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Dynamical Behavior of a Third-Order Difference Equation with Arbitrary Powers

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Abstract. The aim of this paper is to investigate the dynamical behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_n}{\beta + \gamma x_{n-1}^p x_{n-2}^q},\ n = 0, 1, \ldots,$$

where the parameters $\alpha, \beta, \gamma, p, q$ are non-negative numbers and the initial values x_{-2}, x_{-1}, x_0 are positive numbers. Also, some numerical examples are given to verify our theoretical results.

1. Introduction

In the last twenty years, many papers appeared focusing on the investigation of the qualitative analysis of solutions of difference equations (see [2, 3, 4, 7, 8, 9, 14, 15, 17, 22] and the references cited therein). Applications of difference equations have appeared in many areas such as population dynamics, ecology, economics, probability theory, genetics, psychology, physics, engineering, sociology, statistical problems, stochastic time series, number theory, electrical networks, neural net-

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works, queuing problems and so on. Namely, the theory of difference equations gets a central position in applicable analysis. Hence, it is very valuable to study the dynamical behavior of solutions of non-linear rational difference equations.

In our opinion, it is of a great importance to investigate not only non-linear difference equations, but also those equations which contain powers of arbitrary positive numbers (see [3, 4, 6, 7, 11, 13, 21, 23]).

The purpose of this paper is to study the local asymptotic stability of equilibria, the periodic nature and the global behavior of solutions of the following fractional difference equation

$$x_{n+1}=\frac{\alpha x_n}{\beta+\gamma x_{n-1}^{p}x_{n-2}^{q}},\ n\in\mathbb{N} \tag{1.1}$$

where the parameters α,β,γ,p,q are non-negative numbers and the initial values x_{-2},x_{-1},x_0 are positive numbers such that the denominator is always positive.

In [7], El-Owaidy *et al.* investigated the global behavior of the following rational recursive sequence

$$x_{n+1}=\frac{\alpha x_{n-1}}{\beta+\gamma x_{n-2}^{p}},\ n\in\mathbb{N}$$

with non-negative parameters and non-negative initial values.

By generalizing the results of El-Owaidy *et al.* [7], Chen *et al.* [6] studied the dynamical behavior of the following rational difference equation

$$x_{n+1}=\frac{\alpha x_{n-k}}{\beta+\gamma x_{n-l}^{p}},\ n\in\mathbb{N}$$

where $k,l\in\mathbb{N}$, the parameters are positive real numbers and the initial values $x_{-\max\{k,l\}},\dots,x_{-1},x_0\in(0,\infty)$.

Ahmed in [3, 4] investigated the global asymptotic behavior and the periodic character of the difference equations

$$x_{n+1}=\frac{\alpha x_{n-1}}{\beta+\gamma x_{n}^{p}x_{n-2}^{q}},\ n\in\mathbb{N}$$

and

$$x_{n+1}=\frac{\alpha x_{n-1}}{\beta+\gamma\prod\limits_{i=l}^{k}x_{n-2i}^{p_i}},n\in\mathbb{N}$$

where the parameters are non-negative real numbers and the initial values are non-negative real numbers.

In [10], Erdogan *et al.* investigated the dynamical behavior of positive solutions of the following higher order difference equation

$$x_{n+1}=\frac{\alpha x_{n-1}}{\beta+\gamma\sum\limits_{k=1}^{t}x_{n-2k}\prod\limits_{k=1}^{t}x_{n-2k}},n\in\mathbb{N}$$

where the parameters are non-negative real numbers and the initial values are non-negative real numbers.

In [16], Karatas investigated the global behavior of the equilibria of the following difference equation

$$x_{n+1} = \frac{Ax_{n-m}}{B + C\prod\limits_{i=0}^{2k+1} x_{n-i}}, n \in \mathbb{N}$$

where the parameters are non-negative real numbers and the initial values are non-negative real numbers.

If some parameters of Eq.(1.1) are zero, then special cases emerge. If $\alpha = 0$, we have the trivial case. If $\beta = 0$, Eq.(1.1) is reduced to a linear difference equation by the change of variables $x_n = e^{y_n}$. If $\gamma = 0$, Eq.(1.1) is reduced to a linear first order difference equation.

