

## Differential Subordination and Superordination Results associated with Srivastava-Attiya Integral Operator

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ABSTRACT. Differential subordination and superordination results associated with a generalized Hurwitz-Lerch Zeta function in the open unit disk are obtained by investigating appropriate classes of admissible functions. In particular some inequalities for generalized Hurwitz-Lerch Zeta function are obtained.

### 1. Introduction

Let  $\mathcal{H}$  denote the class of analytic functions in the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  and  $\mathcal{S}$  be the subclass of  $\mathcal{H}$  consisting of functions which are univalent in  $\mathbb{D}$ . For  $a \in \mathbb{C}$  and  $n \in \mathbb{N}$  consider

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, z \in \mathbb{D}\},$$

with  $\mathcal{H}_0 = \mathcal{H}[0, 1]$  and  $\mathcal{H}_1 = \mathcal{H}[1, 1]$ . We denote by  $\mathcal{A}$  the class of the functions  $\mathcal{H}[a, 1]$  which are normalized by the condition  $f(0) = 0 = f'(0) - 1$  and have representation of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

Given two functions  $f, g \in \mathcal{H}$ , we say that  $f$  is subordinated to  $g$  or  $g$  is said to

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be superordinate to  $f$ , and write  $f(z) \prec g(z)$ , if there exists a function  $w$  analytic in  $\mathbb{D}$  with  $w(0) = 0$  and  $|w(z)| < 1$  for all  $z \in \mathbb{D}$ , such that  $f(z) = g(w(z))$ . In particular, if  $g$  is univalent in  $\mathbb{D}$ , then  $f(z) \prec g(z)$  if and only if  $f(0) = g(0)$  and  $f(\mathbb{D}) \subset g(\mathbb{D})$ . We denote by  $\mathcal{Q}$  the set of functions  $q$  that are analytic and injective on  $\mathbb{D} \setminus E(q)$ , where

$$E(q) = \left\{ \zeta \in \partial\mathbb{D} : \lim_{z \rightarrow \zeta} q(z) = \infty \right\},$$

and are such that  $q'(\zeta) \neq 0$  for  $\zeta \in \partial\mathbb{D} \setminus E(q)$ . Further let the subclass of  $\mathcal{Q}$  for which  $q(0) = a$  be denoted by  $\mathcal{Q}_a$ ,  $\mathcal{Q}(0) \equiv \mathcal{Q}_0$  and  $\mathcal{Q}(1) \equiv \mathcal{Q}_1$ .

The general Hurwitz-Lerch Zeta function  $\phi(z, s, a)$  is defined by [22, p. 21]

$$(1.2) \quad \phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(a+n)^s} = \frac{1}{a^s} + \frac{z}{(1+a)^s} + \frac{z^2}{(2+a)^s} + \dots,$$

where  $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ ,  $s \in \mathbb{C}$  when  $|z| < 1$ , and  $\operatorname{Re} s > 1$  when  $|z| = 1$ . The general Hurwitz-Lerch Zeta function contains, as its special cases, well-known functions as the Riemann and Hurwitz (or generalized) Zeta function, Lerch Zeta function, the Polylogarithmic function and the Lipschitz-Lerch Zeta function. A generalization of function  $\phi(z, s, a)$  was studied by Lin and Srivastava [16] in the following form

$$(1.3) \quad \Phi_{\mu, \nu}^{(\rho, \sigma)}(z, s, a) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s},$$

where  $\mu \in \mathbb{C}$ ,  $\nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-$ ;  $\sigma \in \mathbb{R}^+$ ,  $\rho < \sigma$ ;  $\rho = \sigma$  and  $s \in \mathbb{C}$  when  $|z| = 1$ , and  $\rho = \sigma$  and  $\operatorname{Re}(s - \mu + \nu) > 1$  when  $|z| > 1$ .

Another generalization of the Hurwitz-Lerch Zeta function  $\phi(z, s, a)$  was studied by Garg *et al.* [11] in the following form

$$(1.4) \quad \Phi_{\lambda, \mu; \nu}(z, s, a) = \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n}{(\nu)_n n!} \frac{z^n}{(n+a)^s},$$

where  $\lambda, \mu, s \in \mathbb{C}$ ,  $\nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-$  when  $|z| < 1$ , and  $\operatorname{Re}(s + \nu - \lambda - \mu) > 1$  when  $|z| = 1$ . Here  $(\alpha)_k$  is the Pochhammer symbol defined by,  $(\alpha)_0 = 1$ ,  $(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$ , ( $k \in \mathbb{N}$ ). Note that for  $\lambda = \nu$  and  $\mu = 1$ , then (1.4) yields the general Hurwitz-Lerch Zeta function  $\phi(z, s, a)$  defined by (1.2) and for  $\lambda = \nu$ , we find that (1.4) reduces to the function  $\phi_{\mu}^*(z, s, a)$  studied by Goyal and Laddha [12] (see also [13]). The generalized Hurwitz-Lerch Zeta function  $\Phi_{\lambda, \mu; \nu}(z, s, a)$  has further been generalized by Srivastava *et al.* [26]. One may refer to monographs by Srivastava and Choi [22, 25] for further details and recent developments on zeta and  $q$ -zeta functions.

