,19,26,28$
$_n15_{138583}$	4	$K(5,8)$	$8,11,14,15,16,18,20,25,31,32$
$_n15_{138638}$	4	$K(5,7)$	$11,12,14,15,16,17,19,26$
$_n15_{140472}$	6	$K(5,8)$	$8,11,12,17,20,28,31,32$
$_n15_{140484}$	5	$K(5,7)$	$8,12,15,16,17,20,26$
$_n15_{140504}$	6	$K(5,8)$	$8,11,12,19,20,28,31,32$
$_n15_{140512}$	5	$K(5,7)$	$8,12,15,16,17,18,26$
$_n15_{142336}$	5	$K(5,8)$	$8,11,12,17,20,25,28,31,32$
$_n15_{143434}$	6	$K(5,8)$	$8,11,12,16,17,20,31,32$
$_n15_{143513}$	5	$K(5,8)$	$8,11,12,19,20,25,28,31,32$
$_n15_{143746}$	6	$K(5,8)$	$8,11,12,16,17,18,28,32$
$_n15_{161040}$	6	$K(5,8)$	$8,11,12,15,18,25,28,32$
$_n15_{162196}$	5	$K(5,8)$	$8,11,14,15,16,18,20,25,28$
$_n15_{163295}$	6	$K(5,8)$	$8,11,12,15,18,20,31,32$
$_n16_{60787}$	4	$K(5,7)$	$11,15,16,17,19,20,26,28$
$_n16_{61891}$	4	$K(5,7)$	$8,12,16,17,19,20,26,28$
$_n16_{61908}$	4	$K(5,7)$	$8,11,15,17,19,20,26,28$
$_n16_{92582}$	6	$K(5,8)$	$8,11,12,16,17,20,25,28$
$_n16_{92599}$	5	$K(5,8)$	$8,11,12,16,17,20,25,28,32$
$_n16_{96834}$	5	$K(5,8)$	$8,11,12,14,16,18,20,25,28$
$_n16_{102142}$	5	$K(7,5)$	$7,12,18,19,24,27,30$
$_n16_{105805}$	5	$K(5,8)$	$8,11,12,14,19,20,25,28,32$
$_n16_{122275}$	5	$K(5,8)$	$8,11,12,15,17,19,25,31,32$
$_n16_{159238}$	6	$K(5,8)$	$8,11,14,15,16,18,19,20$
$_n16_{175397}$	5	$K(7,5)$	$7,18,19,23,27,29,30$
$_n16_{184868}$	7	$K(7,5)$	$22,23,28,29,30$
$_n16_{224338}$	4	$K(5,7)$	$11,15,16,17,18,19,26,28$
$_n16_{258180}$	5	$K(7,5)$	$7,17,18,19,22,29,30$
$_n16_{268063}$	5	$K(5,8)$	$8,11,12,14,16,20,25,31,32$
$_n16_{268772}$	5	$K(5,7)$	$8,15,16,17,18,26,28$
$_n16_{268957}$	5	$K(5,7)$	$8,11,16,17,19,26,28$
$_n16_{268958}$	4	$K(5,7)$	$8,11,15,16,17,19,26,28$
$_n16_{269473}$	5	$K(5,7)$	$8,11,15,16,17,26,28$
$_n16_{281266}$	5	$K(7,5)$	$7,17,18,19,27,29,30$
$_n16_{287959}$	6	$K(7,5)$	$7,17,23,24,28,30$
$_n16_{287969}$	4	$K(5,8)$	$8,12,14,15,16,17,19,20,25,31$
$_n16_{301351}$	5	$K(7,5)$	$7,17,19,22,23,29,30$
$_n16_{301943}$	4	$K(5,7)$	$11,12,15,16,19,20,26,28$
$_n16_{302111}$	4	$K(5,7)$	$8,11,12,16,17,19,20,26$
$_n16_{309481}$	6	$K(5,7)$	$11,14,15,20,26,28$
$_n16_{318483}$	4	$K(5,7)$	$11,12,15,16,17,18,26,28$
$_n16_{320433}$	4	$K(5,7)$	$8,11,12,16,17,18,26,28$

