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# Graph Equations Involving Tensor Product of Graphs

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Abstract. In this paper, we solve the following four graph equations $L^k(G) = H \oplus J$; $M(G) = H \oplus J$; $\overline{L^k(G)} = H \oplus J$ and $\overline{M(G)} = H \oplus J$, where $J$ is $nK_2$ for $n \geq 1$. Here, the equality symbol $=$ means the isomorphism between the corresponding graphs. In particular, we shall obtain all pairs of graphs $(G, H)$, which satisfy the above mentioned equations, upto isomorphism.

## 1. Introduction

We shall consider only finite, simple and undirected graphs. We follow the terminology of Harary [5]. For a graph $G$, let $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$, respectively. As in [5], let $P_n$, $C_n$ and $K_n$ denote a path, cycle and complete graph, on $n$ vertices, respectively. We call a graph with just one vertex is *trivial* and all other graphs are *nontrivial*. The *degree* of a vertex $v$ in a graph $G$, is the number of edges incident to $v$ and is denoted by *deg(v)*. The *maximum degree* of a graph $G$, is the maximum degree among the vertices of $G$ and is denoted by $\Delta(G)$. Let $G$ be a graph. A subgraph $H$ of $G$ is an *induced subgraph of G* if whenever $u$ and $v$ are vertices of $H$ and $uv$ is an edge of $G$, then $uv$ is also an edge of $H$. If $S$ is a nonempty set of vertices of $G$, then the *subgraph of G induced by S*, denoted by $\langle S \rangle$, is the induced subgraph with vertex set $S$. If $X$ is a nonempty set of edges of $G$, then *the subgraph of G induced by X*, denoted by $\langle X \rangle$, is the induced subgraph of $G$, whose vertex set is the set of all vertices of edges in $X$ and whose edge set is $X$. Throughout this paper, the equality sign $=$ means the isomorphism between the corresponding graphs. A graph $G$ is *connected* if there is

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at least one path between every pair of its vertices ; otherwise, $G$ is *disconnected.* A graph which is both connected and nontrivial is a *nontrivial connected graph.* For a connected graph $G$, $nG$ ; $n \geq 1$, is the graph with $n$ components, each being isomorphic to $G$. The *complement* $\overline{G}$ of a graph $G$, is the graph whose vertex set is $V(G)$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. In any graph $G$, the set of vertices adjacent to a vertex $u$ are the *neighbours of* $u$ and the *neighbourhood of* $u$ is the set $N(u) = \{v \in V(G) : uv \in E(G)\}$. The *closed neighbourhood of* $u$ is $N[u] = N(u) \cup \{u\}$. A graph $G$ is *bipartite graph* if $V(G)$ can be partitioned into two subsets $X$ and $Y$, such that every edge of $G$ has one end in $X$ and the other end in $Y$. Further, if each vertex of $X$ is joined to each vertex of $Y$, then such a graph $G$ is a *complete bipartite graph* and is denoted by $K_{m,n}$, where $m = |X|$ and $n = |Y|$. A vertex of a connected graph is a *cutvertex* if its removal produces a disconnected graph. A nontrivial connected graph with no cutvertices is a *block.*

The *tensor product* of two graphs $G_1$ and $G_2$ (see, [3], [11]), is the graph denoted by $G_1 \oplus G_2$, with vertex set $V(G_1 \oplus G_2) = V(G_1) \times V(G_2)$ and any two of its vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent, whenever $u_1$ is adjacent to $u_2$ in $G_1$ and $v_1$ is adjacent to $v_2$ in $G_2$. $G_1$ and $G_2$ are *factors* of $G_1 \oplus G_2$. Other popular names for the tensor product that have appeared in the literature are *Kronecker product, Cross product, Direct product and Conjunction product.* Tensor product of graphs has been extensively studied by many authors, because of their applications and importance in the computer networks, pattern recognitions and computer graphics. For any integer $p \geq 1$, $(\oplus_{i=1}^{p} K_2)$ is the tensor product $(K_2 \oplus K_2 \oplus \ldots \oplus K_2)$, which consists of $p$ factors, each being isomorphic to $K_2$. By definition, $(\oplus_{i=1}^{p} K_2) = 2^{p-1} K_2$. Consequently, $(\oplus_{i=1}^{p} K_2) = nK_2$ if and only if $n = 2^{p-1}$.