Recently, Prajapat and Bulboacă [20] introduced a linear operator  $\mathcal{J}_{\lambda, \mu; \nu}^{s, a}$ , which is defined by means of following Hadamard (or convolution) product

$$(1.5) \quad \mathcal{J}_{\lambda, \mu; \nu}^{s, a}(f)(z) = \mathcal{G}_{\lambda, \mu; \nu}^{s, a}(z) * f(z), \quad z \in \mathbb{D},$$

where  $\lambda, \mu, s \in \mathbb{C}$ ,  $\nu, a \in \mathbb{C} \setminus \mathbb{Z}_0^-$  and  $f \in \mathcal{A}$ , while the function  $\mathcal{G}_{\lambda, \mu; \nu}^{s, a}$  is defined by

$$(1.6) \quad \begin{aligned} \mathcal{G}_{\lambda, \mu; \nu}^{s, a}(z) &= \frac{\nu(1+a)^s}{\lambda\mu} [\Phi_{\lambda, \mu; \nu}(z, s, a) - a^{-s}] \\ &= z + \sum_{n=2}^{\infty} \frac{(\lambda+1)_{n-1}(\mu+1)_{n-1}}{(\nu+1)_{n-1} n!} \left(\frac{1+a}{n+a}\right)^s z^n, \quad z \in \mathbb{D}. \end{aligned}$$

Now, by using (1.6) in (1.5), we get

$$(1.7) \quad \mathcal{J}_{\lambda, \mu; \nu}^{s, a} f(z) = z + \sum_{n=2}^{\infty} \frac{(\lambda+1)_{n-1}(\mu+1)_{n-1}}{(\nu+1)_{n-1} n!} \left(\frac{1+a}{n+a}\right)^s a_n z^n, \quad z \in \mathbb{D}.$$

Note that, the operator  $\mathcal{J}_{\lambda, \mu; \nu}^{s, a}$  is well-defined for  $\lambda, \mu, s \in \mathbb{C}$  and  $\nu, a \in \mathbb{C} \setminus \mathbb{Z}^-$ . Also, the operator  $\mathcal{J}_{\lambda, \mu; \nu}^{s, a}$  generalize several familiar operators studied by Noor and Bukhari [19], Choi *et al.* [8], Wang *et al.* [27], Srivastava and Attiya [23], Cho and Srivastava [7], Jung *et al.* [14], Bernardi [5], Carlson and Shaffer [6], Dziok and Srivastava [9, 10] and Srivastava [24]. Further, general families of convolution operators such as Dziok-Srivastava operator [9] and Srivastava-Wright operator [24] can be obtained from the operator  $\mathcal{J}_{\lambda, \mu; \nu}^{s, a}$  with the use of convolution of analytic functions.

It is readily verified from (1.7) that

$$(1.8) \quad z \left( \mathcal{J}_{\lambda, \mu; \nu}^{s+1, a} f(z) \right)' = (a+1) \mathcal{J}_{\lambda, \mu; \nu}^{s, a} f(z) - a \mathcal{J}_{\lambda, \mu; \nu}^{s+1, a} f(z),$$

$$(1.9) \quad z \left( \mathcal{J}_{\lambda, \mu; \nu}^{s, a} f(z) \right)' = (\lambda+1) \mathcal{J}_{\lambda+1, \mu; \nu}^{s, a} f(z) - \lambda \mathcal{J}_{\lambda, \mu; \nu}^{s, a} f(z),$$

$$(1.10) \quad z \left( \mathcal{J}_{\lambda, \mu; \nu+1}^{s, a} f(z) \right)' = (\nu+1) \mathcal{J}_{\lambda, \mu; \nu}^{s, a} f(z) - \nu \mathcal{J}_{\lambda, \mu; \nu+1}^{s, a} f(z).$$

