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Almost Ricci Soliton and Gradient Almost Ricci Soliton on 3-dimensional f-Kenmotsu Manifolds

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Abstract. The object of the present paper is to study almost Ricci solitons and gradient almost Ricci solitons in 3-dimensional f-Kenmotsu manifolds.

1. Introduction

The study of almost Ricci soliton was introduced by Pigola et. al. [18], where essentially they modified the definition of Ricci soliton by adding the condition on the parameter λ to be a variable function, more precisely, we say that a Riemannian manifold (M^n, g) admits an almost Ricci soliton, if there exists a complete vector field V, called potential vector field and a smooth soliton function $\lambda : M^n \longrightarrow \mathbb{R}$ satisfying

$$Ric + \frac{1}{2}\pounds_V g = \lambda g, \tag{1.1}$$

where Ric and $\pounds$ stand, respectively, for the Ricci tensor and Lie derivative. We shall refer to this equation as the fundamental equation of an almost Ricci soliton (M^n, g, V, λ). It will be called expanding, steady or shrinking, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Otherwise it will be called indefinite. When the vector field V is gradient of a smooth function $f : M^n \longrightarrow \mathbb{R}$ the metric will be called gradient almost Ricci soliton. In this case the preceding equation becomes

$$Ric + \nabla^2 f = \lambda g, \tag{1.2}$$

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where $\nabla^2 f$ stands for the Hessian of f. Sometimes classical theory of tensorial calculus is more convenient to make computations. Then, we can write the fundamental equation in this language as follows:

$$R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}. \tag{1.3}$$

Moreover, if the vector field X is trivial, or the potential f is constant, the almost Ricci soliton will be called trivial, otherwise it will be a non-trivial almost Ricci soliton. We notice that when $n \geq 3$ and $X$is a Killing vector field an almost Ricci soliton will be a Ricci soliton, since in this case we have an Einstein manifold, from which we can apply Schur's lemma to deduce that λ is constant. Taking into account that the soliton function λ is not necessarily constant, certainly comparison with soliton theory will be modified. In particular the rigidity result contained in Theorem 1.3 of [18] indicates that almost Ricci solitons should reveal a reasonably broad generalization of the fruitful concept of classical soliton. In fact, we refer the reader to [18] to see some of this changes.

In the direction to understand the geometry of almost Ricci soliton, Barros and Ribeiro Jr. proved in [2] that a compact gradient almost Ricci soliton with non-trivial conformal vector field is isometric to a Euclidean sphere. In the same paper they proved an integral formula for compact case, which was used to prove several rigidity results, for more details see [2].

The existence of Ricci almost soliton has been confirmed by Pigola et. al. [18] on some certain class of warped product manifolds. Some characterization of Ricci almost soliton on a compact Riemannian manifold can be found in ([1], [2], [3]). It is interesting to note that if the potential vector field V of the Ricci almost soliton (M, g, V, λ) is Killing then the soliton becomes trivial, provided the dimension of $M > 2$. Moreover, if V is conformal then M^n is isometric to Euclidean sphere S^n. Thus the Ricci almost soliton can be considered as a generalization of Einstein metric as well as Ricci soliton.

In [6], authors studied Ricci solitons and gradient Ricci solitons on 3-dimensional normal almost contact metric manifolds. In [10] authors studied compact Ricci soliton. Beside these, A. Ghosh [12] studied K-contact and Sasakian manifolds whose metric is gradient almost Ricci solitons. Conditions of K-contact and Sasakian manifolds are more stronger than normal almost contact metric manifolds in the sense that the 1-form η of normal almost contact metric manifolds are not contact form. The Ricci soliton and gradient Ricci soliton have been studied by several authors such as ([5], [7], [9]) and many others.

The present paper is organized as follows:

After preliminaries, in section 3 we study almost Ricci soliton in 3-dimensional f-Kenmotsu manifolds. Finally, we consider gradient almost Ricci solitons in 3-dimensional f-Kenmotsu manifolds.

2. Preliminaries

Let M be an almost contact manifold, i.e., M is a connected $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) [4]. As usually, denote by Φ the fundamental 2-form of M, $\Phi(X, Y) = g(X, \phi Y)$, $X, Y \in \chi(M)$, $\chi(M)$ being the Lie algebra of differentiable vector fields on M.

For further use, we recall the following definitions ([4], [11], [19]). The manifold M and its structure (ϕ, ξ, η, g) is said to be:

(i) normal if the almost complex structure defined on the product manifold $M \times \mathbb{R}$ is integrable (equivalently $[\phi, \phi] + 2d\eta \otimes \xi = 0$),

(ii) almost cosymplectic if $d\eta = 0$ and $d\Phi = 0$,

(iii) cosymplectic if it is normal and almost cosymplectic (equivalently, $\nabla\phi = 0$, ∇ being covariant differentiation with respect to the Levi-Civita connection).