Chartrand introduced the term *graph valued function* in [1]. Line graph, middle graph and the complement of a graph, are some examples of the graph valued functions. The concept of the line graph is so natural that it has been independently discovered by many authors in the past, (see, [5]). In a graph, if any two distinct edges $x$ and $y$ are incident with a common vertex, then they are *incident edges.* The *line graph* $L(G)$ of a graph $G$, is the graph whose vertex set is the edge set of $G$ and in which two vertices are adjacent, if the corresponding edges are incident in $G$. The *iterated line graph* of $G$, denoted by $L^k(G)$ is defined in a natural way as follows : $L^0(G) = G$, $L^1(G) = L(G)$ and $L^k(G) = L(L^{k-1}(G))$ for $k \geq 1$. The notion of the middle graphs, was first introduced in 1973 in [9] as Semitotal-(line)graphs. Surprisingly, this is also studied independently in 1976 in [4]. The *middle graph* $M(G)$ of a graph $G$ is the graph, whose vertex set is $V(G) \cup E(G)$ and two vertices of $M(G)$ are adjacent if either they are incident edges of $G$ or one is a vertex and the other is an edge of $G$ incident with it.

*Graph equations* are equations in which unknowns are graphs. The term graph equation was first used in [2] . Many problems in graph theory can be formulated in terms of graph equations. In the literature of graph equations, different types of equations have been solved by several authors. For example, (see, [6], [7], [10]). This gives a motivation to solve some more equations involving tensor product graphs, line (or middle) graphs and their complements.

Now, for any integers $k, n \geq 1$, we solve the following four graph equations:

(1) $L^k(G) = H \oplus nK_2$,

(2) $M(G) = H \oplus nK_2$,

(3) $\overline{L^k(G)} = H \oplus nK_2$,

(4) $\overline{M(G)} = H \oplus nK_2$.

A pair of graphs $(G, H)$ satisfying an equation is a *solution* of the equation.

Now, we state two basic results, which shall be used in our later discussion.

**Proposition 1.1.**([3]) $C_{2p+1} \oplus K_2 = C_{2(2p+1)}$ for $p \geq 1$.

**Proposition 1.2.**([8]) *Let $G$ be a connected, bipartite graph. Then $G \oplus nK_2 = 2nG$ for $n \geq 1$.*

Next, we establish the following result for our immediate use.

**Proposition 1.3.** *For any $t$, $n \geq 1$, $C_{2t+1} \oplus nK_2 = nC_{2(2t+1)}$.*

*Proof.* Notice that $C_{2t+1} \oplus nK_2 = n(C_{2t+1} \oplus K_2)$ for $t \geq 1$. By Proposition 1.1, $C_{2t+1} \oplus K_2 = C_{2(2t+1)}$. Therefore, $C_{2t+1} \oplus nK_2 = nC_{2(2t+1)}$. □

Let $G$ and $H$ be any two disjoint graphs. The *union of $G$ and $H$*, denoted by $G \cup H$, has $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The *join of $G$ and $H$*, denoted by $G + H$, has $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G \cup H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$. In order to solve our equations, we need the following result (see, [5, Theorem 8.4]) . A graph $G$ is a line graph if and only if $G$ has none of the nine specified graphs $G_i$ $(1 \leq i \leq 9)$ as an induced subgraph. We mention here only four of nine graphs and their complements as given below:

| | |
|---|---|
| $G_1 = K_{1,3}$ | $\overline{G_1} = K_1 \cup K_3$ |
| $G_2 = (K_1 \cup K_2) + \overline{K_2}$ | $\overline{G_2} = K_2 \cup P_3$ |
| $G_3 = K_5 - x$ (*where* $x \in E(K_5)$) | $\overline{G_3} = K_2 \cup \overline{K_3}$ |
| $G_6 = K_2 + 2K_2$ | $\overline{G_6} = \overline{K_2} \cup K_{2,2}$ |

## 2. The Solution of $L^k(G) = H \oplus nK_2$

First, we establish the following lemma.

**Lemma 2.1.** *Let $G$ be any graph without isolated vertices and let $H_i$, $i \in \{1, 2\}$ be a nontrivial graph with at least one edge. Suppose $L(G) = H_1 \oplus H_2$. Then $\Delta(H_i) \leq 2$.*

*Proof.* On contrary, assume that $\Delta(H_i) \geq 3$ for some $i$. Let us consider $\Delta(H_1) \geq 3$. Then there exists a vertex $u$ in $H_1$ such that $deg(u) \geq 3$. Let $u_1, u_2$ and $u_3$ be any three neighbours of $u$ in $H_1$ and $H_2$ is a nontrivial graph having at least one edge $e = v_1v_2$. Let us consider $M = \langle N[u] \rangle$ and $N = \langle \{e\} \rangle$. Clearly, either the vertices $(u, v_1), (u_1, v_2), (u_2, v_2), (u_3, v_2)$ or $(u, v_2), (u_1, v_1), (u_2, v_1), (u_3, v_1)$ induce a subgraph isomorphic to $K_{1,3}$ in $M \oplus N$. Since $M \oplus N$ is a subgraph of $H_1 \oplus H_2$, it follows that $H_1 \oplus H_2$ contains $K_{1,3}$ as an induced subgraph. Hence $G_1$ is a forbidden induced subgraph of $H_1 \oplus H_2$. From [5, Theorem 8.4], $H_1 \oplus H_2$ is not a line graph of $G$. This is a contradiction to the fact that $H_1 \oplus H_2 = L(G)$. □

Now, we shall solve the equation $L^k(G) = H \oplus nK_2$ for $k, n \geq 1$.