Table 12: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
${}_n16_{320683}$	5	$K(7,5)$	$7, 17, 18, 19, 24, 27, 30$
${}_n16_{334103}$	6	$K(7,5)$	$7, 17, 22, 23, 28, 30$
${}_n16_{346876}$	6	$K(7,5)$	$7, 17, 22, 23, 29, 30$
${}_n16_{354133}$	4	$K(5,7)$	$11, 12, 16, 17, 18, 19, 26, 28$
${}_n16_{358558}$	6	$K(5,8)$	$8, 11, 12, 14, 19, 25, 28, 32$
${}_n16_{364604}$	4	$K(5,8)$	$8, 12, 14, 15, 16, 17, 18, 20, 25, 31$
${}_n16_{375559}$	6	$K(5,7)$	$11, 12, 14, 15, 19, 20$
${}_n16_{378485}$	6	$K(7,5)$	$7, 17, 18, 22, 29, 30$
${}_n16_{378724}$	6	$K(7,5)$	$7, 12, 18, 19, 24, 27$
${}_n16_{380614}$	6	$K(5,8)$	$8, 11, 14, 15, 16, 18, 25, 28$
${}_n16_{390683}$	6	$K(5,7)$	$8, 16, 17, 19, 26, 28$
${}_n16_{391940}$	6	$K(5,7)$	$14, 16, 17, 19, 26, 28$
${}_n16_{404141}$	6	$K(4,7)$	$8, 11, 20$
${}_n16_{412591}$	5	$K(5,7)$	$8, 11, 12, 14, 18, 19, 26$
${}_n16_{428675}$	6	$K(5,8)$	$8, 11, 12, 14, 16, 25, 31, 32$
${}_n16_{428705}$	5	$K(5,7)$	$8, 12, 15, 16, 19, 20, 26$
${}_n16_{428759}$	6	$K(5,8)$	$8, 11, 12, 14, 16, 20, 31, 32$
${}_n16_{429022}$	4	$K(5,8)$	$8, 11, 12, 14, 16, 17, 18, 20, 25, 32$
${}_n16_{429023}$	6	$K(5,8)$	$8, 11, 12, 14, 16, 17, 18, 20$
${}_n16_{429568}$	6	$K(5,6)$	$11, 16, 17, 19$
${}_n16_{431680}$	4	$K(5,7)$	$8, 12, 14, 16, 18, 19, 26, 28$
${}_n16_{431707}$	6	$K(5,7)$	$11, 15, 16, 17, 18, 19$
${}_n16_{431760}$	5	$K(7,5)$	$7, 17, 19, 23, 24, 27, 29$
${}_n16_{431855}$	5	$K(7,5)$	$7, 17, 19, 22, 23, 27, 29$
${}_n16_{435492}$	5	$K(5,6)$	$8, 11, 12, 18, 19$
${}_n16_{439243}$	6	$K(5,6)$	$8, 11, 16, 18$
${}_n16_{439259}$	5	$K(5,6)$	$8, 11, 12, 16, 18$
${}_n16_{439273}$	4	$K(5,7)$	$11, 12, 15, 16, 17, 18, 19, 26$
${}_n16_{444529}$	5	$K(7,5)$	$7, 17, 19, 22, 23, 28, 29$
${}_n16_{445185}$	5	$K(5,8)$	$8, 11, 12, 16, 18, 19, 20, 25, 31$
${}_n16_{456786}$	5	$K(5,7)$	$8, 11, 16, 17, 18, 19, 26$
${}_n16_{456797}$	5	$K(5,8)$	$8, 11, 12, 18, 19, 20, 25, 31, 32$
${}_n16_{459366}$	6	$K(5,8)$	$8, 11, 12, 14, 19, 20, 31, 32$
${}_n16_{459448}$	6	$K(5,8)$	$8, 11, 12, 14, 16, 20, 28, 31$
${}_n16_{459802}$	5	$K(5,7)$	$8, 11, 15, 16, 19, 20, 28$
${}_n16_{460795}$	5	$K(5,8)$	$8, 11, 12, 14, 16, 17, 20, 25, 31$
${}_n16_{461331}$	4	$K(5,8)$	$8, 11, 15, 17, 19, 20, 25, 28, 31, 32$
${}_n16_{463333}$	5	$K(5,7)$	$8, 11, 15, 16, 19, 26, 28$
${}_n16_{470460}$	5	$K(5,7)$	$8, 11, 15, 16, 17, 18, 26$
${}_n16_{471541}$	5	$K(5,7)$	$8, 11, 14, 16, 18, 19, 26$
${}_n16_{473187}$	6	$K(7,5)$	$17, 19, 23, 24, 27, 29$
${}_n16_{476678}$	6	$K(5,8)$	$8, 11, 12, 14, 19, 20, 28, 31$
${}_n16_{479581}$	6	$K(7,5)$	$17, 19, 22, 23, 28, 29$