Note that Eq.(1.1) can be reduced to the following fractional difference equation

$$y_{n+1} = \frac{ry_n}{1 + y_{n-1}^p y_{n-2}^q}, \; n \in \mathbb{N} \tag{1.2}$$

by the change of variables $x_n = (\frac{\beta}{\gamma})^{\frac{1}{p+q}} y_n$ with $r = \frac{\alpha}{\beta}$. So, we shall study Eq.(1.2).

2. Preliminaries

For the sake of completeness and the readers convenience, we are including some basic results (one can see [1, 5, 12, 18, 19, 20] and the references cited therein).

Let I be an interval of real numbers and let $f : I \times I \times I \to I$ be a continuously differentiable function. Then for any condition $x_{-2}, x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}), \; n \in \mathbb{N} \tag{2.1}$$

has a unique positive solution $\{x_n\}_{n=-2}^{\infty}$.

Definition 2.1. An *equilibrium point* of Eq.(2.1) is a point $\overline{x}$ that satisfies

$$\overline{x} = f(\overline{x}, \overline{x}, \overline{x}).$$

The point $\overline{x}$ is also said to a *fixed point* of the function f.

Definition 2.2. Let $\overline{x}$ be a positive equilibrium of (2.1).

(a) $\overline{x}$ is *stable* if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every positive solution $\{x_n\}_{n=-2}^{\infty}$ of (2.1) with $\sum\limits_{i=-2}^{0} |x_i - \overline{x}| < \delta$, $|x_n - \overline{x}| < \varepsilon$, holds for $n \in \mathbb{N}$.

(b) $\overline{x}$ is *locally asymptotically stable* if $\overline{x}$ is stable and there is $\gamma > 0$ such that $\lim x_n = \overline{x}$ holds for every positive solution $\{x_n\}_{n=-2}^{\infty}$ of (2.1) with $\sum\limits_{i=-2}^{0} |x_i - \overline{x}| < \gamma$.

(c) $\overline{x}$ is a *global attractor* if $\lim x_n = \overline{x}$ holds for every positive solution $\{x_n\}_{n=-2}^{\infty}$ of (2.1).

(d) $\overline{x}$ is *globally asymptotically stable* if $\overline{x}$ is both stable and global attractor.

Definition 2.3. The linearized equation of (2.1) about the equilibrium point $\overline{x}$ is

$$y_{n+1} = \zeta_0 y_n + \zeta_1 y_{n-1} + \zeta_2 y_{n-2},\ n \in \mathbb{N} \tag{2.2}$$

where

$$\zeta_0 = \frac{\partial f}{\partial x_n}(\overline{x},\overline{x},\overline{x}),\ \zeta_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x},\overline{x},\overline{x}),\ \zeta_2 = \frac{\partial f}{\partial x_{n-2}}(\overline{x},\overline{x},\overline{x}).$$

The characteristic equation of (2.2) is

$$F(\lambda) = \lambda^3 - \zeta_0\lambda^2 - \zeta_1\lambda - \zeta_2 = 0. \tag{2.3}$$

The following result, known as the *Linearized Stability Theorem*, is very useful in determining the local stability character of the equilibrium point $\overline{x}$ of equation (2.1).

Theorem 2.4. (*The Linearized Stability Theorem*)
Assume that the function F is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\overline{x}$. Then, the following statements are true:

(i) *If all roots of* (2.3) *have absolute value less than one, then the equilibrium point $\overline{x}$ of* (2.1) *is locally asymptotically stable.*

(ii) *If at least one of the roots of* (2.3) *has absolute value greater than one, then the equilibrium point $\overline{x}$ of* (2.1) *is unstable. Also, the equilibrium point $\overline{x}$ of* (2.1) *is called a saddle point if* (2.3) *has roots both inside and outside the unit disk.*

Theorem 2.5. *Assume that α_2, α_1, and α_0 are real numbers. Then, a necessary and sufficient condition for all roots of the equation*

$$\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = 0$$

to lie inside the unit disk is

$$|\alpha_2 + \alpha_0| < 1 + \alpha_1,\ \ |\alpha_2 - 3\alpha_0| < 3 - \alpha_1 \text{ and } \alpha_0^2 + \alpha_1 - \alpha_0\alpha_2 < 1. \tag{2.4}$$

3. Main Results

In this section we prove our main results.