Let  $\Omega$  and  $\Delta$  be any set in  $\mathbb{C}$ , let  $p$  be an analytic function in  $\mathbb{D}$  with  $p(0) = 1$  and let  $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$ . Miller and Mocanu [17] studied implications of the form

$$(1.11) \quad \{ \psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{D} \} \subset \Omega \implies p(\mathbb{D}) \subset \Delta.$$

If  $\Delta$  is a simply connected domain containing the point  $a$  and  $\Delta \neq \mathbb{C}$ , then the Riemann mapping theorem ensures that there is a conformal mapping  $q$  of  $\mathbb{D}$  onto  $\Delta$  such that  $q(0) = a$ . In this case (1.11) can be rewritten as

$$(1.12) \quad \{ \psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{D} \} \subset \Omega \implies p(z) \prec q(z).$$

Further, if  $\Omega$  is a simply connected domain and  $\Omega \neq \mathbb{C}$ , then there is a conformal mapping  $h$  of  $\mathbb{D}$  onto  $\Omega$  such that  $h(0) = \psi(a, 0, 0; 0)$ . If in addition, the function  $\psi(p(z), zp'(z), z^2p''(z); z)$  is analytic in  $\mathbb{D}$ , then (1.12) can be rewritten as

$$(1.13) \quad \psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \implies p(z) \prec q(z).$$

To prove our main results, we need the following definitions and lemmas.

**Definition 1.1.** ([17, Definition 2.3a, p. 27]) Let  $\Omega$  be a set in  $\mathbb{C}$ ,  $q \in \mathcal{Q}$  and  $n$  be a positive integer. The class of admissible functions  $\Psi_n[\Omega, q]$  consists of those functions  $\psi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$  that satisfy the admissibility condition  $\psi(r, s, t; z) \notin \Omega$ , whenever

$$r = q(\zeta), \quad s = k\zeta q'(\zeta) \quad \text{and} \quad \Re\left(\frac{t}{s} + 1\right) \geq k \Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right)$$

for  $z \in \mathbb{D}$ ,  $\zeta \in \partial\mathbb{D} \setminus E(q)$  and  $k \geq n$ . In particular,  $\Psi_1[\Omega, q] \equiv \Psi[\Omega, q]$ .

**Definition 1.2.** ([18, Definition 3, p. 817]) Let  $\Omega$  be a set in  $\mathbb{C}$  and  $q \in \mathcal{H}[a, n]$  with  $q'(z) \neq 0$ . The class of admissible functions  $\Psi'_n[\Omega, q]$  consists of those function  $\psi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$  that satisfy the admissibility condition  $\psi(r, s, t; \zeta) \in \Omega$ , whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m} \quad \text{and} \quad \Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} \Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

for  $z \in \mathbb{D}$ ,  $\zeta \in \partial\mathbb{D} \setminus E(q)$  and  $m \geq n \geq 1$ . In particular,  $\Psi'_1[\Omega, q] \equiv \Psi'[\Omega, q]$ .

**Lemma 1.1.** ([17, Definition 2.3b, p. 28]) Let  $\psi \in \Psi_n[\Omega, q]$  with  $q(0) = a$ . If  $p \in \mathcal{H}[a, n]$  satisfies

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,$$

then  $p(z) \prec q(z)$ .

**Lemma 1.2.** ([18, Theorem 1, p. 887]) Let  $\psi \in \Psi'_n[\Omega, q]$  with  $q(0) = a$ . If  $p \in \mathcal{Q}_a$  and  $\psi(p(z), zp'(z), z^2p''(z); z)$  is univalent in  $\mathbb{D}$ , then

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{D}\}$$

implies  $q(z) \prec p(z)$ .

In this article, for suitable defined classes of admissible functions, involving the operator  $\mathcal{J}_{\lambda, \mu; \nu}^{\sigma, a}$ , we study the implications of the form (1.12)–(1.13). Through the simple algebraic check of admissible functions, we get various subordination, superordination and differential inequalities that would be difficult to obtain directly. Aghalary *et al.* [1], Ali *et al.* [2, 3], Baricz *et al.* [4], Kim and Srivastava [15], Xiang *et al.* [28] and Soni *et al.* [21] have considered similar problem.

### 2. Subordination Results

We define the following class of admissible functions that will be required in our first result.