Table 13: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
$_n16_{489170}$	4	$K(5,7)$	$8,11,12,16,17,18,19,26$
$_n16_{503071}$	5	$K(5,7)$	$11,12,14,15,19,20,26$
$_n16_{503072}$	4	$K(5,7)$	$11,12,15,16,18,19,26,28$
$_n16_{503168}$	4	$K(5,7)$	$11,12,14,15,16,18,19,26$
$_n16_{510204}$	6	$K(7,5)$	$7,12,19,24,27,29$
$_n16_{510585}$	5	$K(5,7)$	$8,11,15,16,17,19,20$
$_n16_{513086}$	5	$K(7,5)$	$7,17,19,24,27,29,30$
$_n16_{517937}$	6	$K(5,8)$	$8,11,12,14,16,17,20,32$
$_n16_{517944}$	5	$K(5,7)$	$8,12,15,16,19,20,28$
$_n16_{521361}$	6	$K(7,4)$	$8,14,23$
$_n16_{523720}$	6	$K(5,6)$	$8,12,17,19$
$_n16_{524345}$	5	$K(5,7)$	$8,11,15,16,18,19,26$
$_n16_{524356}$	4	$K(5,7)$	$8,11,12,14,15,16,19,26$
$_n16_{536514}$	6	$K(5,7)$	$14,16,18,19,20,26$
$_n16_{536524}$	5	$K(5,7)$	$8,12,14,15,18,19,26$
$_n16_{538551}$	5	$K(5,6)$	$12,14,16,18,19$
$_n16_{553103}$	5	$K(7,5)$	$7,17,18,19,22,23,29$
$_n16_{553111}$	6	$K(5,7)$	$11,14,18,20,26,28$
$_n16_{556997}$	6	$K(5,7)$	$11,12,14,20,26,28$
$_n16_{561966}$	5	$K(7,5)$	$7,17,18,19,22,23,28$
$_n16_{567117}$	6	$K(5,7)$	$15,16,17,19,26,28$
$_n16_{571236}$	6	$K(5,7)$	$14,17,18,19,26,28$
$_n16_{573992}$	6	$K(5,7)$	$11,14,18,19,20,26$
$_n16_{580657}$	6	$K(5,8)$	$8,11,12,16,18,19,20,31$
$_n16_{581451}$	6	$K(7,5)$	$17,19,22,23,27,29$
$_n16_{582083}$	6	$K(7,5)$	$7,17,22,23,28,29$
$_n16_{584460}$	6	$K(5,8)$	$8,11,12,18,19,20,31,32$
$_n16_{584571}$	6	$K(5,8)$	$8,11,12,18,19,20,28,31$
$_n16_{584582}$	5	$K(5,7)$	$8,11,12,15,16,18,26$
$_n16_{585589}$	5	$K(5,8)$	$8,11,12,15,18,19,20,25,31$
$_n16_{594233}$	6	$K(5,7)$	$11,17,18,19,26,28$
$_n16_{597144}$	5	$K(5,7)$	$8,11,14,15,16,19,26$
$_n16_{598093}$	5	$K(5,7)$	$8,11,14,15,19,20,26$
$_n16_{601164}$	5	$K(5,8)$	$8,11,12,14,18,19,20,25,31$
$_n16_{601875}$	7	$K(7,5)$	$17,23,27,28,29$
$_n16_{603616}$	6	$K(5,7)$	$16,17,18,19,26,28$
$_n16_{605426}$	5	$K(5,7)$	$8,11,14,15,18,19,26$
$_n16_{606988}$	5	$K(5,6)$	$8,15,16,18,20$
$_n16_{612849}$	6	$K(5,6)$	$8,12,17,18$
$_n16_{613057}$	6	$K(5,7)$	$15,16,18,19,26,28$
$_n16_{613143}$	6	$K(5,6)$	$12,16,18,19$
$_n16_{613788}$	5	$K(5,7)$	$8,11,15,18,19,20,26$
$_n16_{624501}$	6	$K(5,8)$	$8,11,12,16,18,19,25,31$