The manifold M is called locally conformal cosymplectic (respectively, almost cosymplectic) if M has an open covering $\{U_t\}$ endowed with differentiable functions $\sigma_t : U_t \to \mathbb{R}$ such that over each U_t the almost contact metric structure $(\phi_t, \xi_t, \eta_t, g_t)$ defined by

$$\phi_t = \phi, \quad \xi_t = e^{\sigma_t}\xi, \quad \eta_t = e^{-\sigma_t}\eta, \quad g_t = e^{-2\sigma_t}g$$

is cosymplectic (respectively, almost cosymplectic).

Olszak and Rosca [16] studied normal locally conformal almost cosymplectic manifold. They gave a geometric interpretation of f-Kenmotsu manifolds and studied some curvature properties. Among others they proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold.

By an f-Kenmotsu manifold we mean an almost contact metric manifold which is normal and locally conformal almost cosymplectic.

Let M be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure (ϕ, ξ, η, g) satisfying

$$\begin{aligned} \phi^2 &= -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \\ \phi\xi &= 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi), \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \end{aligned} \tag{2.1}$$

for any vector fields $X, Y \in \chi(M)$, where I is the identity of the tangent bundle TM, ϕ is a tensor field of $(1,1)$-type, η is a 1-form, ξ is a vector field and g is a metric tensor field. We say that (M, ϕ, ξ, η, g) is an f-Kenmotsu manifold if the covariant differentiation of ϕ satisfies [15]:

$$(\nabla_X \phi)(Y) = f(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.2}$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$. If $f = \alpha = \text{constant} \neq 0$, then the manifold is a α-Kenmotsu manifold [13]. 1-Kenmotsu manifold is a Kenmotsu manifold ([14], [17]). If $f = 0$, then the manifold is cosymplectic [13]. An f-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi f$.

For an f-Kenmotsu manifold from (2.2) it follows that

$$\nabla_X \xi = f\{X - \eta(X)\xi\}. \tag{2.3}$$

The condition $df \wedge \eta = 0$ holds if $\dim M \geq 5$. In general this does not hold if $\dim M = 3$ [16].

In a 3-dimensional Riemannian manifold, we always have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{2.4}$$

In a 3-dimensional f-Kenmotsu manifold we have [16]

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2f^2 + 2f')(X \wedge Y)Z \\ &\quad -(\frac{r}{2} + 3f^2 + 3f')\{\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z\}, \end{aligned} \tag{2.5}$$

$$S(X,Y) = (\frac{r}{2} + f^2 + f^{'})g(X,Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{2.6}$$

where r is the scalar curvature of M and $f' = \xi(f)$.

From (2.5), we obtain

$$R(X,Y)\xi = -(f^2 + f^{'})[\eta(Y)X - \eta(X)Y], \tag{2.7}$$

and (2.6) yields

$$S(X,\xi) = -2(f^2 + f^{'})\eta(X). \tag{2.8}$$

Example.([8]) We consider the three-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3,\ z \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = z^2 \frac{\partial}{\partial x}, \quad e_2 = z^2 \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1,e_3) &= g(e_2,e_3) = g(e_1,e_2) = 0, \\ g(e_1,e_1) &= g(e_2,e_2) = g(e_3,e_3) = 1. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$.

Then using linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Now, by direct computations we obtain

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -\frac{2}{z}e_2, \quad [e_1, e_3] = -\frac{2}{z}e_1.$$

The Riemannian connection ∇ of the metric tensor g is given by the Koszul's formula which is

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned} \tag{2.9}$$

Using (2.9) we have

$$2g(\nabla_{e_1} e_3, e_1) = 2g(-\frac{2}{z}e_1, e_1),$$

$$2g(\nabla_{e_1} e_3, e_2) = 0 \quad \text{and} \quad 2g(\nabla_{e_1} e_3, e_3) = 0.$$

Hence $\nabla_{e_1} e_3 = -\frac{2}{z}e_1$. Similarly, $\nabla_{e_2} e_3 = -\frac{2}{z}e_2$ and $\nabla_{e_3} e_3 = 0$. (2.9) further yields

$$\begin{aligned} &\nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_1 = \frac{2}{z}e_3, \\ &\nabla_{e_2} e_2 = \frac{2}{z}e_3, && \nabla_{e_2} e_1 = 0, \\ &\nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_1 = 0. \end{aligned}$$

From the above it follows that the manifold satisfies $\nabla_X \xi = f\{X - \eta(X)\xi\}$ for $\xi = e_3$, where $f = -\frac{2}{z}$. Hence we conclude that M is an f-Kenmotsu manifold. Also $f^2 + f' \neq 0$. Hence M is a regular f-Kenmotsu manifold.