**Definition 2.1.** Let  $\Omega$  be a set in  $\mathbb{C}$  and  $q \in \mathcal{Q}_1$ . The class of admissible functions  $\Phi(\Omega, q)$  consists of functions  $\phi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$  that satisfy the admissibility condition  $\phi(u, v, w; z) \notin \Omega$ , whenever  $u = q(\zeta)$ ,  $v = (aq(\zeta) + k\zeta q'(\zeta))/(a + 1)$  and

$$\Re\left(\frac{(a + 1)^2 w - a^2 u}{(a + 1)v - au} - 2a\right) \geq k \Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

for  $z \in \mathbb{D}$ ,  $\zeta \in \partial\mathbb{D} \setminus E(q)$ ,  $k \geq 1$ .

**Theorem 2.1.** Let  $\phi \in \Phi(\Omega, q)$ . If  $f \in \mathcal{A}$  satisfies

$$(2.1) \quad \left\{ \phi\left(J_{\lambda, \mu, \nu}^{s+2, a} f(z), J_{\lambda, \mu, \nu}^{s+1, a} f(z), J_{\lambda, \mu, \nu}^{s, a} f(z); z\right) : z \in \mathbb{D} \right\} \subset \Omega,$$

then  $J_{\lambda, \mu, \nu}^{s+2, a} f(z) \prec q(z)$ ,  $z \in \mathbb{D}$ .

*Proof.* Define the analytic function  $p$  in  $\mathbb{D}$  by

$$(2.2) \quad p(z) = J_{\lambda, \mu, \nu}^{s+2, a} f(z)$$

and assume that  $p \not\prec q$ . Differentiating (2.2) with respect to  $z$  and using (1.8), we get

$$(2.3) \quad J_{\lambda, \mu, \nu}^{s+1, a} f(z) = \frac{zp'(z) + ap(z)}{a + 1}$$

and

$$(2.4) \quad J_{\lambda, \mu, \nu}^{s, a} f(z) = \frac{z^2 p''(z) + (1 + 2a)zp'(z) + a^2 p(z)}{(a + 1)^2}.$$

Define the transformation from  $\mathbb{C}^3$  to  $\mathbb{C}$  by

$$(2.5) \quad u = r, \quad v = \frac{s + ar}{a + 1} \quad \text{and} \quad w = \frac{t + (1 + 2a)s + a^2 r}{(a + 1)^2}.$$

where  $r = p(z)$ ,  $s = zp'(z)$ ,  $t = z^2 p''(z)$ . Let

$$(2.6) \quad \begin{aligned} \psi(r, s, t; z) &= \phi(u, v, w; z) \\ &= \phi\left(r, \frac{s + ar}{a + 1}, \frac{t + (1 + 2a)s + a^2 r}{(a + 1)^2}; z\right), \end{aligned}$$

using (2.2)–(2.4) in (2.6), we get

$$(2.7) \quad \psi\left(p(z), zp'(z), z^2 p''(z); z\right) = \phi\left(J_{\lambda, \mu, \nu}^{s+2, a} f(z), J_{\lambda, \mu, \nu}^{s+1, a} f(z), J_{\lambda, \mu, \nu}^{s, a} f(z); z\right).$$

Hence (1) becomes  $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ . To complete the proof, we need to show that the admissibility condition for  $\phi \in \Phi(\Omega, q)$  is equivalent to the admissibility condition for  $\psi$  as given in Definition 2.1. In view of (2.5), we note that

$$\frac{t}{s} + 1 = \frac{(a+1)^2w - a^2u}{(a+1)v - au} - 2a,$$

then

$$\Re \left( \frac{(a+1)^2w - a^2u}{(a+1)v - au} - 2a \right) = \Re \left( \frac{t}{s} + 1 \right) \geq k \Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right), \quad z \in \mathbb{D}, \quad \zeta \in \partial\mathbb{D} \setminus E(q).$$

By Lemma 1.1, we have  $\psi(p(z), zp'(z), z^2p''(z); z) \notin \Omega$ , which contradicts (2.1). Thus we must have  $p(z) \prec q(z)$ . This completes the proof of Theorem 2.1.  $\square$

If  $\Omega \neq \mathbb{C}$  is a simply connected domain, then  $\Omega = h(\mathbb{D})$  for some conformal mapping  $h$  of  $\mathbb{D}$  onto  $\Omega$ . In this case, the class  $\Phi(h(\mathbb{D}), q)$  is written as  $\Phi(h, q)$ . The following result is an immediate consequence of Theorem 2.1.

**Theorem 2.2.** *Let  $\phi \in \Phi(h, q)$  with  $q(0) = 1$ . If  $f \in \mathcal{A}$  satisfies*

$$(2.8) \quad \phi \left( J_{\lambda, \mu, \nu}^{s+2, a} f(z), J_{\lambda, \mu, \nu}^{s+1, a} f(z), J_{\lambda, \mu, \nu}^{s, a} f(z); z \right) \prec h(z), \quad z \in \mathbb{D},$$

then  $J_{\lambda, \mu, \nu}^{s+2, a} f(z) \prec q(z)$ .