**Theorem 2.2.** *Let $G$ be a graph without isolated vertices and let $H$ be any connected graph. Then $L^k(G) = H \oplus nK_2$ for $k, n \geq 1$, holds if and only if $(G, H)$ is one of the following pairs of graphs:*

1. $(2nP_{m+k}, P_m)$; $m \geq 1$,
2. $(2nC_{2t}, C_{2t})$; $t \geq 2$,
3. $(nC_{4t+2}, C_{2t+1})$; $t \geq 1$.

*Proof.* We first consider the case $k = 1$ and find all pairs of graphs $(G, H)$ satisfying the following equation

$$L(G) = H \oplus nK_2 \quad \text{for} \quad n \geq 1. \tag{2.1}$$

By Lemma 2.1, $H$ is either $P_m$ for $m \geq 1$ or $C_p$ for $p \geq 3$, because $H$ is connected. There are two cases to discuss :
**Case 1.** Assume that $H = P_m$ ; $m \geq 1$. From Proposition 1.2, $L(G) = 2nP_m$. Hence, $G = 2nP_{m+1}$. Consequently, $(2nP_{m+1}\ ,\ P_m)$ is the solution of the equation (2.1).
**Case 2.** Assume that $H = C_p$ ; $p \geq 3$.
We discuss two possibilities depending on $p$ :
**(2.1).** If $p = 2t + 1$ ; $t \geq 1$, then $H$ is an odd cycle. By Proposition 1.3, $L(G) = nC_{2(2t+1)}$. Consequently, $G = nC_{4t+2}$. Thus, $(nC_{4t+2},\ C_{2t+1})$ is the possible solution of our equation (2.1).
**(2.2).** If $p = 2t$ ; $t \geq 2$, then $H$ is an even cycle, which is connected and bipartite. From Proposition 1.2, $L(G) = 2nC_{2t}$. Hence, $G = 2nC_{2t}$. Thus, $(2nC_{2t},\ C_{2t})$ is the solution of our equation (2.1).
Finally, consider $k \geq 2$. In this situation, the solution of the equation $L^k(G) = H \oplus nK_2$ directly follows by the iterated nature of line graphs.

The converse of this theorem is obvious and hence it is omitted. □

For any graph $G$, the *endedge graph* of $G$, denoted by $G^+$, is the graph obtained from $G$ by adjoining an endedge $u_i u_i^{'}$ at each vertex $u_i$ of $G$. Hamada et al., have shown in [4] that $M(G) = L(G^+)$.

Now, we shall solve the equation $M(G) = H \oplus nK_2$ for $n \geq 1$. Theorem 2.2, with $k = 1$, provides three pairs of graphs $(2nP_{m+1},\ P_m)$ for $m \geq 1$, $(2nC_{2t}\ ,\ C_{2t})$ for $t \geq 2$ and $(nC_{4t+2}\ ,\ C_{2t+1})$ for $t \geq 1$, which are the solutions of the equation $L(G) = H \oplus nK_2$ for $n \geq 1$. Among these pairs, only two pairs $(2nK_2, K_1)$ and $(2nP_4, P_3)$ are of the form $(G^+, H)$. In view of the result $M(G) = L(G^+)$, the following corollary is evident.

**Corollary 2.3.** *Let $G$ be any graph and let $H$ be any connected graph. Then the equation $M(G) = H \oplus nK_2$ for $n \geq 1$, holds if and only if $(G, H)$ is either $(2nK_1,\ K_1)$ or $(2nK_2,\ P_3)$.*

## 3. The Solution of $\overline{L^k(G)} = H \oplus nK_2$

We first solve the following equation

$$\overline{L(G)} = H \oplus nK_2 \quad \text{for} \quad n \geq 1. \tag{3.1}$$

**Theorem 3.1.** *Let $G$ and $H$ be any two graphs. Then $\overline{L(G)} = H \oplus nK_2$ for $n \geq 1$, holds if and only if $(G, H)$ is one of the following two possibilities:*

1. *$(K_{1,2mn},\ \overline{K_m})$ for $m \geq 1$.*
2. *$n = 1$ and $(K_{2,m},\ K_m)$ for $m \geq 2$.*