Table 14: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
$_n16_{634316}$	6	$K(5,8)$	$8,11,12,14,16,18,25,31$
$_n16_{648001}$	5	$K(5,8)$	$8,11,12,16,17,20,25,31,32$
$_n16_{668333}$	6	$K(5,8)$	$8,11,12,14,18,19,20,31$
$_n16_{675631}$	6	$K(7,5)$	$7,17,22,23,24,29$
$_n16_{678986}$	5	$K(5,8)$	$8,11,12,14,15,18,20,25,31$
$_n16_{679057}$	6	$K(5,8)$	$8,11,12,15,18,19,25,31$
$_n16_{679078}$	5	$K(5,8)$	$8,11,12,14,15,18,20,25,32$
$_n16_{679080}$	6	$K(5,8)$	$8,11,12,18,19,25,31,32$
$_n16_{679187}$	6	$K(5,7)$	$11,14,17,18,19,26$
$_n16_{679234}$	6	$K(5,7)$	$11,14,17,18,26,28$
$_n16_{679264}$	6	$K(5,7)$	$11,12,14,17,18,26$
$_n16_{680466}$	6	$K(5,8)$	$8,11,14,15,16,18,20,32$
$_n16_{685905}$	6	$K(5,8)$	$8,11,12,17,18,19,25,31$
$_n16_{685993}$	6	$K(7,5)$	$7,12,19,23,28,29$
$_n16_{686390}$	6	$K(5,8)$	$8,11,14,15,16,18,20,28$
$_n16_{686572}$	5	$K(5,6)$	$8,11,12,15,16$
$_n16_{686680}$	6	$K(5,7)$	$14,16,18,19,26,28$
$_n16_{686806}$	6	$K(5,7)$	$11,14,15,17,20,26$
$_n16_{695732}$	6	$K(7,5)$	$7,12,19,22,23,28$
$_n16_{699194}$	6	$K(5,8)$	$8,11,14,15,16,19,20,25$
$_n16_{699299}$	6	$K(5,8)$	$8,11,14,15,16,19,20,28$
$_n16_{703716}$	5	$K(5,8)$	$8,11,12,16,19,20,25,31,32$
$_n16_{723250}$	6	$K(7,5)$	$17,19,23,27,28,29$
$_n16_{723646}$	6	$K(5,8)$	$8,11,14,15,16,18,20,31$
$_n16_{725845}$	6	$K(5,8)$	$8,11,12,14,18,19,25,31$
$_n16_{750859}$	6	$K(5,7)$	$8,14,16,18,26,28$
$_n16_{771177}$	7	$K(8,5)$	$20,21,26,27,32,34,35$
$_n16_{783154}$	7	$K(5,6)$	$8,14,16$
$_n16_{783161}$	6	$K(5,7)$	$8,11,12,15,18,26$
$_n16_{792631}$	7	$K(5,6)$	$8,11,12$
$_n16_{794697}$	6	$K(5,7)$	$11,12,14,15,20,26$
$_n16_{805055}$	4	$K(5,7)$	$8,12,14,16,17,18,26,28$
$_n16_{825321}$	5	$K(5,8)$	$8,11,12,15,18,19,20,25,32$
$_n16_{844110}$	6	$K(5,7)$	$8,12,15,16,17,20$
$_n16_{862981}$	5	$K(5,7)$	$8,11,15,16,19,20,26$
$_n16_{868659}$	6	$K(5,7)$	$8,15,16,19,20,26$
$_n16_{871491}$	6	$K(5,7)$	$11,12,15,16,17,19$
$_n16_{872760}$	7	$K(8,5)$	$21,26,27,30,32,33,35$
$_n16_{873339}$	6	$K(5,7)$	$8,15,16,18,19,26$
$_n16_{873435}$	5	$K(5,7)$	$8,11,12,15,19,20,26$
$_n16_{873722}$	5	$K(5,6)$	$8,12,16,18,20$
$_n16_{880688}$	7	$K(7,5)$	$17,23,28,29,30$
$_n16_{907472}$	6	$K(5,7)$	$15,16,17,19,20,26$