Our next result is an extension of Theorem 2.1 to the case when the behaviour of  $q$  on  $\partial\mathbb{D}$  is not known.

**Corollary 2.1.** *Let  $\Omega \subset \mathbb{C}$  and  $q$  is univalent in  $\mathbb{D}$  with  $q(0) = 1$ . Let  $\phi \in \Phi(\Omega, q_\rho)$ , where  $q_\rho(z) = q(\rho z)$ ,  $0 < \rho < 1$ . If  $f \in \mathcal{A}$  satisfies*

$$(2.9) \quad \phi \left( J_{\lambda, \mu, \nu}^{s+2, a} f(z), J_{\lambda, \mu, \nu}^{s+1, a} f(z), J_{\lambda, \mu, \nu}^{s, a} f(z); z \right) \in \Omega$$

then  $J_{\lambda, \mu, \nu}^{s+2, a} f(z) \prec q(z)$ .

*Proof.* Theorem 2.1 yields that under the hypothesis  $J_{\lambda, \mu, \nu}^{s+2, a} f(z) \prec q_\rho(z)$ ,  $0 < \rho < 1$ . Since  $q$  is univalent in  $\mathbb{D}$ , then by definition  $q_\rho(z) \prec q(z)$ . Hence the result.  $\square$

**Theorem 2.3.** *Let  $h, q \in \mathcal{S}$  with  $q(0) = 1$ . Also set  $q_\rho(z) = q(\rho z)$  and  $h_\rho(z) = h(\rho z)$ , where  $0 < \rho < 1$ . Suppose that  $\phi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$  satisfies one of the following conditions:*

- (i)  $\phi \in \Phi(h, q_\rho)$ , or
- (ii) there exist  $\rho_0 \in (0, 1)$  such that  $\phi \in \Phi(h_\rho, q_\rho)$  for all  $\rho \in (\rho_0, 1)$ .

If  $f \in \mathcal{A}$  and satisfying (2.9), then  $J_{\lambda, \mu, \nu}^{s+2, a} f(z) \prec q(z)$ .

*Proof.* From Corollary 2.1, we have  $J_{\lambda, \mu, \nu}^{s+2, a} f(z) \prec q_\rho(z)$ . Since  $q_\rho(z) \prec q(z)$ , hence

we deduce that  $J_{\lambda,\mu,\nu}^{s+2,a} f(z) \prec q(z)$ . Further, if we let  $J_{\lambda,\mu,\nu}^{s+2,a} f_{\rho}(z) = J_{\lambda,\mu,\nu}^{s+2,a} f(\rho z)$ , then

$$\begin{aligned} &\phi\left(J_{\lambda,\mu,\nu}^{s+2,a} f_{\rho}(z), J_{\lambda,\mu,\nu}^{s+1,a} f_{\rho}(z), J_{\lambda,\mu,\nu}^{s,a} f_{\rho}(z); z\right) \\ &= \phi\left(J_{\lambda,\mu,\nu}^{s+2,a} f(\rho z), J_{\lambda,\mu,\nu}^{s+1,a} f(\rho z), J_{\lambda,\mu,\nu}^{s,a} f(\rho z); \rho z\right) \in h_{\rho}(\mathbb{D}). \end{aligned}$$

It is easy to see that Theorem 2.1 holds, if condition (1) is replaced by

$$(2.10) \quad \phi\left(J_{\lambda,\mu,\nu}^{s+2,a} f(w(z)), J_{\lambda,\mu,\nu}^{s+1,a} f(w(z)), J_{\lambda,\mu,\nu}^{s,a} f(w(z)); w(z)\right) \in \Omega,$$

where  $w$  is a mapping from  $\mathbb{D}$  to  $\mathbb{D}$ . If we take  $w(z) = \rho z$ ,  $0 < \rho < 1$ , in (2.10), then we obtain

$$J_{\lambda,\mu,\nu}^{s+2,a} f_{\rho}(z) \prec q_{\rho}(z), \quad \rho \in (0, 1).$$

Now by letting  $\rho \rightarrow 1^-$ , we obtain that  $J_{\lambda,\mu,\nu}^{s+2,a} f(z) \prec q(z)$ . This completes the proof of theorem.  $\square$

The next theorem yields the best dominant of the differential subordination.