3. Almost Ricci Soliton

In this section we consider almost Ricci solitons on 3-dimensional f-Kenmotsu manifolds. In particular, let the potential vector field V be pointwise collinear with ξ i.e., $V = b\xi$, where b is a function on M. Then from (1.1) we have

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X,Y) = 2\lambda g(X,Y). \tag{3.1}$$

Using (2.3) in (3.1), we get

$$2fb[g(X,Y) - \eta(X)\eta(Y)] + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X,Y) = 2\lambda g(X,Y). \tag{3.2}$$

Putting $Y = \xi$ in (3.2) and using (2.8) yields

$$(Xb) + (\xi b)\eta(X) - 4(f^2 + f')\eta(X) = 2\lambda\eta(X). \tag{3.3}$$

Putting $X = \xi$ in (3.3) we obtain

$$\xi b = 2(f^2 + f') + \lambda. \tag{3.4}$$

Putting the value of ξb in (3.3) yields

$$db = [\lambda + 2(f^2 + f')]\eta. \tag{3.5}$$

Applying d on (3.5) and using $d^2 = 0$, we get

$$0 = d^2 b = [\lambda + 2(f^2 + f')]d\eta. \tag{3.6}$$

Taking wedge product of (3.6) with η, we have

$$[\lambda + 2(f^2 + f')]\eta \wedge d\eta = 0. \tag{3.7}$$

Since $\eta \wedge d\eta \neq 0$ in a 3-dimensional f-Kenmotsu manifold, therefore

$$\lambda + 2(f^2 + f') = 0 \Rightarrow \lambda = -2(f^2 + f'). \tag{3.8}$$

Using (3.8) in (3.5) gives $db = 0$ i.e., $b =$constant. Therefore from (3.2) we have

$$S(X, Y) = (\lambda - fb)g(X, Y) + fb\eta(X)\eta(Y). \tag{3.9}$$

In view of (3.9) we can state the following:

Theorem 3.1. *If in a 3-dimensional f-Kenmotsu manifold the metric g admits almost Ricci soliton and V is pointwise collinear with ξ, then V is constant multiple of ξ and the manifold is η-Einstein of the form* (3.9).

The converse of the above theorem is not true, in general. However if we take $f = constant$, i.e., if we consider a 3-dimensional η-Einstein f-Kenmotsu manifold, then it admits a Ricci soliton. This can be proved as follows:

Let M be a 3-dimensional η-Einstein f-Kenmotsu manifold and $V = \xi$. Then

$$S(X, Y) = \gamma g(X, Y) + \delta\eta(X)\eta(Y), \tag{3.10}$$

where γ and δ are certain scalars.

Now using (2.3)

$$\begin{aligned}(\pounds_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) \\ &= 2f\{g(X, Y) - \eta(X)\eta(Y)\}.\end{aligned}$$

Therefore

$$\begin{aligned}(\pounds_\xi g)(X, Y) + 2S(X, Y) - 2\lambda g(X, Y) &= 2(f + \gamma - \lambda)g(X, Y) \\ &\quad -2(f - \delta)\eta(X)\eta(Y).\end{aligned} \tag{3.11}$$

From equation (3.11) it follows that M admits a Ricci soliton (g, ξ, λ) if $f+\gamma-\lambda=0$ and $\delta = f =$ constant. From (3.10) we have using (2.8), $-2f^2 = \gamma+\delta$. Hence $\gamma = -2f^2 - f =$ constant. Therefore $\lambda = (\gamma+\delta) =$ constant. So we have the following:

Theorem 3.2. *If a 3-dimensional f-Kenmotsu manifold is η-Einstein of the form $S = \gamma g + \delta\eta\otimes\eta$, then a Ricci almost soliton (M, g, ξ, λ) reduces to a Ricci soliton $(g, \xi, (\gamma+\delta))$.*

Now let $V = \xi$. Then (3.1) reduces to

$$(\pounds_\xi g)(X,Y) + 2S(X,Y) = 2\lambda g(X,Y). \tag{3.12}$$

Now, in view of (2.6) we have

$$\begin{aligned}(\pounds_\xi g)(X,Y) \;&=\; -2\Big[\Big(\frac{r}{2}+f^2+f'\Big)g(X,Y)\\ &\quad -\Big(\frac{r}{2}+3f^2+3f'\Big)\eta(X)\eta(Y)\Big] + 2\lambda g(X,Y).\end{aligned} \tag{3.13}$$

$$\begin{aligned}2f\{g(X,Y)-\eta(X)\eta(Y)\} \;&=\; 2\lambda g(X,Y) - 2\Big[\Big(\frac{r}{2}+f^2+f'\Big)g(X,Y)\\ &\quad -\Big(\frac{r}{2}+3f^2+3f'\Big)\eta(X)\eta(Y)\Big].\end{aligned} \tag{3.14}$$