*Proof.* We discuss two cases, depending on the nature of $H$:
**Case 1.** Suppose that $H$ has no edges. Then $H = \overline{K_m}$ for $m \geq 1$. Therefore, $\overline{L(G)} = 2n\overline{K_m}$ for $n \geq 1$. Consequently, $L(G) = K_{2mn}$ and $G = K_{1,2mn}$. In this case, $(K_{1,2mn},\ \overline{K_m})$ is the solution of our equation (3.1).
**Case 2.** Suppose that $H$ has at least one edge. Then for $n \geq 2$, we see that $H \oplus nK_2$ contains an induced subgraph isomorphic to $K_2 \oplus nK_2$. It is easy to check that $K_2 \oplus nK_2$ contains $\overline{G_3}$ as a forbidden induced subgraph of $\overline{L(G)}$. Hence, there is no solution to $H \oplus nK_2 = \overline{L(G)}$.

Now, there are two possibilities, depending on the connectivity of $H$ and $n = 1$:
**(2.1).** Assume that $H$ is disconnected. Since $n = 1$, immediately an induced subgraph isomorphic to $K_1 \cup K_2$ appears in $H$. Further, we see that $(K_1 \cup K_2) \oplus K_2 = \overline{K_2} \cup 2K_2$ appears as an induced subgraph in $H \oplus K_2$ and it also contains a forbidden induced subgraph isomorphic to $\overline{G_3}$. Therefore, $H \oplus nK_2 = \overline{L(G)}$ has no solution.
**(2.2).** Assume that $H$ is connected.
We discuss three cases, depending on the size of $\Delta(H)$ :
**(2.2.1).** $\Delta(H) = 1$.
Since $H$ is connected with at least one edge, it follows that $H = K_2$. Then $\overline{L(G)} = 2K_2$ and hence $L(G) = K_{2,2}$. Therefore, $G = K_{2,2}$.
**(2.2.2).** $\Delta(H) = 2$.
Then $H$ is either a path $P_m$ for $m \geq 3$ or a cycle $C_p$ for $p \geq 3$. We see that $H$ is neither $P_m$ nor $C_p$ for $p \geq 4$. Otherwise, $H \oplus K_2$ contains a forbidden induced subgraph isomorphic to $\overline{G_2}$ and hence $H \oplus K_2 = \overline{L(G)}$ has no solution. In this case, the only possibility for $H$ is $K_3$. Then $\overline{L(G)} = K_3 \oplus K_2$ and hence $G$ is $K_{2,3}$.
**(2.2.3).** $\Delta(H) \geq 3$.
There are two cases to discuss :
Suppose that $H$ is a block. Then $H = K_m$ for $m \geq 4$. Otherwise, $K_1 \cup K_2,\ P_3$ or $K_4 - x$ (where $x \in E(K_4)$), appears as an induced subgraph in $H$. Consequently, $H \oplus K_2$ contains an induced subgraph isomorphic to $\overline{G_3},\ \overline{G_2}$ or $\overline{G_6}$. Therefore, $H \oplus K_2 = \overline{L(G)}$ has no solution. In this case, $H = K_m$. Then $\overline{L(G)} = K_m \oplus K_2$

and hence $G = K_{2,m}$.
Suppose that $H$ is not a block. Since $H$ is connected and $\Delta(G) \geq 3, H$ is a nontrivial graph having at least one cutvertex. Immediately, $P_3$ is an induced subgraph in $H$. Hence, $P_3 \oplus K_2$ appears in $H \oplus K_2$ and hence $\overline{G_2}$ appears as a forbidden induced subgraph in $H \oplus K_2$. Therefore, $\overline{L(G)} = H \oplus K_2$ has no solution.

The converse of this theorem is obvious and hence it is omitted. □

The immediate consequence of the above theorem is the following corollary.

**Corollary 3.2.** *Let $G$ and $H$ be any two graphs. $\overline{L^k(G)} = H \oplus nK_2$ for $k \geq 2$ and $n \geq 1$, holds if and only if $n = 1$ and $(G, H)$ is either $(P_{k+2},\ K_1)$ or $(C_4,\ K_2)$.*

Finally, we determine the solutions of the equation $\overline{M(G)} = H \oplus nK_2$ for $n \geq 1$. Notice that among the solutions of $\overline{L(G)} = H \oplus nK_2$ in Theorem 3.1, none is of the form $(G^+, H)$. Hence, the following corollary is evident.

**Corollary 3.3.** *For any two graphs $G$ and $H$, the equation $\overline{M(G)} = H \oplus nK_2$ for $n \geq 1$, has no solution.*

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