**Theorem 2.4.** *Let  $h \in \mathcal{S}$ . Suppose that  $\phi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$  and the differential equation*

$$(2.11) \quad \phi\left(q(z), \frac{zq'(z) + aq(z)}{a+1}, \frac{z^2q''(z) + (1+2a)zq'(z) + a^2q(z)}{(a+1)^2}; z\right) = h(z)$$

has a solution  $q$  with  $q(0) = 1$ , which satisfies one of the following conditions:

- (i)  $q \in \mathcal{Q}_1$  and  $\phi \in \Phi(h, q)$ ,
- (ii)  $q \in \mathcal{S}$  and  $\phi \in \Phi(h, q_{\rho})$ , for some  $\rho \in (0, 1)$ , or
- (iii)  $q \in \mathcal{S}$  and there exist  $\rho_0 \in (0, 1)$  such that  $\phi \in \Phi(h_{\rho}, q_{\rho})$  for all  $\rho \in (\rho_0, 1)$ .

If  $f \in \mathcal{A}$  satisfies (2.8), then  $J_{\lambda,\mu,\nu}^{s+2,a} f(z) \prec q(z)$ , and  $q$  is the best dominant.

*Proof.* By applying Theorem 2.2 and Theorem 2.3, we deduce that  $q(z)$  is a dominant of (2.8). Since  $q$  satisfies (2.11), it is also a solution of (2.8) and therefore  $q$  will be dominated by all dominants of (2.8). Hence  $q$  is the best dominant.  $\square$

In the particular case  $q(z) = Mz$ ,  $M > 0$ , and in view of Definition 2.1, the class of admissible functions  $\Phi(\Omega, q)$  is denoted by  $\Phi(\Omega, M)$ , as described below.

**Definition 2.2.** Let  $\Omega$  be a set in  $\mathbb{C}$  and  $M > 0$ . The class of admissible function  $\Phi(\Omega, M)$  consists of those functions  $\phi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$  such that

$$(2.12) \quad \phi\left(Me^{i\theta}, \frac{k+a}{a+1}Me^{i\theta}, \frac{L + \{a^2 + (2a+1)k\}Me^{i\theta}}{(a+1)^2}; z\right) \notin \Omega,$$

whenever  $z \in \mathbb{D}$ ,  $\Re(Le^{-i\theta}) \geq Mk(k-1)$ , for all real  $\theta$ , and  $k \geq 1$ .

**Definition 2.3.** Let  $\Omega_1$  be a set in  $\mathbb{C}$  and  $q \in \Omega_1$ . Let  $\tilde{\Phi}(\Omega, q)$  denote the class of admissible functions  $\tilde{\phi} : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$  that satisfy  $\tilde{\phi}(u, v, w; z) \notin \Omega$ , whenever

$$u = q(\zeta), \quad v = \frac{\lambda q(\zeta) + k\zeta q'(\zeta)}{\lambda + 1}$$

and

$$\Re \left\{ \frac{(\lambda + 1)(\lambda + 2)w - \lambda(\lambda + 1)u}{(\lambda + 1)v - \lambda u} - 2\lambda - 1 \right\} \geq k \Re \left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\},$$

for  $z \in \mathbb{D}$ ,  $\zeta \in \partial\mathbb{D} \setminus E(q)$ ,  $k \geq 1$ .

**Theorem 2.5.** Let  $\tilde{\phi} \in \tilde{\Phi}(\Omega, q)$  with  $q(0) = 1$ . If  $f \in \mathcal{A}$  satisfies

$$(2.13) \quad \tilde{\phi} \left( J_{\lambda, \mu, \nu}^{s, a} f(z), J_{\lambda+1, \mu, \nu}^{s, a} f(z), J_{\lambda+2, \mu, \nu}^{s, a} f(z); z \right) \in \Omega, \quad z \in \mathbb{D},$$

then  $J_{\lambda, \mu, \nu}^{s, a} f(z) \prec q(z)$ .

*Proof.* Define an analytic function by  $p(z) = J_{\lambda, \mu, \nu}^{s, a} f(z)$ . Differentiating  $p(z)$  with respect to  $z$ , using (1.9) and following similar steps as in proof of Theorem 2.1, along with Definition 2.3, we get the desired result.  $\square$

**Theorem 2.6.** Let  $\tilde{\phi} \in \tilde{\Phi}(h, q)$  with  $q(0) = 1$ . If  $f \in \mathcal{A}$  satisfies

$$\tilde{\phi} \left( J_{\lambda, \mu, \nu}^{s, a} f(z), J_{\lambda+1, \mu, \nu}^{s, a} f(z), J_{\lambda+2, \mu, \nu}^{s, a} f(z); z \right) \prec h(z), \quad z \in \mathbb{D},$$

then  $J_{\lambda, \mu, \nu}^{s, a} f(z) \prec q(z)$ .