Putting $X = Y = \xi$ in (3.14) yields

$$\lambda = 4(f^2+f'). \tag{3.15}$$

Assuming that $f =$ constant, we get $f' = \xi f = 0$. This implies $\lambda = 4f^2 =$ constant. Thus we can state the following:

Theorem 3.3. *If a 3-dimensional f-Kenmotsu manifold with $f =$constant admits almost Ricci soliton then it reduces to a Ricci soliton.*

4. Gradient Almost Ricci Soliton

This section is devoted to study 3-dimensional f-Kenmotsu manifolds admitting gradient almost Ricci soliton. For a gradient almost Ricci soliton, we have

$$\nabla_Y Df = \lambda Y - QY, \tag{4.1}$$

where D denotes the gradient operator of g.

Differentiating (4.1) covariantly in the direction of X yields

$$\nabla_X\nabla_Y Df = d\lambda(X)Y + \lambda\nabla_X Y - (\nabla_X Q)Y. \tag{4.2}$$

Similarly, we get

$$\nabla_Y \nabla_X Df = d\lambda(Y)X + \lambda \nabla_Y X - (\nabla_Y Q)X. \tag{4.3}$$

and

$$\nabla_{[X,Y]} Df = \lambda[X,Y] - Q[X,Y]. \tag{4.4}$$

In view of (4.2),(4.3) and (4.4), we have

$$\begin{aligned} R(X,Y)Df &= \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df \\ &= (\nabla_Y Q)X - (\nabla_X Q)Y - (Y\lambda)X + (X\lambda)Y. \end{aligned} \tag{4.5}$$

We get from (2.6)

$$QY = \left(\frac{r}{2} + f^2 + f'\right) Y - \left(\frac{r}{2} + 3f^2 + 3f'\right) \eta(Y)\xi, \tag{4.6}$$

Differentiating (4.6) covariantly in the direction of X and using (2.3), we get

$$\begin{aligned} (\nabla_X Q)Y &= \left\{\frac{(Xr)}{2} + 2f(Xf) + (Xf')\right\} Y \\ &- \left\{\frac{(Xr)}{2} + 6f(Xf) + 3(Xf')\right\} \{fg(X,Y)\xi + f\eta(Y)X - 2f\eta(X)\eta(Y)\xi\}. \end{aligned} \tag{4.7}$$

In view of (4.7), we get from (4.5)

$$\begin{aligned} R(X,Y)Df &= \left\{\frac{(Yr)}{2} + 2f(Yf) + (Yf')\right\} X - \left\{\frac{(Xr)}{2} + 2f(Xf) + (Xf')\right\} Y \\ &- \left\{\frac{(Yr)}{2} + 6f(Yf) + 3(Yf')\right\} \{fg(X,Y)\xi + f\eta(X)Y - 2f\eta(X)\eta(Y)\xi\} \\ &+ \left\{\frac{(Xr)}{2} + 6f(Xf) + 3(Xf')\right\} \{fg(X,Y)\xi + f\eta(Y)X - 2f\eta(X)\eta(Y)\xi\} \\ &- (Y\lambda)X + (X\lambda)Y. \end{aligned} \tag{4.8}$$

This implies

$$\begin{aligned} g(R(X,\xi)Df,\xi) &= \left\{\frac{(\xi r)}{2} + 2f(\xi f) + (\xi f')\right\} \eta(X) \\ &- \left\{\frac{(Xr)}{2} + 2f(Xf) + (Xf')\right\} \\ &- (\xi\lambda)\eta(X) + (X\lambda). \end{aligned} \tag{4.9}$$

Also, we have from (2.5)

$$g(R(X,\xi)Df,\xi) = (f^2 + f')\{(Xf) - (\xi f)\eta(X)\}. \tag{4.10}$$

In view of (4.9) and (4.10) we obtain

$$
\begin{aligned}
(f^2+f')\{(Xf)-(\xi f)\eta(X)\} &= \left\{\frac{(\xi r)}{2}+2f(\xi f)+(\xi f')\right\}\eta(X)\\
&- \left\{\frac{(Xr)}{2}+2f(Xf)+(Xf')\right\}\\
&- (\xi\lambda)\eta(X)+(X\lambda). \qquad (4.11)
\end{aligned}
$$

Assuming that the scalar curvature r and f are constants. Then it follows from (4.11) that

$$d\lambda-(\xi\lambda)\eta=0. \qquad (12)$$

Applying d both sides of (12), we get

$$\xi\lambda=0. \qquad (13)$$

Using (13) in (12), we have

$$d\lambda=0. \qquad (14)$$

This implies $\lambda=$ constant. Thus we can state the following:

Theorem 4.1. *If a 3-dimensional f-Kenmotsu manifold admits gradient almost Ricci soliton then it reduces to a Ricci soliton provided the scalar curvature r and f are constants.*

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