Theorem 3.1. *We have the following cases for the equilibrium points of Eq.*(1.2).

(i) $\overline{y}_0 = 0$ *is always the equilibrium point of Eq.*(1.2).

(ii) *If* $r > 1$, *then Eq.*(1.2) *has the positive equilibrium* $\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$.

(iii) *If* $r < 1$ *and* $\frac{1}{p+q}$ *is an even positive integer, then Eq.*(1.2) *has the positive equilibrium* $\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$ *which is always in the interval* $(0,1)$.

Proof. The proof is easily obtained from the definition of equilibrium point. □

In the following theorems, we investigate the local asymptotic behavior of the equilibria and the global behavior of solutions of Eq.(1.2) with $r, p, q > 0$ and positive initial conditions.

Theorem 3.2. *For Eq.*(1.2), *we have the following results.*

(i) *If* $r < 1$, *then the zero equilibrium point is locally asymptotically stable.*

(ii) *If* $r > 1$, *then the zero equilibrium point is locally unstable.*

(iii) *If* $r = 1$, *then the zero equilibrium point is non-hyperbolic point.*

(iv) *Assume that* $r > 1$ *and let* $q < \frac{r}{r-1}(-\frac{1}{2} + \frac{1}{2}\sqrt{5 - 4p(\frac{r-1}{r})})$. *Then the positive equilibrium point* $\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$ *is locally asymptotically stable if either*

$$q \leq p \tag{3.1}$$

or

$$p < q < p + 2\frac{r}{r-1}. \tag{3.2}$$

(v) *Assume that* $r \in (0,1)$ *such that* $\frac{1}{p+q}$ *is an even positive integer. Then the positive equilibrium point* $\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$ *is unstable.*

Proof. The linearized equation associated with Eq.(1.2) about zero equilibrium has the form

$$z_{n+1} - r z_n = 0, \; n = 0, 1, ... \tag{3.3}$$

The characteristic equation of (3.3) about the zero equilibrium is

$$\lambda^3 - r\lambda^2 = 0. \tag{3.4}$$

So, the proof of (i), (ii) and (iii) follows immediately from Linearized Stability Theorem.

For the proof (iv) suppose that $r > 1$, then the linearized equation associated with Eq.(1.2) about $\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$ is

$$z_{n+1} - z_n + \frac{p(r-1)}{r} z_{n-1} + \frac{q(r-1)}{r} z_{n-2} = 0, \; n = 0, 1, ... \tag{3.5}$$

The associated characteristic equation about the equilibrium $\overline{y}_1 = (r-1)^{\frac{1}{p+q}}$ is

$$\lambda^3 - \lambda^2 + \frac{p(r-1)}{r}\lambda + \frac{q(r-1)}{r} = 0. \tag{3.6}$$

According to Theorem (2.2) and (3.6), we have $\alpha_2 = -1$, $\alpha_1 = \frac{p(r-1)}{r}$ and $\alpha_0 = \frac{q(r-1)}{r}$.

If $q \le p$, then we have

$$-1 + q(\frac{r-1}{r}) < 1 + p(\frac{r-1}{r}).$$

Otherwise, if

$$q < p + 2\frac{r}{r-1},$$

then

$$q(\frac{r-1}{r}) < p(\frac{r-1}{r}) + 2.$$

This implies that

$$-1 + q(\frac{r-1}{r}) < 1 + p(\frac{r-1}{r}).$$

Therefore, in all cases

$$|\alpha_2 + \alpha_0| < 1 + \alpha_1.$$

As $q < \frac{r}{r-1}(-\frac{1}{2} + \frac{1}{2}\sqrt{5 - 4p(\frac{r-1}{r})})$, we get the following two results:
Firstly: Multiplying both sides by $\frac{r-1}{r}$, we get

$$q(\frac{r-1}{r}) + \frac{1}{2} < \frac{1}{2}\sqrt{5 - 4p(\frac{r-1}{r})}.$$

That is,

$$q^2(\frac{r-1}{r})^2 + q(\frac{r-1}{r}) + \frac{1}{4} < \frac{1}{4}(5 - 4p(\frac{r-1}{r})).$$