**Definition 2.4.** Let  $\Omega$  be a set in  $\mathbb{C}$  and  $q \in \Omega_1$ . The class of admissible functions  $\Theta(\Omega, q)$  consists of functions  $\phi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$  that satisfy the admissibility condition  $\phi(u, v, w; z) \notin \Omega$ , whenever

$$u = q(\zeta), \quad v = \frac{k}{(a + 1)} \frac{\zeta q'(\zeta)}{q(\zeta)} + q(\zeta) \quad (q(\zeta) \neq 0)$$

and

$$\Re \left( (a + 1) \left\{ \frac{wv - u^2}{v - u} - 3u \right\} \right) \geq k \Re \left( 1 + \frac{\zeta q''(\zeta)}{q'(\zeta)} \right),$$

for  $z \in \mathbb{D}$ ,  $\zeta \in \partial\mathbb{D} \setminus E(q)$  and  $k \geq 1$ .

**Theorem 2.7.** Let  $\phi \in \Theta(\Omega, q)$ . If  $f \in \mathcal{A}$  satisfies

$$(2.14) \quad \left\{ \phi \left( \frac{J_{\lambda, \mu, \nu}^{s+2, a} f(z)}{J_{\lambda, \mu, \nu}^{s+3, a} f(z)}, \frac{J_{\lambda, \mu, \nu}^{s+1, a} f(z)}{J_{\lambda, \mu, \nu}^{s+2, a} f(z)}, \frac{J_{\lambda, \mu, \nu}^{s, a} f(z)}{J_{\lambda, \mu, \nu}^{s+1, a} f(z)}; z \right) : z \in \mathbb{D} \right\} \subset \Omega,$$

then

$$\frac{J_{\lambda, \mu, \nu}^{s+2, a} f(z)}{J_{\lambda, \mu, \nu}^{s+3, a} f(z)} \prec q(z).$$



*Proof.* Define the analytic function

$$(2.15) \quad p(z) = \frac{J_{\lambda, \mu, \nu}^{s+2, a} f(z)}{J_{\lambda, \mu, \nu}^{s+3, a} f(z)}$$

and assume that  $p \not\prec q$ . Differentiating (2.15) with respect to  $z$  and making use of (1.9) and following similar steps as in proof of Theorem 2.1, along with Definition 2.4, we get the desired result.  $\square$

### 3. Superordination Results

In this section we investigate differential superordination results. First, we consider the following class of admissible function.

**Definition 3.1.** Let  $\Omega$  be a set in  $\mathbb{C}$  and  $q \in \mathcal{H}_1$  with  $q'(z) \neq 0$ . The class of admissible functions  $\Phi'(\Omega, q)$  consists of functions  $\phi : \mathbb{C}^3 \times \mathbb{D} \rightarrow \mathbb{C}$  that satisfy  $\phi(u, v, w; \zeta) \in \Omega$ , whenever

$$u = q(z), \quad v = \frac{zq'(z) + maq(z)}{m(a+1)}$$

and

$$\Re \left\{ \frac{(a+1)^2 w - a^2 u}{(a+1)v - au} - 2a \right\} \leq \frac{1}{m} \Re \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$

for  $z \in \mathbb{D}$ ,  $\zeta \in \partial\mathbb{D}$  and  $m \geq 1$ .

**Theorem 3.1.** Let  $\phi \in \Phi'(\Omega, q)$ . If  $f \in \mathcal{A}$ ,  $J_{\lambda, \mu, \nu}^{s+2, a} f(z) \in \Omega_1$  and

$$\phi \left( J_{\lambda, \mu, \nu}^{s+2, a} f(z), J_{\lambda, \mu, \nu}^{s+1, a} f(z), J_{\lambda, \mu, \nu}^{s, a} f(z); z \right)$$

is univalent in  $\mathbb{D}$ , then

$$(3.1) \quad \Omega \in \phi \left( J_{\lambda, \mu, \nu}^{s+2, a} f(z), J_{\lambda, \mu, \nu}^{s+1, a} f(z), J_{\lambda, \mu, \nu}^{s, a} f(z); z \right), \quad z \in \mathbb{D},$$

implies  $q(z) \prec J_{\lambda, \mu, \nu}^{s+2, a} f(z)$ .