Table 15: Unknotting number of some knots

Knot H-T Notation	Unknotting Number	Torus Knot	Crossings changed
${}_{n}16_{917298}$	6	$K(5,6)$	$8,12,18,19$
${}_{n}16_{921769}$	7	$K(7,5)$	$17,19,23,24,29$
${}_{n}16_{931575}$	7	$K(8,5)$	$16,20,21,26,33,34,35$
${}_{n}16_{931934}$	6	$K(5,6)$	$8,12,14,16$
${}_{n}16_{935396}$	6	$K(5,6)$	$8,12,15,16$
${}_{n}16_{935425}$	5	$K(5,6)$	$8,12,15,16,20$
${}_{n}16_{938383}$	5	$K(5,6)$	$8,12,16,17,20$
${}_{n}16_{964288}$	6	$K(5,7)$	$11,14,15,17,18,26$
${}_{n}16_{992662}$	7	$K(8,5)$	$21,26,28,30,32,33,35$
${}_{n}16_{992899}$	7	$K(8,5)$	$15,20,21,26,30,34,35$
${}_{n}16_{993506}$	7	$K(8,5)$	$20,21,26,28,32,33,35$
${}_{n}16_{996468}$	7	$K(7,5)$	$7,12,23,28,29$
${}_{n}16_{996934}$	7	$K(8,5)$	$20,26,28,30,32,33,35$
${}_{n}16_{997310}$	7	$K(5,6)$	$8,14,15$
${}_{n}16_{999418}$	5	$K(5,7)$	$8,12,14,16,17,18,28$
${}_{n}16_{1000335}$	6	$K(7,5)$	$7,17,19,23,24,29$
${}_{n}16_{1000337}$	5	$K(5,8)$	$8,11,12,14,17,18,20,25,32$
${}_{n}16_{1003404}$	6	$K(7,5)$	$7,17,18,23,28,29$
${}_{n}16_{1008903}$	7	$K(8,5)$	$16,20,21,26,27,30,34$

Table 16: Unknotting number of some knots

6. Conclusion

It is easy to observe that by identifying the position of a knot K in some unknotting sequence, we can give its unknotting number. Since we have minimal unknotting crossing data for all torus knots, we have infinitely many unknotting sequences for torus knots. We can find unknotting number of infinitely many knots which are in these unknotting sequences of torus knots. Here, we have considered only 4 torus knots and got unknotting numbers of more than 600 knots. One can try to find unknotting numbers for different knots by considering other torus knots and their minimal unknotting crossing data. Even though we can ensure the unknotting number of infinitely many knots, the only difficulty that we may face is the identification of knot type. The probability of getting a new knot having crossing number less than or equal to 16, reduces with increase in the number of crossings of a torus knot.

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