Then,

$$q^2(\frac{r-1}{r})^2 + p(\frac{r-1}{r}) + q(\frac{r-1}{r}) < 1.$$

Therefore,

$$\alpha_0^2 + \alpha_1 - \alpha_0\alpha_2 < 1.$$

Secondly: As $\alpha_1 < 1 - \alpha_0^2 - \alpha_0$, we get

$$\begin{aligned} 1 + 3q(\frac{r-1}{r}) + p(\frac{r-1}{r}) &< 1 + 3q(\frac{r-1}{r}) + 1 - q(\frac{r-1}{r}) - q^2(\frac{r-1}{r})^2 \\ &= 2 + 2q(\frac{r-1}{r}) - q^2(\frac{r-1}{r})^2. \end{aligned}$$

Note that

$$2q(\frac{r-1}{r}) - q^2(\frac{r-1}{r})^2 < 1.$$

Otherwise, $\left(q(\frac{r-1}{r}) - 1\right) \leq 0$, which is either a contradiction or contradicts the given assumption. Then, we have that

$$1 + 3q(\frac{r-1}{r}) + p(\frac{r-1}{r}) < 3.$$

This implies that

$$1 + 3q(\frac{r-1}{r}) < 3 - p(\frac{r-1}{r}).$$

Therefore,

$$|\alpha_2 + \alpha_0| < 3 - \alpha_1.$$

Applying Theorem (2.2), we get the result. This completes the proof (iv).

For the proof (v) we assume that $r < 1$, then the linearized equation associated with Eq.(1.2) about $\overline{y}_2 = (r-1)^{\frac{1}{p+q}}$ is

$$t_{n+1} - t_n + \frac{p(r-1)}{r} t_{n-1} + \frac{q(r-1)}{r} t_{n-2} = 0, \; n = 0, 1, \ldots.$$

Therefore, the characteristic equation about the equilibrium $\overline{y}_2$ is

$$\lambda^3 - \lambda^2 + \frac{p(r-1)}{r}\lambda + \frac{q(r-1)}{r} = 0.$$

If we set the function as follows;

$$g(\lambda) = \lambda^3 - \lambda^2 + \frac{p(r-1)}{r}\lambda + \frac{q(r-1)}{r},$$

then, it is clear that

$$g(1) = \frac{(p+q)(r-1)}{r} < 0$$

and

$$\lim_{\lambda\to\infty} g(\lambda) = \infty.$$

So, $g(\lambda)$ has a root in the interval $(1, \infty)$. This completes the proof. □

Theorem 3.3. *Every solution of Eq.(1.2) is bounded.*

Proof. Let $\{y_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1.2). For the sake of contradiction, assume that the solution is not bounded from above. Then, there exists a subsequence $\{y_{n_m+1}\}_{m=0}^{\infty}$ such that

$$\lim_{n\to\infty} n_m = \infty, \; \lim_{n\to\infty} y_{n_m+1} = \infty$$

and

$$y_{n_m+1} = \max\{y_n : n \leq n_m\}, \text{ for all } m \geq 0.$$

From Eq.(1.2) we have

$$y_{n_m+1} = \frac{ry_{n_m}}{1 + y^p_{n_m-1}y^q_{n_m-2}} \to \infty \quad \text{as } m \to \infty.$$

So, we obtain $y_{n_m} \to \infty$. Similarly, we can obtain $y_{n_m-1} \to \infty$ and $y_{n_m-2} \to \infty$ as $m \to \infty$. Hence, for sufficiently large m

$$0 \leq y_{n_m+1} - y_{n_m} = \frac{y_{n_m}(r - 1 - y^p_{n_m-1}y^q_{n_m-2})}{1 + y^p_{n_m-1}y^q_{n_m-2}} < 0,$$

which is a contradiction. This completes the proof. □

Theorem 3.4. *Assume that $r > 1$. Then the following statements are true:*

(i) *Let $\{y_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1.2) such that for some $n_0 \in \mathbb{N}$, either*

$$y_n > \overline{y}_1 = \sqrt[p+q]{r-1} \quad \textit{for} \quad n > n_0$$

or

$$y_n < \overline{y}_1 = \sqrt[p+q]{r-1} \quad \textit{for} \quad n > n_0.$$

Then, for $n \geq n_0 + 2$, the sequence $\{y_n\}$ is monotonic and

$$\lim_{n\to\infty} y_n = \overline{y}_1.$$

(ii) *Let $\{y_n\}_{n=-2}^{\infty}$ be a non-oscillatory solution of Eq.(1.2) and consider the positive equilibrium $\overline{y}_1$. Then, the extreme in each semicycle about $\overline{y}_1$ occurs at either the second term or the third.*