If  $\Omega \neq \mathbb{C}$  is simply connected domain, then  $\Omega = h(\mathbb{D})$  for some conformal mapping  $h(z)$  of  $\mathbb{D}$  onto  $\Omega$ , and then the class  $\Phi'(h(\mathbb{D}), q)$  is written as  $\Phi'(h, q)$ . Proceeding as in the previous section, the following result is an immediate consequence of Theorem 3.1.

**Theorem 3.2.** Let  $q \in \mathcal{H}_1$ ,  $h$  be univalent in  $\mathbb{D}$  and  $\phi \in \Phi'(h, q)$ . If  $f(z) \in \mathcal{A}$ ,  $J_{\lambda, \mu, \nu}^{s+2, a} f(z) \in \Omega_1$  and

$$\phi \left( J_{\lambda, \mu, \nu}^{s+2, a} f(z), J_{\lambda, \mu, \nu}^{s+1, a} f(z), J_{\lambda, \mu, \nu}^{s, a} f(z); z \right)$$

is univalent in  $\mathbb{D}$ , then

$$(3.2) \quad h(z) \prec \phi \left( J_{\lambda, \mu, \nu}^{s+2, a} f(z), J_{\lambda, \mu, \nu}^{s+1, a} f(z), J_{\lambda, \mu, \nu}^{s, a} f(z); z \right)$$

implies that

$$q(z) \prec J_{\lambda, \mu, \nu}^{s+2, a} f(z).$$

The following theorems prove the existence of the best subordination of (3.1) and (3.3) for an appropriate  $\phi$ . The proof is similar of Theorem 2.4 and is therefore omitted.

**Theorem 3.3.** Let  $h \in \mathcal{H}_1$  and  $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \rightarrow \mathbb{C}$ . Suppose that the differential equation

$$\phi \left( q(z), \frac{zq'(z) + aq(z)}{a+1}, \frac{z^2q''(z) + (1+2a)zq'(z) + a^2q(z)}{(a+1)^2}; z \right) = h(z)$$

has a solution  $q(z) \in \mathcal{Q}_1$ . If  $\phi \in \Phi'(h, q)$ ,  $f \in \mathcal{A}$ ,  $J_{\lambda, \mu, \nu}^{s+2, a} f(z) \in \mathcal{Q}_1$  and

$$\phi \left( J_{\lambda, \mu, \nu}^{s+2, a} f(z), J_{\lambda, \mu, \nu}^{s+1, a} f(z), J_{\lambda, \mu, \nu}^{s, a} f(z); z \right)$$

is univalent in  $\mathbb{D}$ , then

$$h(z) \prec \phi \left( J_{\lambda, \mu, \nu}^{s+2, a} f(z), J_{\lambda, \mu, \nu}^{s+1, a} f(z), J_{\lambda, \mu, \nu}^{s, a} f(z); z \right)$$

implies

$$q(z) \prec J_{\lambda, \mu, \nu}^{s+2, a} f(z),$$

and  $q$  is the best subordinant.

By combining Theorem 2.2 and Theorem 3.2, we obtain the following sandwich-type theorem.

**Corollary 3.1.** Let  $h_1, q_1 \in \mathcal{H}_1$ ,  $h_2$  is univalent in  $\mathbb{D}$ . Suppose also that  $q_2 \in \mathcal{Q}_1$  with  $q_1(0) = q_2(0) = 1$  and  $\phi \in \Phi(h_2, q_2) \cap \Phi'(h_1, q_1)$ . If  $f(z) \in \mathcal{A}$ ,  $J_{\lambda, \mu, \nu}^{s+2, a} f(z) \in \mathcal{H} \cap \mathcal{Q}_1$  and

$$\phi \left( J_{\lambda, \mu, \nu}^{s+2, a} f(z), J_{\lambda, \mu, \nu}^{s+1, a} f(z), J_{\lambda, \mu, \nu}^{s, a} f(z); z \right)$$

is univalent in  $\mathbb{D}$ , then

$$h_1(z) \prec \phi \left( J_{\lambda, \mu, \nu}^{s+2, a} f(z), J_{\lambda, \mu, \nu}^{s+1, a} f(z), J_{\lambda, \mu, \nu}^{s, a} f(z); z \right) \prec h_2(z),$$

implies

$$q_1(z) \prec J_{\lambda, \mu, \nu}^{s+2, a} f(z) \prec q_2(z).$$

Similarly, for well defined classes of admissible functions, we can establish dual results of Theorem 2.5 and Theorem 2.7 and would obtain further sandwich type results. These consideration can be fruitfully worked out and we skip the details in this regards.

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