Proof. Assume that for some $n > n_0$

$$y_n > \overline{y}_1 = \sqrt[p+q]{r-1}$$

holds. That is, for $n \geq n_0 + 2$ we have

$$y_{n+1} = \frac{ry_n}{1 + y^p_{n-1}y^q_{n-2}} < \frac{ry_n}{1 + \overline{y}_1^{p+q}} = y_n.$$

Hence, we obtain that $\{y_n\}$ is monotonic for $n \geq n_0 + 2$. Let $\lim_{n\to\infty} y_n = l$.

For the sake of contradiction, assume that $l > \overline{y}_1$. Then, we obtain

$$l(1 + l^{p+q}) = rl,$$

from which we see that $l = \sqrt[p+q]{r-1}$ which contradicts the fact that $\overline{y}_1$ is the only positive equilibrium point.

The other case is similar and will be omitted. (ii) We prove only in case of positive semicycles. The proof for negative semicycles are similar and will be omitted. Assume that for some $N \geq 0$, the first three terms in a positive smicycle are y_N, y_{N+1} and y_{N+2}. Then

$$y_N \geq \overline{y}_1, \quad y_{N+1} > \overline{y}_1 \quad y_{N+2} > \overline{y}_1$$

and

$$\begin{aligned} y_{N+3} &= \frac{ry_{N+2}}{1+y_{N+1}^p y_N^q} < \frac{ry_{N+2}}{1+\overline{y}_1^{p+q}} = y_{N+2}, \\ y_{N+4} &= \frac{ry_{N+3}}{1+y_{N+2}^p y_{N+1}^q} < \frac{ry_{N+3}}{1+\overline{y}_1^{p+q}} = y_{N+3}, \end{aligned}$$

as desired. □

Theorem 3.5. *Assume that $r < 1$, then the zero equilibrium point of Eq.(1.2) is globally asymptotically stable.*

Proof. We know by Theorem 3.2 that, the zero equilibrium point of Eq.(1.2) is locally asymptotically stable. Hence, it suffices to show that

$$\lim_{n\to\infty} y_n = 0$$

for any positive solution $\{y_n\}_{n=-2}^{\infty}$ of Eq.(1.2). Let $\{y_n\}_{n=-2}^{\infty}$ be a positive solution of Eq.(1.2). Then we have for all $n \geq 0$.

$$0 < y_{n+1} = \frac{ry_n}{1+y_{n-1}^p y_{n-2}^q} < ry_n.$$

By induction we obtain

$$y_n < r^n y_0.$$

For $r < 1$, we get

$$\lim y_n = 0.$$

This completes the proof. □

Theorem 3.6. *If $p+2 \geq q$, then Eq.(1.2) has no prime period-2 solutions. If $q > p+2$ and $r > \frac{q-p}{q-p-2}$, then Eq.(1.2) has prime period-2 solutions.*

Proof. Assume that a prime two periodic solution exists in the following form

$$\ldots, x, y, x, y, \ldots$$

of Eq.(1.2). From Eq.(1.2) , we get the following equalities:

$$x = \frac{ry}{1+x^p y^q} \quad \text{and} \quad y = \frac{rx}{1+y^p x^q}.$$

That is,

$$ry - x = x^{p+1} y^q \quad \text{and} \quad rx - y = x^q y^{p+1}.$$

This implies that

$$(\frac{x}{y})^{p+1-q} = \frac{ry-x}{rx-y}.$$

Now if we set $\lambda = \frac{x}{y}$, then we get

$$r - \lambda = \lambda^{p+1-q}(r\lambda - 1). \tag{3.7}$$

As $\lambda^{p+1-q} > 0$ always, we obtain the relation $\frac{1}{r} < \lambda < r$. We consider the following cases:

Case 1. $p+2 \geq q$. We shall show that Eq.(3.7) has no positive real roots except for $\lambda = 1$. If $p+2-q = 0$, then from Eq.(3.7) we get $\lambda = 1$. Now suppose that $p+2-q > 0$. It is clear that $\lambda = 1$ is a root of Eq.(3.1). Consider the function

$$h(\lambda) = r\lambda^{p+2-q} - \lambda^{p+1-q} + \lambda - r.$$

The derivative of the function h is

$$(p+2-q)r\lambda^{p+1-q} - (p+1-q)\lambda^{p-q} + 1.$$

For all values of $\lambda \geq 0$, we have

$$(p+1-q)\lambda^{p-q}(r\lambda - 1) + r\lambda^{p+1-q} + 1 > 0.$$

That is, h is an increasing function. Therefore, $\lambda = 1$ is the unique zero of the function h.

Case 2 $p+2-q < 0$. From Eq.(3.7) we get

$$\lambda^{q-p} - r\lambda^{q-p-1} + r\lambda - 1 = 0.$$

Let

$$g(\lambda) = \lambda^{q-p} - r\lambda^{q-p-1} + r\lambda - 1 = 0.$$

Using simple analysis, if $r > \frac{q-p}{q-p-2}$, then the function g has a zero λ_0 other than $\lambda = 1$.

Now, by a simple calculation, x and y satisfy the relation

$$x^2 - y^2 = x^q y^{p+2} - x^{p+2} y^q.$$

If we set $x_{-2} = x_0 = x$ and $x_{-1} = y$. Then

$$x_1 = \frac{rx}{1 + y^p x^q} = \frac{rx}{1 + (\frac{x^2}{y^2} - 1 + x^{p+2}y^{q-2})} = \frac{ry^2}{x(1 + x^p y^q)} = y$$

and

$$x_2 = \frac{ry}{1 + x^p y^q} = \frac{rx}{1 + (\frac{y^2}{x^2} - 1 + x^{q-2}y^{p+2})} = \frac{rx^2}{y(1 + x^q y^p)} = x.$$

This completes the proof. □

4. Numerical Examples

In this section, we will give some interesting numerical examples in order to verify the theoretical results of this paper.

Example 4.1. Figure 1 shows that if $r = 1.09$ and $p = q = 1$, then the solution $\{y_n\}_{n=-2}^{\infty}$ of Eq.(1.2) with initial conditions $y_{-2} = 2$, $y_{-1} = 1.8$, $y_0 = 1$ converges (increasingly) to $\overline{y}_1 = \sqrt{1.09 - 1} = 0.3$.

Example 4.2. Figure 2 shows that if $r = 1.1$, $p = 0.25$ and $q = 0.25$, then the solution $\{y_n\}_{n=-2}^{\infty}$ of Eq.(1.2) with initial conditions $y_{-2} = 2$, $y_{-1} = 0.8$ and $y_0 = 1$ converges (decreasingly) to $\overline{y}_1 = (0.1)^2 = 0.01$.

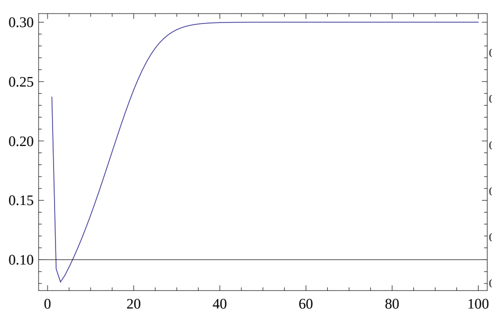

Figure 1: $y_{n+1} = \frac{1.09y_n}{1+y_{n-1}y_{n-2}}$

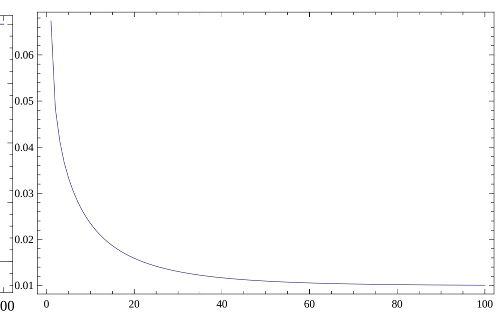

Figure 2: $y_{n+1} = \frac{1.1y_n}{1+y_{n-1}^{0.25}y_{n-2}^{0.25}}$

Example 4.3. Figure 3 shows that if $r = 1.8$, $p = 2$ and $q = 0.25$, then the solution $\{y_n\}_{n=-2}^{\infty}$ of Eq.(1.2) with initial conditions $y_{-2} = 2$, $y_{-1} = 0.8$ and $y_0 = 1$ oscillates about the equilibrium point $\overline{y}_1 = (0.8)^{\frac{1}{2.25}} = 0.905$.

Example 4.4. Figure 4 shows that if $r = 0.32$, $p = 0.01$ and $q = 0.49$ ($\frac{1}{p+q} = 2$), then the equilibrium point $\overline{y}_2 = (0.32 - 1)^{\frac{1}{0.5}} \simeq 0.46$ of Eq.(1.2) is unstable.

Example 4.5. Figure 5 shows that if $r = 3.5$ and $p = 1$ and $q = 4$ ($r = 3.5 > \frac{q-p}{q-p-2} = 3$), then the solution $\{y_n\}_{n=-2}^{\infty}$ of Eq.(1.2) with initial conditions $y_{-2} = (12)^{1/5}$, $y_{-1} = 0.5(12)^{1/5}$, $y_0 = (12)^{1/5}$ is a period-2 solution.

Example 4.6. Figure 6 shows that if $r = 2.5$ and $p = 1$ and $q = 4$ ($r = 2.5 < \frac{q-p}{q-p-2} = 3$), then the solution $\{y_n\}_{n=-2}^{\infty}$ of Eq.(1.2) with initial conditions $y_{-2} = (12)^{1/5}$, $y_{-1} = 0.5(12)^{1/5}$, $y_0 = (12)^{1/5}$ is not a periodic solution.

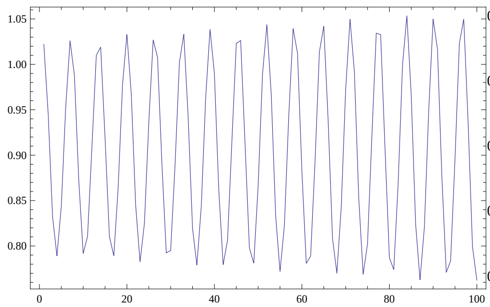

Figure 3: $y_{n+1} = \frac{1.8y_n}{1+y_{n-1}^2 y_{n-2}^{0.25}}$

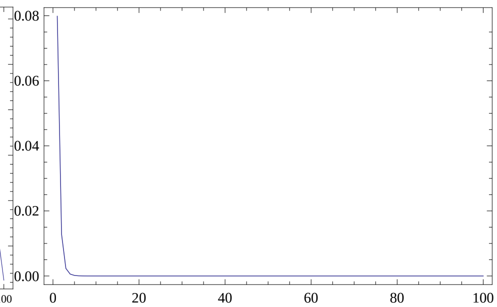

Figure 4: $y_{n+1} = \frac{0.32y_n}{1+y_{n-1}^{0.01} y_{n-2}^{0.49}}$

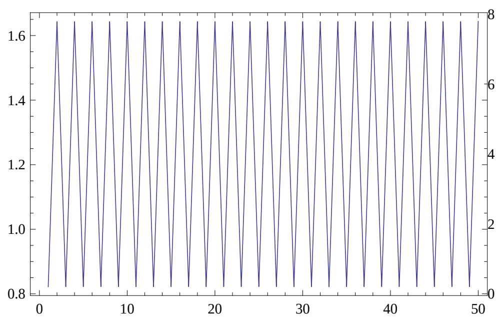

Figure 5: $y_{n+1} = \frac{3.5y_n}{1+y_{n-1} y_{n-2}^4}$

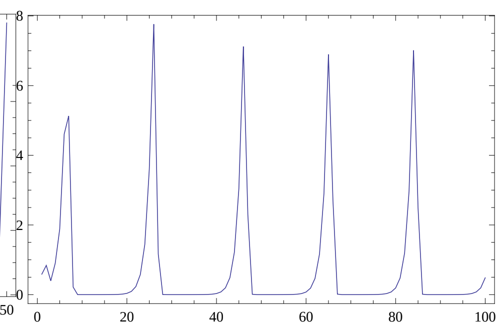

Figure 6: $y_{n+1} = \frac{2.5y_n}{1+y_{n-1} y_{n-2}^4}$

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